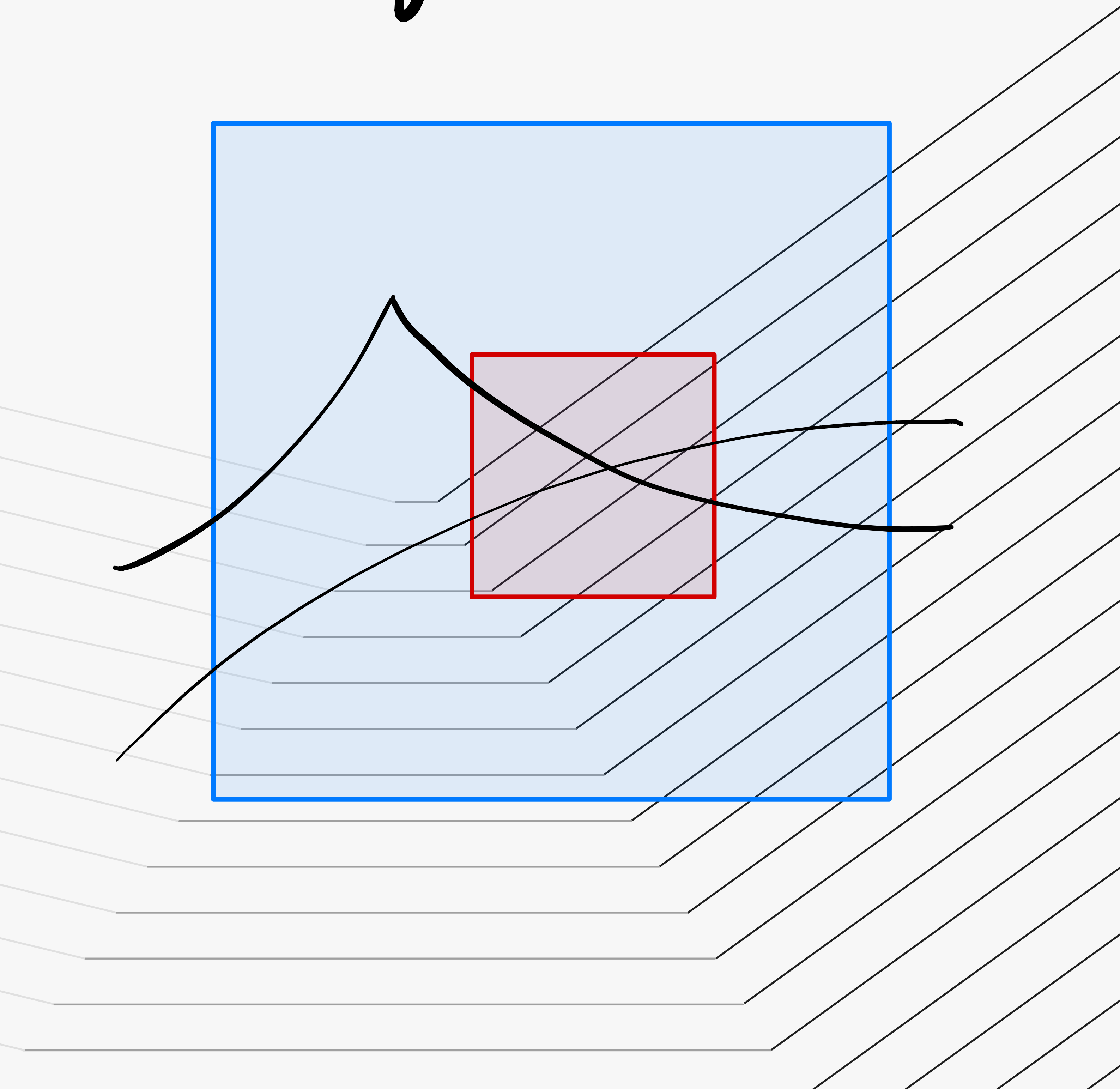
Lecture 2.

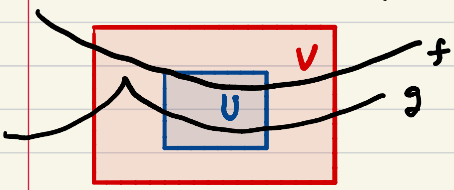
Presheaf & Sheaf.



Outline
- Presheaf, germ and stalk.
- sheafication.
- sheaf homomorphism $\begin{cases} \text{injective} \\ \text{surjective} \\ \text{isomorphism} \\ \text{exact sequence} \end{cases}$
- Examples.

- Throughout, X will be a topological space.

- Motivation: consider open sets $U \subseteq V \subseteq \mathbb{R}^n$.



f smooth on V, then certainly f is smooth on U.
g is not smooth on V but it is smooth on U.

So the information of smooth functions on U is <u>richer</u> than on V.

- Def (Presheaf of abelian group/ring/module)
A presheaf $\mathcal{F}$ over X associates each open $U \subseteq X$ a group/ring/module $\mathcal{F}(U)$ s.t. the following holds:

① For $\forall$ $V \subseteq U$, $\exists$ a homomorphism
$$r^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$$
s.t. $r^U_U = id$.

约定: $\mathcal{F}(\phi) = 0$.

② For $\forall$ $W \subseteq V \subseteq U$ one has
$$r^V_W \circ r^U_V = r^U_W.$$

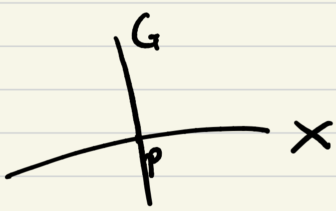
- $\forall$ $s \in \mathcal{F}(U)$ is called a section of $\mathcal{F}$ on U.
put $r^U_V(s) =: s|_V$.

- Example
① Constant presheaf. Let G be a group. Define a presheaf $\mathcal{F}$ in the following way: for $\forall$ open U let
$$\mathcal{F}(U) := G.$$
For $\forall$ $\phi \neq V \subseteq U$, let $r^U_V = id$

not a sheaf in general

$\mathcal{F}_x = G$ $\forall$ $x \in X$

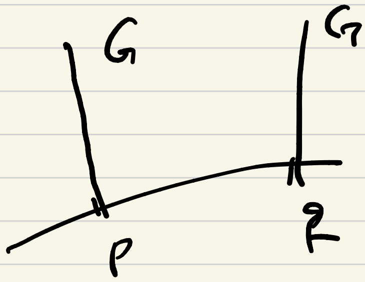
② Skyscraper: let $p \in X$. Assume X Hausdorff. Let G be a group.
$$\mathcal{F}(U) := \begin{cases} G, & p \in U \\ 0 & p \notin U. \end{cases}$$



$$\mathcal{F}_x = \begin{cases} 0 & x \neq p \\ G & x = p \end{cases}$$

③ Two skyscrapers : Given $p \neq q \in X$. Assume X Hausdorff.

Let G be a group.



$$\mathcal{F}(U) := \begin{cases} G, & p \text{ or } q \in U \\ 0 & p \,\&\, q \notin U. \end{cases}$$

But this is not sheaf.

$$\mathcal{F}_x = \begin{cases} 0 & x \neq p \text{ or } q \\ G & x = p \\ G & x = q \end{cases}$$

④ presheaf of functions.

Let $X := \Omega \subseteq \mathbb{R}^n$ domain in $\mathbb{R}^n$.

For $\forall$ open $U \subseteq \Omega$, let

$$\mathcal{F}(U) := \{ \text{continuous functions on } U \}.$$

$\uparrow$ C^1, smooth, analytic, bounded.

⑤ presheaf of holomorphic/meromorphic functions.

Let $X := \Omega \subseteq \mathbb{C}^n$ be a domain in $\mathbb{C}^n$.

For $\forall$ open $U \subseteq \Omega$, let

$$\mathcal{O}(U) := \{ \text{holo funct. on } U \}.$$

$$\mathcal{O}_x = \{ \text{convergent power series around } x \}$$

$$\mathcal{M}(U) := \{ \text{meromorphic funct. on } U \}.$$

$$\mathcal{O}^*(U) := \{ \text{holomorphic funct. w/ no zero on } U \}$$

$$\mathcal{B}(U) := \{ \text{bounded holomorphic function on } U \}.$$

⑥ Ideal presheaf. Let $X := \mathbb{C}$.

$$I(U) := \begin{cases} \mathcal{O}(U), & 0 \notin U \\ \{ f \in \mathcal{O}(U) \mid f(0) = 0 \}, & 0 \in U \end{cases}$$

$$I_x = \mathcal{O}_x, \quad x \neq 0.$$

$I(U)$ is an ideal of $\mathcal{O}(U)$.

$$I_x = \{ f \in \mathcal{O}_x \mid f(0) = 0 \}$$

• **Def** A sheaf $\mathcal{F}$ is a presheaf which satisfies the following two additional axioms

(A) If for $s, t \in \mathcal{F}(U)$ $\exists$ open cover $U = \bigcup_{i \in I} U_i$ s.t. $s|_{U_i} = t|_{U_i}$ for $\forall i \in I$ then $s = t$.

(B) For open cover $U = \bigcup_{i \in I} U_i$ & $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for $\forall\, i, j \in I$ $\exists$ $s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$ $\forall i$.

▲ (A) means that $\forall$ section is uniquely determined by its local information

(B) says that $\forall$ compatible local sections glues together to a global section.

In the above examples, check that

③ two-skyscraper is not a sheaf.

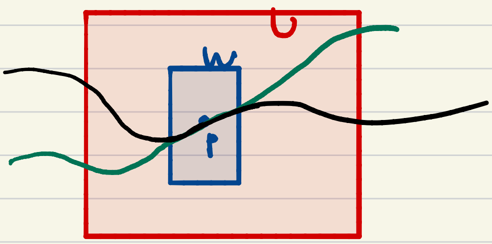
⑤ $\mathcal{B}$ is not a sheaf

• Germ & Stalk. Let $\mathcal{F}$ be a presheaf.

Let $p \in X$ be a point. For $\forall$ two open U & V both containing p We say $s \in \mathcal{F}(U)$ & $t \in \mathcal{F}(V)$ have the same germ at p if $\exists\, W \subseteq U \cap V$ containing p s.t. $s|_W = t|_W$.

In other words, a germ at p is an equivalence class of sections: $\{ s_U \in \mathcal{F}(U) \mid p \in U \} / \sim$ where

$$s_U \sim s_V \Leftrightarrow \exists_{p \in} W \subseteq U \cap V \text{ s.t. } s_U|_W = s_V|_W.$$



e.g. $0 \in \Omega \subseteq \mathbb{C}^n$ connected.

For f & $g \in \mathcal{O}(\Omega)$ w/ the same germ at 0, one must have $f = g$ on Ω

In general, $f \in \mathcal{O}(U)$ & $g \in \mathcal{O}(V)$ have the same germ at $0 \in U \cap V$ iff f & g have the same power series expansion at 0.

$\mathcal{F}_p := \{ \text{germ at } p \}$. This is called the stalk of $\mathcal{F}$ at p.

- Note that for $\forall$ open U containing p, there is a natural map $\mathcal{F}(U) \longrightarrow \mathcal{F}_p$
$$s \longmapsto s_p.$$
s_p is the germ of s at p.

- <u>Ex</u> ① $\mathcal{O}_0 = \{ \text{convergent power series around } 0 \}$.
  ② Compute the stalk of all the previous examples.

- <u>Sheafication</u>

Let $\mathcal{F}$ be a presheaf. Then the sheafication of $\mathcal{F}$, denoted by $\mathcal{F}^+$ is given by

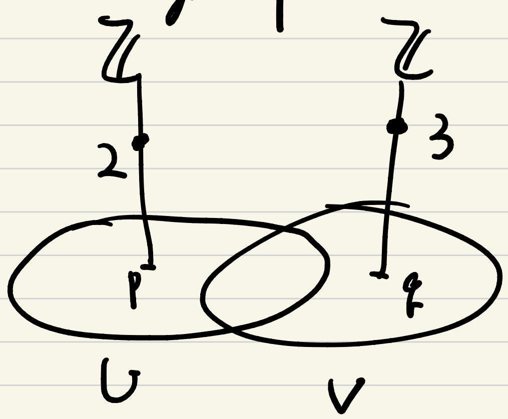
$$\mathcal{F}^+(U) := \left\{ \bigsqcup_{p\in U} s(p) \;\middle|\; \begin{array}{l} \text{for } \forall p \in U \;\; \exists \; p \in V \subseteq U \\ \& \; s \in \mathcal{F}(V) \text{ s.t. } s_q = s(q), \forall q \in V \end{array} \right\}.$$

<u>e.x.</u> Show that $\mathcal{F}^+$ is a sheaf.

So there is a natural inclusion $\mathcal{F}(U) \xhookrightarrow{i} \mathcal{F}^+(U)$.

- e.x. When $\mathcal{F}$ is a sheaf, then one actually has $\mathcal{F}(U) \xrightarrow[\sim]{i} \mathcal{F}^+(U)$

- Example: For two skyscraper



$(U,2)$ $(V,3)$ cannot be glued in $\mathcal{F}(U\cup V)$

$(U,2)$ & $(V,3)$ give rise to a section in $\mathcal{F}^+(U\cup V)$

- In what follows, whenever we meet a presheaf, we will replace it by its sheafication, i.e., we will only deal w/ sheaves in the rest of this course.

- Sheaf morphisms. Let $\mathcal{F}$ & $\mathcal{G}$ be two sheaves. We say φ is a sheaf morphism from $\mathcal{F}$ to $\mathcal{G}$ if for $\forall$ open U, $\exists$ morphism

$$\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$ s.t. the following diagram commutes for $\forall$ $V \subseteq U$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle r_V^U}\downarrow & & \downarrow{\scriptstyle r_V^U} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

We call $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ if φ_U is inclusion for all U.

- Given a sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$.

① $\ker\varphi$ is given by $\ker\varphi(U) := \ker \varphi_U$.

e.x. $\ker\varphi$ is indeed a subsheaf of $\mathcal{F}$.

② $\operatorname{Im}\varphi$ is the sheafication of the presheaf given by $\{\operatorname{Im}\varphi_U\}_U$

- Given a subsheaf $\mathcal{F} \subseteq \mathcal{G}$.

$\mathcal{G}/\mathcal{F}$ is the sheafication associated to $\left\{ \mathcal{G}(U)/\mathcal{F}(U) \right\}_U$

- Def ① We say $\varphi : \mathcal{F} \to \mathcal{G}$ injective if $\ker\varphi = 0$.

② We say $\varphi : \mathcal{F} \to \mathcal{G}$ surjective if $\mathcal{G}/\operatorname{Im}\varphi = 0$.

TFAE

- Prop : ① $\varphi: \mathcal{F} \to \mathcal{G}$ is injective

  ② $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for $\forall\, U$

  ③ $\varphi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is injective for $\forall\, x \in X$.

- Prop : TFAE

  ① $\varphi: \mathcal{F} \to \mathcal{G}$ is surjective

  ② For $\forall\, \tau \in \mathcal{G}(U)$, $\exists$ open cover $U = \cup U_i$ & $s_i \in \mathcal{F}(U_i)$ s.t. $\tau|_{U_i} = \varphi_{U_i}(s_i)$.

  ③ $\varphi_x: \mathcal{F}_x \to \mathcal{G}_x$ is surjective.

- We end this lecture by defining exactness.

  let $\mathcal{F}, \mathcal{G}, \mathcal{Q}$ be sheaves.

  Then $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{Q} \to 0$ is called an short exact sequence if

  ① α is injective

  ② β is surjective

  ③ $\ker\beta = \operatorname{Im}\alpha$ (as sheaves)

- Prop. $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{Q} \to 0$ is exact iff

  $0 \to \mathcal{F}_x \xrightarrow{\alpha_x} \mathcal{G}_x \xrightarrow{\beta_x} \mathcal{Q} \to 0$ is exact for $\forall x$.

*选讲: Coherent sheaves.