Lecture 7 Siegel modular varieties and Shimura varieties of PEL type

§1. Hodge structures

X proj. smooth$/\mathbb{C}$

Let $R = \mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$, a subring of $\mathbb{R}$.

imagine to be $H^n(X(\mathbb{C})^{an}, R)$

A R-Hodge structure is a finite projective R-module V, equipped with a bigrading

$H^q(X, \Omega^p)$ is

$$V_{\mathbb{C}} := V \otimes_R \mathbb{C} \simeq \bigoplus_{p,q} V^{pq} \quad \leftarrow \text{analogous to } H^n(X(\mathbb{C})^{an}, \mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X/\mathbb{C})$$

s.t. $V^{p,q} = \overline{V^{q,p}}$ (for the complex conjugation on $\mathbb{C}$: $V \otimes_R \mathbb{C} \longrightarrow V \otimes_R \mathbb{C}$)

$$v \otimes z \longmapsto v \otimes \bar{z}$$

The pairs (p,q) for which $V^{p,q} \neq 0$ are called types of V, counted with mult. $\dim_{\mathbb{C}} V^{pq}$

Say the Hodge structure is of pure wt n if for all types (p,q) of V, $n = p+q$.

In this case, the bigrading decomposition is the same as the filtration

$$F^p V := \bigoplus_{r \geq p} V^{rs}, \quad \text{as } V^{pq} = F^p V \cap \overline{F^q V}.$$

The bigrading $V_{\mathbb{C}} \simeq \bigoplus_{p,q} V^{pq}$ can be interpreted as an action of

$\mathbb{G}_m \times \mathbb{G}_m$ on $V_{\mathbb{C}}$ s.t. (z,w) acts on V^{pq} via $z^{-p} w^{-q}$

or $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \longrightarrow GL(V_{\mathbb{C}})$ an algebraic group homomorphism.

complex conjugation

induces $(z,w) \mapsto (\bar{w}, \bar{z})$

Taking $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariants, $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}} \longrightarrow GL(V_{\mathbb{R}})$ homomorphism of alg gps$/\mathbb{R}$.

$\mathbb{S} =: \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$

[Weil restriction: $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} G(A) = G(A \otimes_{\mathbb{R}} \mathbb{C})$ for any $\mathbb{R}$-algebra A.

(Rmk: $(\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}})_{\mathbb{C}} \simeq \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$

So, $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$ is a real form of $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$.)

Cor: $\mathbb{S}(\mathbb{R}) \longrightarrow \mathbb{S}(\mathbb{C}) \simeq \mathbb{C}^\times \times \mathbb{C}^\times \curvearrowright V^{pq}$

$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$, $\quad z \longmapsto (z, \bar{z})$ acts by $z^{-p} \bar{z}^{-q}$

Summary: Giving an R-Hodge structure on V is equivalent to giving an $\mathbb{R}$-homomorphism $h: \mathbb{S} \to GL(V_{\mathbb{R}})$.

<u>Example</u>: $V = \mathbb{Z}(1) = 2\pi i \mathbb{Z}$. Hodge type $(-1,-1)$, $V_{\mathbb{C}} \simeq V^{-1,-1}$, weight -2

$$h: \mathbb{S}(\mathbb{R}) \longrightarrow GL(V_{\mathbb{R}}) = \mathbb{R}^\times$$
$$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \qquad z \longmapsto z\bar{z}$$

The $2\pi i$ comes from $\mathbb{Z}(1) \simeq H_1(\mathbb{G}_m)$ & $\mathbb{C}/2\pi i\mathbb{Z} \overset{\exp}{\simeq} \mathbb{C}^\times$

<u>Example</u>: $V = H_1(E(\mathbb{C}), \mathbb{Z}) \overset{\alpha}{\simeq} \mathbb{Z}^{\oplus 2}$

$V_{\mathbb{C}} = H_1(E(\mathbb{C}), \mathbb{C}) \simeq H_1^{dR}(E/\mathbb{C})$, type $(-1,0), (0,-1)$

$$0 \to \underbrace{H^0(E^\vee, \Omega^1_{E^\vee})}_{F^0 V_{\mathbb{C}}} \to H_1^{dR}(E/\mathbb{C}) \longrightarrow \mathrm{Lie}(E/\mathbb{C}) \to 0$$

$H^{0,-1}$ $\qquad$ $\underline{\underline{H^{-1,0}}} \circlearrowleft \mathbb{S}(\mathbb{R}) \ni z$ auto by mult by z.

$F^{-1} H_1^{dR}(E/\mathbb{C}) = H_1^{dR}(E/\mathbb{C})$

$h: \mathbb{S}(\mathbb{R}) \longrightarrow Gl_2(V_{\mathbb{R}}) \iff$ giving $V_{\mathbb{R}}$ a structure of complex vector space

$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ $\qquad$ then $V_{\mathbb{R}}/V$ recovers the elliptic curve

<u>Theorem</u>: $\{$ <u>$\mathbb{Z}$-Hodge structures</u> of rank 2 and type $(-1,0), (0,-1)\}$

$\updownarrow$ bijection

$\{$Elliptic curves$/\mathbb{C}\}$.

<u>Definition</u> A <u>polarization</u> on an $\mathbb{R}$-Hodge structure V of weight n is a morphism of Hodge structures $\psi: V \otimes V \longrightarrow \mathbb{R}(-n)$

s.t. $(2\pi i)^n \psi(x, h(i)y)$ is symmetric and positive definite

(<u>Rmk</u>: X compact Kähler of dim d, $(H^n(X,\mathbb{R})_{prim}, Q)$ $n \leq d$ is a polarized $\mathbb{R}$-Hodge structure of wt n.)

$$Q([\alpha],[\beta]) := \int_X \omega^{d-n} \wedge \alpha \wedge \beta.$$

<u>Note</u>: $\psi(x, h(i)y) = \psi(h(-i)x, y) = (-1)^n \psi(h(i)x, y)$ as ψ is morphism and $h(i)$ is trivial on $\mathbb{R}(-n)$.

$\psi(x, h(i)y) = \psi(y, h(i)x)$ symm.

$\Rightarrow \psi$ is $(-1)^n$-symmetric.

<u>Fact</u>: $\{$Polarized $\mathbb{Z}$-Hodge structures V of rank $2g$ & type $(-1,0)^g$ & $(0,-1)\}$

$\updownarrow$

$\{$ g-dim'l Abelian varieties$/\mathbb{C}\}$

Defn: An abelian variety A over S is a proper smooth group S-scheme all fibers are geometrically connected.

$$h : \underset{\mathbb{C}^\times}{\overset{\|}{\mathbb{S}(\mathbb{R})}} \longrightarrow GSp(V_{\mathbb{R}}, \psi) \simeq GSp_{2g}(\mathbb{R})$$

$h(\mathbb{C}^\times)$ gives $V_{\mathbb{R}}$ a structure of complex vector space

$\rightsquigarrow A(\mathbb{C}) := V_{\mathbb{R}}/V$ (as complex tori)

& ψ gives rise to an ample line bundle on A (see Mumford.)

§2. Siegel modular varieties.

Quick blackboxs : A/S abelian variety, $\quad A \underset{\pi}{\overset{e}{\rightleftarrows}} S \quad$ (Reference : Mumford, AV.)

$$\mathrm{Pic}(A) : Sch/_S \longrightarrow Sets$$

$$T \longmapsto \{\text{line bundles } L \text{ over } A_T \text{ with an isom. } e_T^* L \simeq \mathcal{O}_T\}$$

$\mathrm{Pic}(A)$ is represented by a smooth group scheme$/S$

& $A^\vee := \mathrm{Pic}^\circ(A) =$ connected component of $\mathrm{Pic}(A)$

Fact : $\mathrm{Pic}^\circ(A)$ is an abelian variety, the universal line bundle

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ A \times \mathrm{Pic}^\circ(A) \end{array} \qquad \text{i.e. } \mathcal{P}|_{A \times \{L\}} \simeq L.$$

Reading this the other way $\rightsquigarrow A \longrightarrow \mathrm{Pic}(\mathrm{Pic}^\circ(A))$ induces $A \simeq A^{\vee\vee}$.

Another explicit construction : If L is a (relatively) ample line bundle over A

then
$$\begin{aligned} A &\longrightarrow \mathrm{Pic}^\circ(A) \\ x &\longmapsto x^* L \otimes L^{-1} \end{aligned}$$

Fact : (when $A/\bar{k}$,) L ample $\iff \{x \in A \ ; \ x^* L \simeq L\} =: K_L$ is finite

& thus $\lambda_L : A \twoheadrightarrow A/K_L \simeq A^\vee$

Moreover
$$\begin{array}{ccc} m^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1} & \xleftarrow{\text{pullback}} & \mathcal{P} \\ | & & | \\ A \times A & \longrightarrow & A \times A^\vee \end{array}$$

$$\Rightarrow \lambda_L : A \to A^\vee \rightsquigarrow \underset{\substack{\| \\ \lambda_L.}}{\lambda_L^\vee} : A \simeq A^{\vee\vee} \to A^\vee$$

<u>Polarization</u>: Isogeny $\lambda : A \to A^\vee$ s.t. $\lambda^\vee : A \simeq A^{\vee\vee} \to A^\vee$ is the same as λ

Conversely, every morphism $\lambda : A \to A^\vee / \mathbb{C}$ s.t. $\lambda \simeq \lambda^\vee$

$\Rightarrow \lambda = \lambda_L$ for a line bundle L.

<u>Caveat</u>: Not true if A/k for non-algebraically closed field.

<u>Integral version</u>: Let Λ be a $\mathbb{Z}$-module of rank $2g$

and $\psi : \Lambda \times \Lambda \to \mathbb{Z}$ a nondegenerate alternating pairing

<u>Base case</u>: ψ is perfect, i.e. $\Lambda = \mathbb{Z}^{\oplus 2g}$, $\psi = \begin{pmatrix} & I_g \\ -I_g & \end{pmatrix}$

$n \in \mathbb{N}$

$$\mathcal{A}_{g,n} : Sch/\mathbb{Q} \longrightarrow \mathcal{S}ets$$

$$S \longmapsto \mathcal{A}_{g,n}(S) = \left\{ \begin{array}{l} (A, \lambda, i) \text{. abelian scheme } A \text{ of dim } g/S \\ \lambda : A \xrightarrow{\sim} A^\vee \text{ polarization} \\ i : (\mathbb{Z}/N\mathbb{Z})^{2g}_S \xrightarrow{\simeq} A[N] \quad \text{isom. s.t.} \\ \text{on each connected comp. } S, \text{ there exists an isom. } \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N \\ \text{s.t.} \begin{array}{ccccc} (\mathbb{Z}/N\mathbb{Z})^{2g} \times (\mathbb{Z}/N\mathbb{Z})^{2g} & & \xrightarrow{\psi} & & \mathbb{Z}/N\mathbb{Z} \\ \simeq\downarrow i \qquad\quad \simeq\downarrow i & & & & \downarrow \wr \\ A[N] \times A[N] & \xrightarrow{1 \times \lambda} & A[N] \times A^\vee[N] & \xrightarrow[\text{Weil pairing}]{} & \mu_N \end{array} \end{array} \right\}$$

is represented by a quasi-proj. sm. scheme $/\mathbb{Q}$. (Mumford)

<u>Caveat</u>: Over Spec $\mathbb{Q}$, there's no isom $\mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N$

(such isomorphism exists over $\mathbb{Q}(\zeta_N)$.)

So $\mathcal{A}_{g,n}$ has no $\mathbb{Q}$-points, even though it is a smooth $\mathbb{Q}$-scheme.

In fact $\pi_0^{geom}(\mathcal{A}_{g,n}) = \pi_0 \left({}_{GSp_{2g}(\mathbb{Q})} \backslash \mathcal{H}_g^{\pm} \times GSp_{2g}(\mathbb{A}_f) / \widehat{\Gamma}(N) \right)$

$$\xrightarrow{\text{similitudes}} \mathbb{Q}_{>0}^{\times} \backslash \mathbb{A}_f^{\times} / (1+N\hat{\mathbb{Z}})^{\times} \simeq (\mathbb{Z}/N\mathbb{Z})^{\times} \circlearrowleft \text{ acted by } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$$

$$\begin{tikzcd} \rightsquigarrow \mathcal{A}_{g,n} \arrow[d] \\ \operatorname{Spec}\mathbb{Q}(\mu_N) \arrow[d] \\ \operatorname{Spec}\mathbb{Q} \end{tikzcd}$$

There's no $\mathbb{Q}$-point on $\mathcal{A}_{g,n}$. b/c there's no morphism $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Q}(\mu_N)$ ($\Leftrightarrow \mathbb{Q}(\mu_N) \to \mathbb{Q}$.)

<u>Remark</u>: $\mathcal{A}_{g,n}$ is a connected $\mathbb{Q}$-scheme but <u>not</u> geometrically connected.

- Slightly more general, $\Lambda_{\mathbb{Q}} \simeq \mathbb{Q}^{2g}$ $\psi: \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ antisymmetric perfect $/\mathbb{Q}$

$K \subseteq GSp(\Lambda \otimes \mathbb{A}_f, \psi)$ open compact

$$\mathcal{M}_K : Sch/\mathbb{Q} \longrightarrow Sets$$

$$S \longmapsto \mathcal{M}_K(S) = \left\{ \begin{array}{l} (A, \lambda, \eta K) \text{ quasi-isogeny class} \\ A \text{ abelian scheme of dim } g \\ \lambda: A \longrightarrow A^{\vee} \text{ quasi-polarization} \\ \text{On each connected component of } S, \text{ fixing a geometric pt } \bar{s} \in S \\ \eta K \text{ is a } \pi_1(S,\bar{s})\text{-stable, } K\text{-orbit of isoms.} \\ \eta: \Lambda_{\mathbb{Q}} \otimes \mathbb{A}_f \xrightarrow{\sim} \hat{V}(A_{\bar{s}}) \text{ and } c: \mathbb{A}_f \xrightarrow{\sim} \mathbb{A}_f(1) \\ \text{s.t. } \begin{tikzcd} \Lambda_{\mathbb{Q}} \otimes \mathbb{A}_f \times \Lambda_{\mathbb{Q}} \otimes \mathbb{A}_f \arrow[rr, "\psi"] \arrow[d, "\simeq\ \eta"] & & \mathbb{A}_f \arrow[d, "= c"] \\ \hat{V}(A_{\bar{s}}) \times \hat{V}(A_{\bar{s}}) \arrow[r, "1\times\lambda"] & \hat{V}(A_{\bar{s}}) \times \hat{V}(A_{\bar{s}}^{\vee}) \arrow[r, "\text{Weil}", "\text{pairing}"'] & \mathbb{A}_f \end{tikzcd} \end{array} \right\}$$

stalk at $\bar{s}$ so $\pi_1(S,\bar{s}) \circlearrowright \hat{V}(A_{\bar{s}})$

<u>Note</u>: K also acts on the isom. $c: \mathbb{A}_f \simeq \mathbb{A}_f(1)$ through the similitude factor.

$\mathcal{M}_K$ is rep'd by a smooth quasi-proj $\mathbb{Q}$-scheme.

<u>Theorem</u>. Let $\mathfrak{h}_g^{\pm} := \{ Z \in Sym_g(\mathbb{C}) : \operatorname{Im}(Z) > 0 \text{ or } \operatorname{Im}(Z) < 0 \}$

$\uparrow$ means totally positive

Then $\mathcal{M}_K(\mathbb{C}) \simeq GSp_{2g}(\mathbb{Q}) \backslash \mathfrak{h}_g^{\pm} \times GSp_{2g}(\mathbb{A}_f) / K$.

Proof: Given (A, λ, η), fix an isom. $H_1(A(\mathbb{C})^{an}, \mathbb{Q}) \overset{\beta}{\simeq} \Lambda_{\mathbb{Q}}$,

s.t. polarization $\lambda \longleftrightarrow \psi$.

$$\Lambda_{\mathbb{Q}} \otimes \mathbb{A}_f \xrightarrow[\simeq]{\eta} H_1^{et}(A(\mathbb{C}), \mathbb{A}_f) \simeq H_1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f \overset{\beta}{\simeq} \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f$$

isomorphism preserving λ-Weil pairing vs. ψ up to scalar

$$\omega_{A^\vee/\mathbb{C}} \hookrightarrow H_1^{dR}(A/\mathbb{C}) \simeq H_1(A(\mathbb{C})^{an}, \mathbb{Q}) \otimes \mathbb{C} \simeq \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$$

This defines a polarized Hodge structure on $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$

$\rightsquigarrow h : \mathbb{S} \longrightarrow GSp(\Lambda_{\mathbb{R}}, \psi) \simeq GSp_{2g}(\mathbb{R})$.

All such h is $GSp_{2g}(\mathbb{R})$-conjugate to

$$h_0 : \mathbb{S} \longrightarrow GSp_{2g}(\mathbb{R})$$

$$x+iy \longmapsto \begin{pmatrix} xI_g & -yI_g \\ yI_g & xI_g \end{pmatrix}$$

$h(i) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rightsquigarrow$ check $(A\cdot i + B)(C\cdot i + D)^{-1} \in \mathfrak{h}_g^{\pm}$.

§3 Unitary Shimura variety

Let E be an imaginary quadratic ext'n.

Let V be a Hermitian space of dim n over E,

that is, $\langle \cdot, \cdot \rangle : V \times V \longrightarrow E$ non-degenerate Hermitian form

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \quad , \quad \langle ax, by \rangle = a\bar{b}\langle x, y \rangle \quad \text{for } x, y \in V, \ a, b \in E.$$

Fix $\delta \in E^{c=-1}$. This determines an embedding $E \subseteq \mathbb{C}$ s.t. $\delta \in \mathbb{R}_{>0} i$

Then the Hermitian form $\langle \cdot, \cdot \rangle$ induces an alternating form

$$\{\cdot, \cdot\} : V \times V \longrightarrow \mathbb{Q} \quad \leftarrow \text{This is } \mathbb{Q}$$

$$\{x, y\} := \mathrm{Tr}_{E/\mathbb{Q}}(\delta \cdot \langle x, y \rangle)$$

check $\{x, y\} = -\{y, x\}$

$$\overset{\shortparallel}{\mathrm{Tr}_{E/\mathbb{Q}}}(\delta \cdot \langle x, y \rangle) = \mathrm{Tr}_{E/\mathbb{Q}}(\delta \cdot \overline{\langle y, x \rangle}) \overset{c(\delta) = -\delta}{=} \mathrm{Tr}_{E/\mathbb{Q}}(-\delta \cdot \langle y, x \rangle) = -\{y, x\}$$

Fact: $\left\{ \begin{array}{c} \text{non-degenerate Herm. forms} \\ \langle \cdot, \cdot \rangle \end{array} \right\} \overset{\text{bij.}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{non-degenerate alternating forms } \{\cdot, \cdot\} : V \times V \to \mathbb{Q} \\ \text{satisfying } \{ax, y\} = \{x, \bar{a}y\} \end{array} \right\}$

Note: This bijection depends on the choice of δ.

Consider group $GU(V)$: for S an $\mathbb{Q}$-algebra

$$GU(V)(S) := \left\{ (g,c) \in \underset{S}{GL}(V \underset{\mathbb{Q}}{\otimes} S) \times S^\times \;\middle|\; \begin{array}{l} \forall\, x,y \in V \\ \langle gx, gy \rangle = c \langle x,y \rangle \\ \Updownarrow \\ \{gx,gy\} = c\{x,y\} \end{array} \right\}$$

similitude unitary group

Have $\quad 1 \longrightarrow U(V) \longrightarrow GU(V) \xrightarrow{c} \mathbb{G}_m \longrightarrow 1$

We are interested in understanding the Shimura variety for $GU(V)$.

Fix an open compact subgroup $K \subseteq GU(V)(\mathbb{A}_f)$

Previously, we've talked about the group, then data at all finite places, now at archimedean place

At ∞, $V_{\mathbb{R}}$ has signature (a,b) $\quad n = a+b$

i.e. $\exists$ a basis of $V_{\mathbb{R}}$ s.t. the Herm. form is $\begin{pmatrix} I_a & \\ & -I_b \end{pmatrix}$

$h : \mathbb{S} \to GL(V_{\mathbb{R}}) = GL_n(\mathbb{C})$ $\quad$ Here $E \otimes \mathbb{R} \simeq \mathbb{C}$ uses the embedding determined by δ

$z \longmapsto \begin{pmatrix} z I_a & \\ & \bar{z} I_b \end{pmatrix}$

Then $\{x, h(i)y\} = \mathrm{Tr}_{E_{\mathbb{R}}/\mathbb{R}}(\delta \cdot \langle x, h(i)y \rangle)$

$c \in \mathbb{R}_{>0}$ $\quad$ conjugate-linear in second factor

(for $x,y \in E_{\mathbb{R}}^{\oplus a}$, this is $\mathrm{Tr}_{E_{\mathbb{R}}/\mathbb{R}}(c \cdot i \cdot (-i) \cdot \langle x,y \rangle)$

so positive definite. Similarly for $E_{\mathbb{R}}^{\oplus b}$.)

Moduli functor: $\quad M_K : \mathrm{Sch}^{\text{loc.noe}}_{/E} \longrightarrow \mathrm{Sets}$

This is E instead; viewed canonically as a subfield of $\mathbb{C}$ using the one given by δ.

$$S \longmapsto M_K(S) = \left\{ \begin{array}{l} (A, i, \lambda, \eta) : \text{up to quasi-isogenies} \\ \cdot\, A \text{ abelian variety of dim } n \text{ over } S \text{ satisfying a signature condition} \\ \cdot\, i : E \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \text{ an embedding} \\ \cdot\, \lambda : A \to A^\vee \text{ a polarization (only requiring to be quasi-isog.)} \\ \quad \text{s.t. the Rosati involution induces complex conj on } \mathcal{O}_E \\ \cdot\, \text{On each connected component of } S, \text{ fixing a geom. point } \bar{s} \end{array} \right.$$

Important remark:

Weil pairing can't see the Hermitian form; so we need to turn the Herm. form $\langle , \rangle$ into a $\mathbb{Q}$-valued symplectic form $\{ , \}$ to compare w/ Weil pairing. This corresponds to a choice of embedding $GU(V) \hookrightarrow GSp_{2g}$.

η is $\pi_1(S, \bar{s})$-stable K-orbit of $\mathcal{O}_E$-linear isom.

$$\eta : V \otimes \mathbb{A}_f \xrightarrow{\sim} \hat{V}(A_{\bar{s}}) \text{ and } \mathbb{A}_f \xrightarrow{\sim} \mathbb{A}_f(1)$$

s.t.

$$\begin{tikzcd} V_{\mathbb{A}_f} \times V_{\mathbb{A}_f} \arrow[r, "\{\cdot,\cdot\}"] \arrow[d, "\eta", "\wr"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{A}_f \arrow[d, "\wr"] \\ \hat{V}(A_{\bar{s}}) \times \hat{V}(A_{\bar{s}}) \arrow[r, "\text{Weil pairing}"] & \mathbb{A}_f(1) \end{tikzcd}$$

Rosati involution Given an endomorphism $\theta : A \to A$, the polarization induces

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d, "\lambda", "\simeq"'] & A \arrow[d, "\lambda", "\simeq"'] \\ A^\vee \arrow[r, "\theta^\vee"] & A^\vee \end{tikzcd}$$

θ_λ is the Rosati involution (quasi-isogeny)

Signature condition: $0 \to \omega_{A^\vee/S} \to H_1^{dR}(A/S) \to \mathrm{Lie}_{A/S} \to 0 \quad \circlearrowleft \mathcal{O}_S \otimes \mathcal{O}_E \simeq \mathcal{O}_S \oplus \mathcal{O}_S$

b/c we chose S to be an E-scheme

$$\underbrace{\circlearrowleft\, \mathcal{O}_E \otimes \mathcal{O}_S}_{\text{locally free } \mathcal{O}_E \otimes \mathcal{O}_S\text{-module of rank } n} \qquad x \otimes a \longmapsto (ax, \bar{a}x)$$

According to the decomposition on the right, get decomposition $0 \to \omega_{A^\vee/S, j} \to H_1^{dR}(A/S)_j \to \mathrm{Lie}_{A/S, j} \to 0$

$j=1$. $\mathcal{O}_E$-linear

$j=2$ $\mathcal{O}_E$-conj-linear

Require: $\mathrm{rank}\,(\mathrm{Lie}_{A/S,1}) = a$, $\mathrm{rank}\,(\mathrm{Lie}_{A/S,2}) = b$.

(corresponding to the condition for h earlier.)

$\circlearrowleft$ corresponds to the eigenspace of $h(\mathbb{C})$, acting by z.

Remark: The polarization λ induces a perfect pairing

$$\lambda : H_1^{dR}(A/S) \times H_1^{dR}(A/S) \longrightarrow \mathcal{O}_S$$

But the Rosati involution condition implies that under the decomposition into $j=1,2$.

we get

$$\begin{array}{ccccc} \lambda : H_1^{dR}(A/S)_1 & \times & H_1^{dR}(A/S)_2 & \longrightarrow & \mathcal{O}_S \\ \cup\!\vert & & \cup\!\vert & & \\ \omega_{A^\vee/S,1} & & \omega_{A^\vee/S,2} & & \end{array}$$

rank $n-a=b$ $\qquad$ rank $n-b=a$

$\omega_{A^\vee/S,1}$ & $\omega_{A^\vee/S,2}$ are exact annihilator of each other.

Theorem When $K \leq GU(V)(\mathbb{A}_f)$ is sufficiently small, $\mathcal{M}_K$ is represented by a smooth variety of dimension $a \cdot b$ over E.

<u>Remark</u>: Similar to the argument for modular curve & Siegel moduli variety, can "almost" prove $\mathcal{M}_K(\mathbb{C}) = G(\mathbb{Q}) \backslash \left(X \times \left(G(\mathbb{A}_f)/K \right) \right)$

where $X = GU_{\mathbb{R}}(a,b)(\mathbb{R}) / G(U_{\mathbb{R}}(a) \times U_{\mathbb{R}}(b))(\mathbb{R}) \overset{\text{if } a \neq b.}{=} U_{\mathbb{R}}(a,b) / U_{\mathbb{R}}(a) \times U_{\mathbb{R}}(b)$

Not quite correct. When $n = a+b$ is even, this is okay.

When $n = a+b$ is odd, $\mathcal{M}_K(\mathbb{C})$ = finite identical copies of this.

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