Lecture 6 Galois representations associated to modular forms

§1. Kodaira-Spencer isomorphism

For modular curves $\begin{array}{c}\mathcal{E}\\ \downarrow \pi \\ Y_1(N)/\mathbb{Q}\end{array}$. Then

$$0 \to \underset{\displaystyle \omega_{\mathcal{E}/Y_1(N)}}{\underset{\wr\wr}{\pi_* \Omega^1_{\mathcal{E}/Y_1(N)}}} \to H^1_{dR}(\mathcal{E}/Y_1(N)) \to \underset{\displaystyle \omega^{-1}_{\mathcal{E}/Y_1(N)}}{\underset{\wr\wr}{R^1\pi_* \mathcal{O}_{\mathcal{E}}}} \to 0$$

by Poincaré duality

$$\begin{tikzcd}
\nabla : H^1_{dR}(\mathcal{E}/Y_1(N)) \arrow[r] \arrow[d, equal] & H^1_{dR}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k} \arrow[d, equal] \\
\omega_{\mathcal{E}/Y_1(N)} \arrow[dr] \arrow[d, dash] \arrow[r, "\text{Griffith transversality}"] & \omega_{\mathcal{E}/Y_1(N)} \otimes \Omega^1_{Y_1(N)/k} \arrow[d, dash] \\
\omega^{-1}_{\mathcal{E}/Y_1(N)} & \omega^{-1}_{\mathcal{E}/Y_1(N)} \otimes \Omega^1_{Y_1(N)/k}
\end{tikzcd}$$

Moreover,

$$\begin{tikzcd}
\omega_{\mathcal{E}/Y_1(N)} \arrow[r, "\nabla"] \arrow[dr] & H^1_{dR}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k} \arrow[d, "pr"] \\
& gr^0 H^1_{dR}(\mathcal{E}/Y_1(N)) \otimes \Omega^1_{Y_1(N)/k}
\end{tikzcd}$$

This map is a map of coherent sheaves
<u>no</u> differential maps

$$pr \circ \nabla(a\omega) = pr(\omega \otimes da + a \cdot \nabla(\omega)) = a \cdot pr \circ \nabla(\omega)$$

b/c $pr(\omega) = 0$

$$\Rightarrow \omega \longrightarrow \omega^{-1} \otimes \Omega^1_{Y_1(N)/k}$$

$$\Rightarrow KS : \omega^{\otimes 2} \longrightarrow \Omega^1_{Y_1(N)/k}$$

<u>Theorem</u> (Kodaira-Spencer isomorphism) KS induces an isomorphism

$$KS : \omega^{\otimes 2} \xrightarrow{\cong} \Omega^1_{X_1(N)/\mathbb{Q}}\underbrace{(\log D)}_{\text{with log pole at cusps}} = \Omega^1_{X_1(N)/\mathbb{Q}}(D)$$

Moreover, this isomorphism even extends to $\mathbb{Z}[\frac{1}{N}]$.

<u>Rmk</u>: In terms of K-S isomorphism, maybe the "correct" definition of modular forms is

$$S_k(\Gamma_1(N)) := H^0(X_1(N), \omega^{\otimes k}(-D)) = H^0(X_1(N), \omega^{\otimes k-2} \otimes \Omega^1_{X_1(N)})$$

$$\& \ M_k(\Gamma_1(N)) = H^0(X_1(N), \omega^{\otimes k-2} \otimes \Omega^1_{X_1(N)}(D))$$

See the next section for more.

§ 2 Eichler-Shimura isomorphism

- $\begin{array}{c}\mathcal{E}\\ \downarrow \pi \\ Y_1(N)\end{array}$ $\quad \mathcal{H}^1_{dR}(\mathcal{E}/Y_1(N))$ is locally free of rank 2 carrying an integrable connection.

  For $k \geq 2$, get symmetric power $\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}$, with integrable connection.

- On the other hand, consider $R^1\pi_*\underline{\mathbb{C}}^{an}$, which is locally const. sheaf on $Y_1(N)^{an}$

  $\rightsquigarrow \mathcal{L}_k := \mathrm{Sym}^{k-2}(R^1\pi_*\underline{\mathbb{C}}^{an})$ loc. const sheaf $/ Y_1(N)^{an}$

  & $\quad 0 \to \mathcal{L}_k \to \mathrm{Sym}^{k-2}\mathcal{H}^{1,an}_{dR} \xrightarrow{\nabla_{GM}} \mathrm{Sym}^{k-2}\mathcal{H}^{1,an}_{dR} \otimes \Omega^1_{Y_1(N)^{an}} \to 0$

So
$$H^1(Y_1(N)^{an}, \mathcal{L}_k) \cong H^1(Y_1(N)^{an}, \mathrm{Sym}^{k-2}\mathcal{H}^{1,an}_{dR} \otimes \Omega^\bullet_{Y_1(N)^{an}})$$
$$\cong H^1(Y_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \otimes \Omega^\bullet_{Y_1(N)})$$

<u>Theorem</u> (Eichler-Shimura) There is a natural isomorphism (Hecke equivariant)
$$H^1(Y_1(N)^{an}, \mathcal{L}_k) \cong M_k(\Gamma_1(N)) \oplus \overline{S_k(\Gamma_1(N))}$$
$\quad \wr\wr \leftarrow$ choosing $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$
$$H^1_{et}(Y_1(N)_{\mathbb{C}}, \mathcal{L}_k^{et}) \qquad \mathcal{L}_k^{*} := \mathrm{Sym}^{k-2}(R^1\pi_*\mathbb{Q}_\ell)$$

Let's ignore the issues at the cusp (should have used a log version of above.)

* The Hodge filtration $\omega - \omega^{-1}$ on $\mathcal{H}^1_{dR}$ induces a natural filtration

$$\begin{tikzcd}
\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} : \arrow[d, "\nabla"] & \omega^{k-2} \arrow[dr, "\cong"] \arrow[r, dash] & \omega^{k-4} \arrow[dr, "\cong"] \arrow[r, dash] & \omega^{k-6} \cdots \arrow[r, dash] & \cdots \arrow[dr, "\cong"] \arrow[r, dash] & \omega^{2-k} \\
\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \otimes \underbrace{\Omega^1_{X_1(N)}(\log C)}_{\cong\, \omega^2} : & \omega^k \arrow[r, dash] & \omega^{k-2} \arrow[r, dash] & \omega^{k-4} \cdots \arrow[r, dash] & \cdots \arrow[r, dash] & \omega^{4-k}
\end{tikzcd}$$

Fil^{k-2} $\quad$ "$\omega^{k-3} \otimes \omega^{-1}$" $\quad$ "$\omega^{k-2} \otimes (\omega^{-1})^{\otimes 2}$" $\quad$ "$(\omega^{-1})^{\otimes k-2}$"

Fil^{k-3} $\cdots$

Griffith transversality.

$$\Rightarrow \left[\mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \to \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR} \otimes \Omega^1_{X_1(N)}(\log C)\right] \simeq \left[\omega^{2-k} \to \omega^k\right]$$

Taking cohomology $\rightsquigarrow$

$$0 \to H^0(X_1(N), \omega^k) \to H^1_{dR}(X_1(N), \mathrm{Sym}^{k-2}\mathcal{H}^1_{dR}) \to \underbrace{H^1(X_1(N), \omega^{2-k})}_{\substack{\parallel \\ H^0(X_1(N),\omega^k)^\vee}} \to 0$$

§3. Eichler-Shimura relations

We continue to ignore cusps / Eisenstein component

$$\begin{array}{c} E \\ \downarrow \pi \\ M_K \end{array} \qquad \mathcal{L}_k := \mathrm{Sym}^{k-2} R^1\pi_* \overline{\mathbb{Q}}_\ell \quad \text{locally free étale sheaf on } M_K \text{ of rk } k-1.$$

<u>Main subject</u>: $\boxed{H^1_{et}(M_{K,\overline{\mathbb{Q}}}, \mathcal{L}_k)}$

<u>Recall</u>: $S_k(K) = \bigoplus_{\substack{\pi \\ \pi_\infty \simeq DS_k^+}} \mathbb{C}v_k \otimes \pi_f^K$

So Eichler-Shimura relations imply

$$H^1_{et}(M_{K,\overline{\mathbb{Q}}}, \mathcal{L}_k) \simeq \bigoplus_{\substack{\pi \\ \pi_\infty \simeq DS_k^+}} (\mathbb{C}v_k \otimes \pi_f^K)^{\oplus 2}$$

one from $H^0(M_K^*, \omega^k)$
one from $H^0(M_K^*, \omega^k)^\vee$

$\circlearrowleft$
$\mathrm{Gal}_{\mathbb{Q}}$ commutes with all Hecke actions

$$\simeq \bigoplus_{\pi, \pi_\infty \simeq DS_k^+} \pi_f^K \otimes \rho_\pi \quad \text{for some } \rho_\pi : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_\ell)$$

<u>Theorem</u>: $\rho_\pi : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_\ell)$ satisfies the following conditions

If p is a prime s.t. π_p is an unramified PS

then ρ_π is unram @ p & $\mathrm{Tr}(\rho_\pi(\mathrm{Frob}_p)) = a_p$.

$$\det(\rho_\pi(\mathrm{Frob}_p)) = \omega_\pi(p)^{-1} = p^{k-1} \cdot \chi_\pi(p).$$

* $\exists$ open compact subgroup $K = \prod_p K_p \subseteq GL_2(\mathbb{A}_f)$

s.t. π appears in $S_k(K)$ in the sense that $\pi_f^K \neq 0$

In particular, if π_p is unram PS, take $K_p = GL_2(\mathbb{Z}_p)$.

Geometric input: Geometry of modular curve $/\mathbb{F}_p$.

$\mathbb{Z}_{(p)} \backslash M_{K^p}$ classifies (E, η^p) : E elliptic curve $/S$ $\quad S: \mathbb{Z}_{(p)}$-scheme

$\eta^p : \hat{\mathbb{Z}}^{(p)\oplus 2} \xrightarrow{\sim} \hat{T}^{(p)}(E)$, $\pi_1(S, \bar{s})$-stable K^p-orbit of isoms.

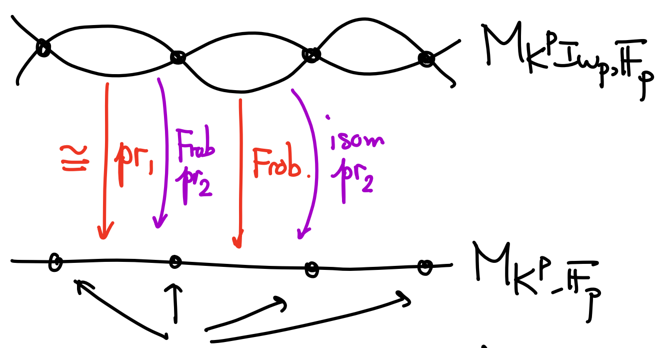
$\mathbb{Z}_{(p)} \backslash M_{K^p Iw_p}$ classifies (E, η^p, C), $(E, \eta^p) \in M_{K^p}(S)$

$C \subseteq E[p]$ subgroup scheme of order p.

$$\begin{tikzcd} & M_{K^p Iw_p} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{Z}_{(p)} \backslash M_{K^p} & & M_{K^p} \end{tikzcd} \qquad \begin{tikzcd} & (E, \eta^p, C) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (E, \eta^p) & & (E/C, \eta^{p\prime} : \hat{\mathbb{Z}}^{(p)\oplus 2} \xrightarrow{\sim} \hat{T}^{(p)}(E) \to \hat{T}^{(p)}(E/C)) \end{tikzcd}$$

Study the special fiber $/\mathbb{F}_p$.

Picture:



$M^{ss}_{K^p, \mathbb{F}_p}$, where E^{univ} is supersingular

$$M^{ord}_{K^p, \mathbb{F}_p} = M_{K^p, \mathbb{F}_p} \setminus M^{ss}_{K^p, \mathbb{F}_p}$$

For each $x \in M^{ord}_{K^p, \mathbb{F}_p}(\bar{\mathbb{F}}_p)$, E_x is ordinary $\quad$ i.e. $E_x[p](\bar{\mathbb{F}}_p) = \mathbb{Z}/p\mathbb{Z}$

$$\rightsquigarrow \quad 0 \to \mu_p \to E_x[p] \to \mathbb{Z}/p\mathbb{Z} \to 0$$

In general, $\quad 0 \to E^{ord}[p]^{conn} \to E^{ord}[p] \to E^{ord}[p]^{et} \to 0$

Then $M^{ord}_{K^p Iw_p, \mathbb{F}_p} = X_1 \sqcup X_2$

$$X_1^{ord} = \{ (E, \eta, C) \mid C = E^{ord}[p]^{conn} \}$$

$\downarrow pr_1 \qquad$ such C is unique. $\Rightarrow$ pr_1 is an isom.

$$M^{ord}_{K^p, \mathbb{F}_p} = \{ (E, \eta) \}$$

Similarly, $X_2^{ord} = \{ (E, \eta, C) \mid C \neq E^{ord}[p]^{conn} \}$

isom. $\downarrow pr_2 \qquad \mapsto \qquad$ can recover E from E/C

$M^{ord}_{K^p,\mathbb{F}_p} = \{(E',\eta')\}$ $\quad (E/C,\eta')$ by $(E/C)/(E/C)[p]^{conn}$.

- $$\begin{tikzcd} & M^{ord}_{K^pIw_p,\mathbb{F}_p} = X_1^{ord} \sqcup X_2^{ord} \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ M^{ord}_{K^p,\mathbb{F}_p} \arrow[rr, dashed, "Frob"] & & M^{ord}_{K^p,\mathbb{F}_p} \end{tikzcd}$$

$$(E,\eta) \longmapsto (E/E[p]^{conn},\eta')$$

Claim: $E/E[p]^{conn} \simeq E^{(p)} = E \times_{S,Frob} S$

$$\begin{tikzcd} E \arrow[r, "Frob"] \arrow[rr, bend right, "p"'] & E^{(p)} \arrow[r, "Ver"] & E \end{tikzcd} \Rightarrow Ker(Frob) \subseteq E[p]$$

& $d(Frob): \omega_{E/S} \to \omega_{E^{(p)}/S}$ is zero map.

But if $C \neq E[p]^{conn}$, C is étale $\Rightarrow E \to E/C$ étale X

So claim holds.

So $$\begin{tikzcd} M^{ord}_{K^p,\mathbb{F}_p} \arrow[rr, bend right, "Frob_p"'] & X_1^{ord} \arrow[l, "pr_1"', "\sim"] \arrow[r, "pr_2"] & M^{ord}_{K^p,\mathbb{F}_p} \end{tikzcd}$$

Similarly, $$\begin{tikzcd} M^{ord}_{K^p,\mathbb{F}_p} & X_1^{ord} \arrow[l, "pr_1"] \arrow[r, "pr_2"', "\simeq"] & M^{ord}_{K^p,\mathbb{F}_p} \arrow[ll, bend left, "Frob_p"] \end{tikzcd}$$

Write X_i = closure of X_i^{ord}, the property above extends to X_1 & X_2

Étale cohomology facts

$$\begin{tikzcd} & M_{K^pIw_p,\mathbb{Z}_{(p)}} \arrow[dl] \arrow[dr] & \\ M_{K^p,\mathbb{Z}_{(p)}} & & M_{K^p,\mathbb{Z}_{(p)}} \\ & M_{K^pIw_p,\mathbb{F}_p} = X_1 \sqcup X_2 \arrow[uu, hook] \arrow[dl] \arrow[dr] & \\ M_{K^p,\mathbb{F}_p} \arrow[uu, hook] & & M_{K^p,\mathbb{F}_p} \arrow[uu, hook] \end{tikzcd}$$

This is finite flat

$$\begin{tikzcd}
T_p : H^1_{ét}(M_{K^p,\overline{\mathbb{Q}}}, \mathcal{L}_k) \arrow[r, "pr_1^*"] \arrow[d, phantom, "\cong" rotate=90] & H^1_{ét}(M_{K^p I_{w_p}, \overline{\mathbb{Q}}}, \mathcal{L}_k) \arrow[r, "pr_{2,*}"] & H^1_{ét}(M_{K^p,\overline{\mathbb{Q}}}, \mathcal{L}_k) \arrow[d, phantom, "\cong" rotate=90] \\
H^1_{ét}(M_{K^p,\overline{\mathbb{F}}_p}, \mathcal{L}_k) \arrow[r, "pr_1^*"] & H^1_{ét}(X_1 \sqcup X_{2,\overline{\mathbb{F}}_p}, \mathcal{L}_k) \arrow[r, "pr_{2,*}"] & H^1_{ét}(M_{K^p,\overline{\mathbb{F}}_p}, \mathcal{L}_k)
\end{tikzcd}$$

So $T_p = \mathrm{Frob}_p^* + \mathrm{Frob}_{p*}$.

Moreover, $\mathrm{Frob}_p^* \circ \mathrm{Frob}_{p*} = p^{k-1} = \underset{\substack{\uparrow \\ \text{from } M_{K^p,\overline{\mathbb{F}}_p} \text{ being a curve}}}{p} \cdot p^{k-2}$ ← from $\underline{\underline{\mathrm{Sym}}}^{k-2} R^1 f_* \overline{\mathbb{Q}}_\ell$

§4. Hasse invariants for modular forms

Let S be an $\mathbb{F}_p$-scheme.

There's a Frobenius endomorphism $S \xrightarrow{Fr_S} S$

If A is an abelian S-scheme, we have a relative Frobenius

$$\begin{tikzcd}
A \arrow[r, dashed, "Fr_{A/S}"] \arrow[rr, bend left, "Fr_A"] \arrow[dr, "\pi"'] & A^{(p)} \arrow[r, "Fr_S"] \arrow[d, "\pi^{(p)}"] & A \arrow[d, "\pi"] \\
& S \arrow[r, "Fr_S"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum s_n^p x^{np} & & \\
& \sum s_n^p x^n \arrow[ul, dashed, mapsto] & \sum s_n x^n \arrow[l, mapsto] \arrow[ull, bend right, mapsto]
\end{tikzcd}$$

"Frobenius on the fiber"

Fact: $\mathrm{Ker}(Fr_{A/S}) \subseteq A[p]$, so we have a factorization

$$\begin{tikzcd}
A \arrow[r, "Fr_{A/S}"] \arrow[rr, bend right, "\times p"'] & A^{(p)} \arrow[r, "V"] & A
\end{tikzcd} \quad /S \qquad \text{all } S\text{-morphisms}$$

called Verschiebung.

Then $H^1_{dR}(A/S) \xrightarrow{V^*} H^1_{dR}(A^{(p)}/S) \cong H^1_{dR}(A/S) \otimes_{\mathcal{O}_S, Fr_S} \mathcal{O}_S$

General Fact: $\mathrm{Im}\, V^*$ is precisely $\omega_{A^{(p)}/S} \cong \omega_{A/S} \otimes_{\mathcal{O}_S, Fr_S} \mathcal{O}_S$

Example: modular curve

$p \nmid N$

$$\begin{tikzcd}
\mathcal{E} \arrow[d] & E \arrow[l] \arrow[d] \\
\mathcal{Y}_1(N) \arrow[d] & Y_1(N) \arrow[l] \arrow[d] \\
\mathrm{Spec}\, \mathbb{Z}_{(p)} & \mathrm{Spec}\, \mathbb{F}_p \arrow[l]
\end{tikzcd}$$

← special fiber.

to be discussed next week.

$\mathcal{X}_1(N)$ & $X_1(N)$ are "compactifications"

$M_k(\Gamma_1(N)) := H^0(\mathcal{X}_1(N), \omega^{\otimes k})$

$M_k(\Gamma_1(N), \mathbb{F}_p) := H^0(X_1(N), \omega_{\mathbb{F}_p}^{\otimes k})$

* Applying the above construction to $E \to Y_1(N)$

get $V^*: H^1_{dR}(E/Y_1(N)) \twoheadrightarrow \omega^{(p)}_{E/Y_1(N)} \cong \omega^{\otimes p}$

$\cup$

ω Hasse invariant map

when pulling back a line bundle along Frobenius, the transition maps got p^{th}-powered, so $\omega^{\otimes p}$

$h := V^* \in \mathrm{Hom}_{\mathcal{O}_{Y_1(N)}}(\omega, \omega^{\otimes p}) = \Gamma(\mathcal{O}_{Y_1(N)}, \omega^\vee \otimes \omega^{\otimes p}) \cong \Gamma(\mathcal{O}_{Y_1(N)}, \omega^{p-1})$

This is called the <u>Hasse invariant</u>; it is a weight $p-1$ mod p modular form.

<u>Fact</u>: The q-expansion for h is just 1. <u>Fact</u>: h has no repeated zeros.

<u>Fact</u>: h is the reduction mod p of Eisenstein series E_{p-1}. (somewhat coincidental)

<u>Lemma</u>: The zero locus of h, $Z(h)$, is precisely the locus of $Y_1(N)$ where E is supersingular.

Proof: At a point $x \in Y_1(N)(\overline{\mathbb{F}}_p)$, the elliptic curve $E_{\bar{x}}$ is ordinary

$\Rightarrow E_{\bar{x}}[p] \cong \mathbb{Z}/p\mathbb{Z} \oplus \mu_p$ as group scheme.

$\mathrm{Fr}_{E_{\bar{x}}/\bar{x}} = \mathrm{id}$ $\qquad$ $\mathrm{Fr}_{E_{\bar{x}}/\bar{x}} = 0$

$\Rightarrow V = 0$ $\qquad$ V is an isom.

<u>Note</u>: $\omega_{E_{\bar{x}}/\bar{x}} \cong \omega_{E_{\bar{x}}[p]/\bar{x}} \circlearrowleft V$ is an isom

$\Rightarrow h$ doesn't vanish at this point.

Conversely, $V^*: \omega_{E_{\bar{x}}/\bar{x}} \longrightarrow \omega_{E_{\bar{x}}^{(p)}/\bar{x}}$ is an isom. $\Rightarrow$ Ker V is an étale group scheme

$\Rightarrow E_{\bar{x}}$ must be ordinary. $\square$