Lecture 1 Adelic interpretation of modular forms and automorphic representations

§1 Adelic description of modular curves

Let $N \geq 4$, consider $\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),\ N \mid c,\ N \mid d-1 \right\}$

Then the modular curve is $Y_1(N)(\mathbb{C}) := \Gamma_1(N) \backslash \mathfrak{h}$ where $\mathfrak{h} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

Let $\mathbb{A}_f :=$ finite adeles of $\mathbb{Q}$.

Theorem 1 There is an isomorphism
$$\boxed{\Gamma_1(N) \backslash \mathfrak{h} \xrightarrow{\cong} GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f) / \widehat{\Gamma_1(N)}}$$

where $\mathfrak{h}^{\pm} := \mathbb{C} \setminus \mathbb{R}$. $\widehat{\Gamma_1(N)} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\underbrace{\hat{\mathbb{Z}}}_{=\prod_p \mathbb{Z}_p}) \;\middle|\; c, d-1 \in N\hat{\mathbb{Z}} \right)$

Need a black box: Strong Approximation: $SL_2(\mathbb{Q})$ is dense in $SL_2(\mathbb{A}_f)$

(In general, if G is a simply-connected simple group over a number field F,

and v is a place of F s.t. $G(F_v)$ is not compact,

then $G(F)$ is dense in $G(\mathbb{A}_F^{(v)})$ adeles away from v.

Intuitively: $\operatorname{vol}\left(G(F) \backslash G(\mathbb{A}_F)\right) < +\infty$
if we quotient by $G(F_v)$, we must get something bad. $\Big\} \Rightarrow G(F)$ dense in $G(\mathbb{A}_F^{(v)})$.

E.g. $G = SL_n$, Sp_{2n}, $D^{Nm=1}$, or $SU(V)$

D division alg over F; V hermitian space for E/F)

Cor: $\boxed{\begin{matrix} SL_2(\mathbb{A}_f) \\ GL_2(\mathbb{A}_f) \end{matrix}}$ $\begin{matrix} \subseteq SL_2(\mathbb{Q}) \cdot \widehat{\Gamma_1(N)} \\ = GL_2(\mathbb{Q}) \cdot \widehat{\Gamma_1(N)} \end{matrix}$

off by $\mathbb{A}_f^\times$ but $\mathbb{A}_f^\times = \mathbb{Q}^\times \cdot \hat{\mathbb{Z}}^\times$, can first modify an element in $GL_2(\mathbb{A}_f)$ into $SL_2(\mathbb{A}_f)$

Remark: This argument needs the class group of $\mathbb{Z}$ to be trivial

Proof of Theorem 1: Consider $GL_2(\mathbb{Q}) \backslash \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f) / \widehat{\Gamma_1(N)}$

By Cor, every coset can be represented by $(z, 1) \in \mathfrak{h}^{\pm} \times GL_2(\mathbb{A}_f)$

The ambiguity lies in $\widetilde{\Gamma_1(N)} := GL_2(\mathbb{Q}) \cap \widehat{\Gamma_1(N)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \;\middle|\; N \mid c-1, d \right\}$

So that the above double coset is $\widetilde{\Gamma_1}(N)\backslash \mathfrak{h}^{\pm}$

$SL_2(\mathbb{Z})$ has index 2 in $GL_2(\mathbb{Z})$ as $\det(GL_2(\mathbb{Z})) = \mathbb{Z}^\times = \{\pm 1\}$

$\Rightarrow \widetilde{\Gamma_1}(N)\backslash \mathfrak{h}^{\pm} = \Gamma_1(N)\backslash \mathfrak{h}$. $\square$

Remark: $GL_2(\mathbb{R})$ acts transitively on $\mathfrak{h}^{\pm}$

$\Rightarrow \mathfrak{h}^{\pm} \cong GL_2(\mathbb{R}) / SO_2 \cdot \mathbb{R}^\times$ ← (almost) max'l compact mod center

$\frac{ai+b}{ci+d} \leftarrow\!\shortmid \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\rightsquigarrow \Gamma_1(N)\backslash \mathfrak{h} \cong GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) / \hat{\Gamma}_1(N) \times \underbrace{K_\infty}_{= SO_2 \cdot \mathbb{R}^\times}$$

Key benefit: Can replace $\hat{\Gamma}_1(N)$ by any open compact subgroup $K_f \subseteq GL_2(\mathbb{A}_f)$

$$\rightsquigarrow Sh_{GL_2}(K_f)(\mathbb{C}) := GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}) / K_f \times K_\infty$$

"Naïve" generalization: For reductive group G over $\mathbb{Q}$ & $K_f \subseteq G(\mathbb{A}_f)$ open compact

$$\rightsquigarrow Sh_G(K)(\mathbb{C}) := G(\mathbb{Q})\backslash G(\mathbb{A}) / K_f \times K_\infty$$ ← "almost" max'l compact mod center

§2. Adelic description of modular forms

- Now, we will transfer modular forms to the adelic side.

Issue: modular form f is not Γ-invariant.

For computational convenience: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $z \in \mathfrak{h} \rightsquigarrow j(g,z) := cz+d$

Then for $g, h \in GL_2(\mathbb{R})$, $\Rightarrow j(gh,z) = j(g,hz)\cdot j(h,z)$

check: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rightsquigarrow gh = \begin{pmatrix} & \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$

$j(gh,z) = (c\alpha+d\gamma)z + (c\beta + d\delta)$, $j(g,hz) = c \cdot \frac{\alpha z + \beta}{\gamma z + \delta} + d$

case of $(\Gamma_0(N), \chi)$ is left as an exercise.

$$\left\{ f : \mathfrak{h} \to \mathbb{C},\ f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) = j(\gamma,z)^k f(z),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}$$

$$= \left\{ f : GL_2(\mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{l} f(\gamma g) = j(\gamma, g(i))\, f(g),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \\ \text{and } f(g z r(\theta)) = f(g),\ z \in \mathbb{R}^\times,\ r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{array} \right\}$$

f is left $\Gamma_0(N)$-equivariant but right K_∞-invariant

$\rightsquigarrow$ Define $F_f : GL_2(\mathbb{R}) \to \mathbb{C}$

$$F_f(g) := \det(g)^{k-1} j(g,i)^{-k} f(g) = \det(g)^{k-1} (c'i + d')^{-k} f(g) \quad \text{if } g = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

exponent here is compatible w/ Hecke operators
will give a geometric explanation in Lecture 4

then for $\gamma \in \Gamma_1(N)$
$$F_f(\gamma g) = \det(g)^{k-1} j(\gamma g, i)^{-k} f(\gamma g)$$
$$= \det(g)^{k-1} j(\gamma, gi)^{-k} j(g,i)^{-k} f(\gamma g) = \det(g)^{k-1} j(g,i)^{-k} f(g) = F_f(g)$$

However,
$$F_f(g z r(\theta)) = \det(gz)^{k-1} \cdot j(g z r(\theta), i)^{-k} f(g z r(\theta))$$
$$= \det(g)^{k-1} z^{2k-2} \cdot j(g, z r(\theta)(i))^{-k} \cdot j(z r(\theta), i)^{-k} f(g)$$
$$= \det(g)^{k-1} z^{2k-2} \cdot j(g,i)^{-k} z^{-k} \cdot (i \sin\theta + \cos\theta)^{-k} f(g)$$
$$= e^{ik\theta} z^{k-2} F_f(g)$$

Note: F_f is left $\Gamma_0(N)$-invariant but right K_∞-equivariant.

Next, upgrade to adelic setup using $\Gamma_0(N) \backslash \mathfrak{h} \simeq GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \widehat{\Gamma}_0(N) \times K_\infty$

$$\left\{ f : \mathfrak{h} \to \mathbb{C},\ f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z),\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \right\}$$

$$\overset{\text{bijection}}{\longleftrightarrow} \left\{ F : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / \widehat{\Gamma}_0(N) \to \mathbb{C} \text{ s.t. } F(g z r(\theta)) = e^{-ik\theta} z^{k-2} F(g) \text{ for } z \in \mathbb{R}^\times,\ r(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \right\}$$

multiplied at ∞-component

Caveat: We have not discussed how to translate the holomorphicity yet. (later in Lecture 3)

§3 Automorphic forms

Definition A Hecke character is a character $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$

For example, if $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a Dirichlet character then $\omega : \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \longrightarrow \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \mathbb{R}_{>0}^\times \cong \prod_p \mathbb{Z}_p^\times \xrightarrow{\psi} \mathbb{C}^\times$ is a Hecke character

An <u>automorphic form</u> on $GL_2(\mathbb{A}_\mathbb{Q})$ (with central character ω) is a function

$$\phi : GL_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}. \quad \text{s.t.}$$

(1) (automorphy) $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G(\mathbb{Q})$

(2) (central character) for $z \in Z(\mathbb{A}_\mathbb{Q})$, $\phi(gz) = \omega(z)\phi(g)$

(3) (smoothness) $\exists$ open compact subgp $K_f \subseteq GL_2(\mathbb{A}_f)$ s.t.

(level structure) $\phi(gk_f) = \phi(g) \quad \forall k_f \in K_f$

& $k_\infty \longmapsto \phi(gk_\infty)$ is smooth in $k_\infty \in GL_2(\mathbb{R})$

(4) (K_∞-finite) (weight)
- <u>version 1</u>: $\phi = \sum \phi_k$ (finite sum; k integers) s.t. $\phi_k(g\, r(\theta)) = e^{-ik\theta} \cdot \phi_k(g)$.
- <u>version 2</u>: $\langle \phi(k\, r(\theta)) \,;\, \theta \in [0, 2\pi] \rangle$ is a finite dim'l $\mathbb{C}$-vector space.

(5) ($\mathfrak{z}$-finite) (holomorphy and more) Let $C :=$ Casimir operator (discussed in lecture 3)

acting as differential operators on the ∞-component

<u>version 1</u>: ϕ is a finite sum of C-eigenvectors

<u>version 2</u>: $\langle \phi, C\phi, C^2\phi, \cdots \rangle$ is a finite dim'l $\mathbb{C}$-vector space

(6) (asymptotics) (growth condition near cusps) for every $c > 0$ and a compact set Ω of $G(\mathbb{A})$, $\exists$ consts C, N

s.t. $\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) \leq C|a|^N \quad \forall g \in \Omega,\ a \in \mathbb{A}^\times$ with $|a| > C$

(say ϕ is <u>slowly increasing</u>)

We say that ϕ is a <u>cusp form</u> if and only if

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0 \quad \text{for almost all } g$$

Denote $\mathcal{A}_{cusp}(GL_2(\mathbb{Q}); \omega) :=$ space of cuspidal automorphic forms on $GL_2(\mathbb{A}_\mathbb{Q})$ with central character ω. $\leftarrow$ ∞-dim'l huge space

<u>Exercise</u>: When $\phi = \phi_f$ comes from modular forms, show that the cusp form condition

is equivalent to the cusp form condition on modular forms.

Fact: $\mathcal{A}_{cusp}(GL_2(\mathbb{Q});\omega) \subseteq L^2_{cusp}({}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A});\omega)$ dense (if ω is unitary)

$$\text{where } \langle \phi, \phi' \rangle_{L^2} := \int_{GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})} \phi(g)\overline{\phi'(g)}\,dg$$

Remark: If G is a general reductive group/$\mathbb{Q}$, can define automorphic forms as

$$\phi : G(\mathbb{Q})\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$$

* admitting a central char, i.e. $Z :=$ center of G, $\exists\, \omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to S^1$ cont.

$$\phi(zg) = \omega(z)\phi(g)$$

* $K_f \times K_\infty$-smooth ($K_\infty :=$ max'l compact mod center)
* K_∞-finite: $\langle K_\infty \cdot \phi \rangle$ is finite dim'l. note: K_∞-compact mod center $\Rightarrow$ all irred. rep'ns are fin. dim'l.
* $\mathfrak{z}$-finite. $\langle Z(U(\mathfrak{g}))\cdot\phi\rangle$ finite dim'l
* ϕ is slowly increasing

Say ϕ is called cuspidal, if $\forall\, N \subseteq G$ unipotent subgp/$\mathbb{Q}$

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng) = 0 \quad \text{for almost all } g.$$

§4. Automorphic representations

$$\mathcal{A}_{cusp}(GL_2(\mathbb{A});\omega) \hookrightarrow L^2_{cusp}(GL_2(\mathbb{A}),\omega)$$

if ω is unitary, i.e. $\omega(\mathbb{Q}^\times\backslash\mathbb{A}^\times) \subseteq S^1 \subseteq \mathbb{C}^\times$

$\circlearrowleft$ $GL_2(\mathbb{A}_f)$, by right translation (not quite $GL_2(\mathbb{R})$ ...) let's ignore this for the moment.

Fact: $\mathcal{A}_{cusp}(GL_2(\mathbb{A}),\omega) = \bigoplus_\pi \pi$ direct sum of irreducible rep'ns of "$GL_2(\mathbb{A})$" with multiplicity one

special to $GL_N(\mathbb{A}_F)$

Each π decomposes into $\pi_\infty \otimes \bigotimes_p' \pi_p$, with each π_p irred rep'n of $GL_2(\mathbb{Q}_p)$

"$GL_2(\mathbb{R})$" ↑ will explain $\otimes'$ in next lecture.

Conversely, given π_p, π_∞, rep'ns of $GL_2(\mathbb{Q}_p)$ & "$GL_2(\mathbb{R})$",

we say $\pi := \bigotimes_v' \pi_v$ is <u>automorphic</u> if π appears in $\mathcal{A}_{cusp}(GL_2(\mathbb{A}), \omega)$

<u>Remark</u>: Being automorphic is a <u>very strong</u> condition:

* it's almost equivalent to asking all $\rho_p : \mathrm{Gal}_{\mathbb{Q}_p} \to GL_2(\mathbb{Q}_\ell)$'s come from restricting a global $\rho : \mathrm{Gal}_{\mathbb{Q}} \to GL_2(\mathbb{Q}_\ell)$, <u>up to conjugation</u>
* even given a single π_p, we may not find other π_v's s.t. $\pi_p \otimes \bigotimes' \pi_v$ is automorphic (b/c $\mathrm{tr}\, \rho_p(\mathrm{Frob}_p)$ is usually an algebraic integer if ρ_p comes from ρ.

<u>Remark</u>: The smoothness (at non-archimedean places) condition implies

$$\mathcal{A}_{cusp}(GL_2(\mathbb{A}), \omega) = \bigcup_{\substack{K_f \subseteq GL_2(\mathbb{A}_f) \\ \text{open compact}}} \mathcal{A}_{cusp}(GL_2(\mathbb{A}), \omega)^{K_f}$$

$$= \bigcup_{K_f} \left\{ \text{smooth functions } \phi : GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K_f \to \mathbb{C} \ \underset{(2)(4)(5)(6)}{\text{satisfying}} \right\}$$

$$\bigoplus_{\pi} \pi = \bigcup_{\substack{K_f = \prod_p K_p \\ K_p \subseteq GL_2(\mathbb{Q}_p) \\ \text{open cpt} \\ \& \ K_p = GL_2(\mathbb{Z}_p) \text{ for almost all } p}} \bigoplus_{\substack{\pi \text{ autom.} \\ \text{cusp.}}} \pi_\infty \otimes \underbrace{\bigotimes_p (\pi_p)^{K_p}}.$$

↑ see in next lecture that for all but finitely many p, $\dim(\pi_p)^{K_p} = 1$.

<u>Remark</u>: Any π that contribute to $\mathcal{A}_{cusp}(GL_2(\mathbb{A}), \omega)^{K_f} \hookleftarrow S_k(K_f)$ must satisfy $\pi_p^{K_p} \neq 0 \ \forall p$.

<u>Next</u>: explain Hecke action on $\mathcal{A}_{cusp}(GL_2(\mathbb{A}), \omega)^{K_f}$ via local rep'n theory.

§5 Representation theory of $GL_2(\mathbb{Q}_p)$

<u>Definition</u> Let F_v be a finite ext'n of $\mathbb{Q}_p$.

Let G be an algebraic group over F_v. Write $G_v := G(F_v)$

A representation $\pi_v/\mathbb{C}$ of G_v is <u>smooth</u> if

$$\forall x \in \pi_v,\ \exists \text{ open compact } K_v \subseteq G_v \text{ s.t. } K_v \cdot x = x$$

$$\left(\Longleftrightarrow \pi_v = \bigcup_{K_v} \pi_v^{K_v} \right)$$

It is called <u>admissible</u> if $\forall$ open compact $K_v \subseteq G_v$, $\dim \pi_v^{K_v} < \infty$

<u>Definition</u>: Say π_v is <u>unitary</u> if there exists a non-degenerate Hermitian form on π_v s.t. $(gx, gy) = (x, y)$ $\forall x, y \in \pi_v$.

All local components π_v from $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)$ are unitary.

* Fix an open compact subgroup $K_v \subseteq G_v$

$\forall g \in G_v$, can define $[K g K] : \pi_v^{K_v} \longrightarrow \pi_v^{K_v}$ as

write: $K_v g K_v = \bigsqcup_i g_i K_v$ as coset decomposition

then $[K_v g K_v](x) := \sum_i g_i(x)$

<u>Alternative point of view</u>: Consider the Hecke algebra

$$\mathcal{H}(G_v, K_v) = \mathbb{C}_c[K_v \backslash G_v / K_v] := \{ f : G_v \to \mathbb{C},\ \text{bi-}K_v\text{-inv, compactly supported funcs} \}$$

Fix a Haar measure μ on G_v s.t. $\mu(K_v) = 1$.

$\rightsquigarrow \mathcal{H}(G_v, K_v)$ is an algebra under convolution:

$$f_1 * f_2(g) := \int_{h \in G_v} f_1(h) f_2(h^{-1} g)\, dh$$

& $\mathcal{H}(G_v, K_v)$ acts on $\pi_v^{K_v}$ via "integration"

$$f * x := \int_{g \in G_v} f(g)\, \pi_v(g)(x)\, dg$$

When $f = \mathbb{1}_{[K_v g K_v]}$, $f *$ action is the same as $[K_v g K_v]$ def'd earlier.