Classical $\to$ spectral representability theorem:

Thm SAG 18.1.0.2: Let $X : \mathrm{CAlg}^{cn} \to \mathcal{S}$ be a functor. Then X is representable by a spectral DM stack if and only if it is nilcomplete, infinitesimally cohesive, admits a cotangent complex, and the functor $X|_{\mathrm{CAlg}^{\heartsuit}}$ is representable by a classical DM stack.

Heuristic principle:

Spectral / Derived algebraic geometry = classical algebraic geometry + deformation theory.

4. Schlessinger's criterion

Def: $X, Y : \mathrm{CAlg}^{cn} \to \mathcal{S}$ functors, $f : X \to Y$ natural transformation. The relative de Rham space is the functor

$$(X/Y)_{dR} : \mathrm{CAlg}^{cn} \longrightarrow \mathcal{S}$$
$$R \longmapsto \operatorname*{colim}_{\substack{I \\ \text{all nilpotent ideals of } \pi_0(R)}} X(\pi_0(R)/I) \times_{Y(\pi_0(R)/I)} Y(R).$$

Example: If f is locally almost of finite presentation, then

$$(X/Y)_{dR}(R) = Y(R) \times_{Y(R^{red})} X(R^{red}).$$

Def: A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of formal spectral DM stacks is a formal thickening if

(1) The induced map $\mathfrak{X}^{red} \to \mathfrak{Y}^{red}$ is an equivalence

(2) It is representable by closed immersions which are lafp.

Representability of the de Rham space by formal thickenings:

SAG 18.2.3.1: $X, Y : \mathrm{CAlg}^{cn} \to \mathcal{S}$ functors, $f : X \to Y$. Assume

(0) X is representable by a formal spectral DM stack $\mathfrak{X}$.

(1) Y is nilcomplete, infinitesimally cohesive and admits a cotangent complex.

(2) $L_{X/Y} \in \mathrm{QCoh}(X)$ is 1-connective and almost perfect

(3) Y is a sheaf for the étale topology.

Then the relative de Rham space $(X/Y)_{dR}$ is representable by a formal thickening of $\mathfrak{X}$.

e.g field K, $\mathrm{Spec}\, K \to Y$

Here the theorem applies only if this is a closed point.

We need a more technical version in order to drop the 1-connectivity assump.

Existence of formal charts (SAG 18.2.5.1). Assume

(0) $X = \mathrm{Sp\acute{e}t}\, B$, $B \in \mathrm{CAlg}^{cn}$

(1) Same as before

(2) Y is formally complete along f, i.e. $\forall R \in \mathrm{CAlg}^{cn}$

$$\underset{I}{\mathrm{colim}}\, X(R/I) \xrightarrow{\sim} \underset{I}{\mathrm{colim}}\, Y(R/I)$$

all nilpotent ideals of $\pi_0(R)$

(3) We are given a morphism $\alpha : \mathcal{F} \to L_{X/Y}$ in $\mathrm{QCoh}(X)$ where $\mathcal{F}$ is perfect of tor-amplitude ≤ 0, $\mathrm{cofib}(\alpha)$ is 1-connective and almost perfect.

Then f factors as $X \xrightarrow{f'} U \xrightarrow{f''} Y$ where

- $U \simeq \mathrm{Spf}(A)$, A adic $\mathbb{E}_\infty$-ring
- f' is a formal thickening
- $\alpha \simeq (f'^* L_{U/Y} \to L_{X/Y})$

5. Artin-Lurie representability theorem

Thm SAG 18.3.0.1: $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ functor, $f: X \to \mathrm{Spec}\, R$, where R is a noetherian $\mathbb{E}_\infty$-ring and $\pi_0 R$ is a $\underbrace{\text{Grothendieck ring}}_{\text{e.g. f.g. rings}}$. Let $n \geq 0$.

Then X is representable by a spectral DM n-stack locally of finite presentation over R if and only if the following hold:

(1) $\forall$ discrete comm ring A, the space $X(A)$ is n-truncated.

(2) X is a sheaf for the étale topology

(3) X is nilcomplete, infinitesimally cohesive and integrable.

(4) X admits a connective cotangent complex L_X

(5) f is locally almost of finite presentation.

Main technical tool: Popescu's smoothing theorem in commutative algebra.

Recall: A map $\phi: A \to B$ of noetherian comm rings is called geometrically regular if it is flat, and $\forall$ prime ideal $\mathfrak{p} \subset A$, $\forall$ finite extension K of the residue field $\kappa(\mathfrak{p})$, the comm ring $K \otimes_A B$ is regular.

A comm ring A is a Grothendieck ring if it is noetherian, and $\forall$ prime ideal $\mathfrak{p} \subset A$, the map $A_\mathfrak{p} \to \hat{A}_\mathfrak{p}$ is geometrically regular.

Popescu's smoothing theorem: $\phi: A \to B$ map of noetherian rings. TFAE:

(1) ϕ is geometrically regular

(2) B can be realized as a filter colimit of smooth A-algebras.

Four major steps in the proof of Artin-Lurie representability theorem:

Step 1: Existence of formal charts (Schlessinger's criterion)

Step 2: Formal charts $\rightsquigarrow$ approximately étale charts

Step 3: Approximately étale charts $\rightsquigarrow$ étale charts

Step 4: Conclude by induction on n.

as in the representability theorem

Step 2: SAG 16.3.1.1: $X, Y: \mathrm{CAlg}^{cn} \to \mathcal{S}$ functors, $q: X \to Y$, $Y = \mathrm{Spec}\, R$

Assume: (1) X is infinitesimally cohesive, nilcomplete, integrable

(2) q is locally almost of finite presentation

(3) q admits a cotangent complex $L_{X/Y}$.

Let $f: \mathrm{Spec}\, \kappa \to X$, κ finitely generated field extension of some residue field of R. Then f factors as $\mathrm{Spec}\, \kappa \to \mathrm{Spec}\, B \to X$ where B is afp over R, and $\pi_1(\kappa \otimes_B L_{\mathrm{Spec}\, B/X}) = 0$.

Idea of proof: Let $\hat{X} := (\mathrm{Spec}\, \kappa / X)_{dR}$

Choose fiber seq $\mathcal{F} \xrightarrow{\alpha} L_{\mathrm{Spec}\, \kappa/\hat{X}} \longrightarrow \mathcal{G}$ in Mod_κ

where $\mathcal{F}$ is perfect of Tor-amplitude ≤ 0, and $\mathcal{G}$ is 1-connective.

Thm of $\exists$ of formal charts $\Rightarrow$ factorization $\mathrm{Spec}\, \kappa \longrightarrow \underbrace{U = \mathrm{Spf}\, A}_{\text{formal thickening}} \longrightarrow \hat{X}$

X is integrable and nilcomplete $\rightsquigarrow$ $\mathrm{Spec}\, \kappa \to \mathrm{Spec}\, A \to X$

Then we use Popescu's smoothing theorem to approximate A by an afp $\mathbb{E}_\infty$-ring B.

Step 3: SAG 18.3.2.1: Same assumptions as in Step 2 + assume $L_{X/Y}$ connective.

Given $f: \mathrm{Spec}\, A \to X$ $\quad A \in \mathrm{CAlg}^{cn}$, $\quad \mathfrak{p}$ prime ideal of $\pi_0 A$, κ res field.

Then $\exists$ étale A-algebra A', prime ideal $\mathfrak{p}'$ of A' lying over $\mathfrak{p}$ s.t.

$\mathrm{Spec}\, A' \to X$ factors as $\mathrm{Spec}\, A' \to \mathrm{Spec}\, B \to X$ where B is afp over R,

and $L_{\mathrm{Spec}\, B/X}$ vanishes.

Idea of proof: Apply Step 2 to

$$\begin{tikzcd}
\mathrm{Spec}\, \kappa \arrow[r] \arrow[dr] & \mathrm{Spec}\, A \arrow[r] & X \\
& \mathrm{Spec}\, B \arrow[ur] &
\end{tikzcd}$$

where B is afp over R, and $\pi_1(\kappa \otimes_B L_{\mathrm{Spec}\, B/X}) = 0$.

$\rightsquigarrow$ modify B $\quad L_{\mathrm{Spec}\, B/X}$ 2-connective

$\rightsquigarrow$ modify B $\quad L_{\mathrm{Spec}\, B/X} \simeq 0$

Using Popescu's smoothing theorem, we find étale A-algebra A' with

$\kappa \otimes_A A' \neq 0$, s.t. $\mathrm{Spec}\, A' \to X$ factors through B.

Step 4: $\Leftarrow$: we proceed by induction on n.

Condition (1) is equiv to

(1') $\forall$ discrete commutative ring A, the map $X(A) \to \mathrm{Map}_{\mathrm{CAlg}}(R, A)$ has n-truncated homotopy fibers.

Now for $n=-2$: (1'): $\forall$ discrete comm ring A, $X(A) \xrightarrow{\sim} \mathrm{Map}_{\mathrm{CAlg}^{cn}}(R, A)$.

Use the classical $\to$ spectral representability theorem.

$n \geq -1$: Let $X_0 = \coprod$ all $\mathrm{Spec}\, B_\alpha \to X$ s.t. B_α afp $/R$ $L_{\mathrm{Spec}\, B_\alpha / X} = 0$.

$X_\bullet$ Čech nerve of $X_0 \to X$.

Induction hypothesis $\Rightarrow$ Each X_m is representable by a spectral DM stack.

$L_{X_0/X} \simeq 0 \Rightarrow$ all $L_{X_m/X_{m'}} \simeq 0 \Rightarrow$ all $X_m \to X_{m'}$ étale

$\Rightarrow$ colimit $|X_\bullet|$ is a spectral DM stack.

To show $|X_\bullet| \simeq X$, it suffices to show that $X_0 \to X$ is an effective epi of étale sheaves, which follows from Step 3.

6. Generalization of the representability theorem to Artin stacks.

Spectral representability theorem for Artin stacks:

$X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ functor, $f: X \to \mathrm{Spec}\, R$, where R is a noetherian E_∞-ring and $\pi_0 R$ is a $\underbrace{\text{Grothendieck ring}}_{\text{e.g. f.g. rings}}$. Let $n \geq 0$.

Then X is representable by a spectral Artin n-stack locally of finite presentation over R if and only if the following hold:

(1) $\forall$ discrete comm ring A, the space $X(A)$ is n-truncated.

(2) X is a sheaf for the étale topology

(3) X is nilcomplete, infinitesimally cohesive and integrable.

(4) X admits a (almost-connective) cotangent complex L_X

(5) f is locally almost of finite presentation.

6. Applications of the representability theorem

6.1 Mapping stack

$X, Y, Z : \mathrm{CAlg}^{cn} \to \mathcal{S}$ functors

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

We have a functor $\underline{\mathrm{Map}}_{/Z}(X, Y) \in \mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})_{/Z}$ equipped with an evaluation map $e: X \times_Z \underline{\mathrm{Map}}_{/Z}(X, Y) \to Y$ with the following universal property: $\forall\, W \in \mathrm{Fun}(\mathrm{CAlg}^{cn}, \mathcal{S})_{/Z}$, composition with e induces a homot equiv

$$\mathrm{Map}(W, \underline{\mathrm{Map}}_{/Z}(X, Y)) \longrightarrow \mathrm{Map}(W \times_Z X, Y)$$

SAG 19.1.6.1: Assume f is representable, proper, flat, lafp,

g is representable, quasi-compact, quasi-separated,

then $\underline{\mathrm{Map}}_{/Z}(X, Y) \longrightarrow Z$ is representable.

6.2: QCoh Perf Vect Pic