Rem: Every spectral DM-stack X determines a functor of points

$$h_X : \mathrm{CAlg}^{cn} \longrightarrow \mathcal{S}$$
$$R \longmapsto \mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, M).$$

It makes sense to consider $\mathrm{QCoh}(X)$ for any functor $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$.

An object $\mathcal{F} \in \mathrm{QCoh}(X)$ can be viewed as a rule which assigns to each point $\eta \in X(R)$ an R-module $\mathcal{F}(\eta)$, which depends functorially on R in the following sense:

if $\phi: R \to R'$ is a map in CAlg^{cn}, and η' denote the image of η in $X(R')$, then we have a canonical equivalence $R' \otimes_R \mathcal{F}(\eta) \simeq \mathcal{F}(\eta')$.

See SAG 6.2 for the precise categorical formulation.

VI. Representability theorems

Idea: Derived stacks arise naturally in moduli problems via the representability theorem.

1. Global cotangent complexes

$\mathcal{X}$ ∞-topos. Applying the abstract cotangent complex formalism in HA, we obtain an absolute cotangent complex functor $L: \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X}) \to \mathrm{Mod}(\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X}))$.

$X = (\mathcal{X}, \mathcal{O}_X)$ spectrally ringed ∞-topos

The absolute cotangent complex of X is $L_X := L_{\mathcal{O}_X} \in \mathrm{Mod}_{\mathcal{O}_X}$.

$\phi: X = (\mathcal{X}, \mathcal{O}_X) \longrightarrow Y = (\mathcal{Y}, \mathcal{O}_Y)$ morphism of spectrally ringed ∞-topoi.

The relative cotangent complex of ϕ is $L_{X/Y} := \mathrm{cofib}(\phi^* L_Y \to L_X)$.

SAG 17.1.2: $\phi: X \to Y$ morphism of non-connective spectral DM stacks. Then $L_{X/Y}$ is a quasi-coherent sheaf on X.

Cotangent complexes make sense also for general functors from CAlg^{cn} to $\mathcal{S}$:

Let $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ be a functor. Let Mod^X_{cn} denote the ∞-category of triples (A, η, M), where $A \in \mathrm{CAlg}^{cn}$, $\eta \in X(A)$, $M \in \mathrm{Mod}_A^{cn}$.

Given a natural transformation $\alpha: X \to Y$ between functors $X, Y: \mathrm{CAlg}^{cn} \to \mathcal{S}$.

Consider the functor $F: \mathrm{Mod}^X_{cn} \to \mathcal{S}$

$$(A, \eta, M) \mapsto \mathrm{fib}\left(X(A \oplus M) \to X(A) \times_{Y(A)} Y(A \oplus M) \right)$$

over the point determined by η

We say that α admits a cotangent complex if there is $L_{X/Y} \in \mathrm{QCoh}(X)$ s.t. $\forall A \in \mathrm{CAlg}^{cn}$, $\eta \in X(A)$, the induced functor $F_\eta : \mathrm{Mod}_A^{cn} \to \mathcal{S}$ is corepresented by $M_\eta := \eta^* L_{X/Y}$, which is almost connective (i.e. n-connective for $n << 0$).

2. Necessary conditions for the representability theorem

2.1 Cohesive functor

Def: $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ functor. X is cohesive if $\forall$ pullback diagram

$$\begin{array}{ccc} A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

in CAlg^{cn} for which the maps $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ are surjective, the induced diagram

$$\begin{array}{ccc} X(A') & \longrightarrow & X(A) \\ \downarrow & & \downarrow \end{array}$$

$$X(B') \longrightarrow X(B)$$

is a pullback in $\mathcal{S}$.

SAG 16.1.3.1: Locally spectrally ringed topos $\rightsquigarrow$ cohesive functor

In particular, spectral DM stack $\rightsquigarrow$ cohesive functor.

Def: A functor $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ is infinitesimally cohesive if in the above definition, we further assume that the surjections $\pi_0 A \to \pi_0 B$ and $\pi_0 B' \to \pi_0 B$ have nilpotent ideals.

Remark: $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$, $R \in \mathrm{CAlg}^{cn}$, $\tilde{R}$ square-zero extension of R by a connective R-module M given by a derivation $d: L_R \to M[1]$. We have a commutative diagram of spaces

$$\begin{array}{ccc} X(\tilde{R}) & \longrightarrow & X(R) \\ \downarrow & & \downarrow \\ \eta \in X(R) & \longrightarrow & X(R \oplus M[1]) \end{array}$$

Suppose X is infinitesimally cohesive and admits a cotangent complex.
Let ν denote the composite map $\eta^* L_X \to L_R \to M[1]$.
Then η can be lifted to a point of $X(\tilde{R})$ if and only if $\nu = 0$ in $\mathrm{Ext}^1_R(\eta^* L_X, M)$.

2.2 Nilcomplete functors

Def: A functor $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ is nilcomplete if $\forall R \in \mathrm{CAlg}^{cn}$,

$X(R) \to \lim X(\tau_{\leq n} R)$ is a homotopy equivalence.

SAG 17.3.2.3: Connective locally spectrally ringed ∞-topos $\rightsquigarrow$ nilcomplete functor

In particular, spectral DM stacks $\rightsquigarrow$ nilcomplete functor.

2.3 Integrable functor

Def: A functor $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ is integrable if for any local noetherian $\mathbb{E}_\infty$-ring A which is complete wrt its maximal ideal $\mathfrak{m} \subset \pi_0 A$, the inclusion $\mathrm{Spf}\, A \hookrightarrow \mathrm{Sp\acute{e}t}\, A$ induces a homotopy equivalence

$$X(A) \simeq \mathrm{Map}(\mathrm{Sp\acute{e}t}\, A, X) \xrightarrow{\sim} \mathrm{Map}(\mathrm{Spf}\, A, X).$$

Def: Let $n \geq 0$. A spectral DM n-stack is a spectral DM stack s.t. $\forall$ comm ring R, the mapping space $\mathrm{Map}_{\mathrm{SpDM}}(\mathrm{Sp\acute{e}t}\, R, X)$ is n-truncated.

A spectral algebraic space is a spectral DM 0-stack.

SAG 17.3.4.2: Spectral DM n-stack $\rightsquigarrow$ integrable functor.

2.4 Finiteness conditions on morphisms

Recall: A morphism $\phi: A \to B$ of connective $\mathbb{E}_\infty$-rings is locally of finite presentation if the functor $C \mapsto \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$ commutes with filtered colimits. It is almost of finite presentation if $C \mapsto \mathrm{Map}_{\mathrm{CAlg}_A}(B, C)$ preserves filtered colimits when restricted to n-truncated connective A-algebras for each $n \geq 0$.

Def: A morphism $f: X \to Y$ of spectral DM stacks is locally (resp. locally almost) of finite presentation if for every commutative diagram

$$\begin{tikzcd} \mathrm{Sp\acute{e}t}\, B \arrow[r, "\acute{e}t"] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Sp\acute{e}t}\, A \arrow[r, "\acute{e}t"] & Y \end{tikzcd}$$

the E_∞-ring B is locally (resp. almost) of finite presentation over A.

For the representability theorem, we need to generalize the above definition to functors.

Def: $X, Y: \mathrm{CAlg}^{cn} \to \mathcal{S}$ functors, $f: X \to Y$

It is locally of finite presentation if for any filtered colimit $A = \mathrm{colim}\, A_\alpha$ of connective E_∞-rings, the canonical map $\mathrm{colim}\, X(A_\alpha) \to X(A) \times_{Y(A)} \mathrm{colim}\, Y(A_\alpha)$ is a homotopy equiv.

It is locally almost of finite presentation if in the above definition, we restrict to filtered colimit of m-truncated connective E_∞-rings for every $m \geq 0$.

SAG 17.4.2: $X, Y: \mathrm{CAlg}^{cn} \to \mathcal{S}$. $f: X \to Y$. Assume f has a cotangent complex. If f is locally (resp. locally almost) of finite presentation, then $L_{X/Y}$ is perfect (almost perfect). The converse holds with an additional assumption on π_0.

3. Representability theorem: from classical to spectral algebraic geometry.

Thm SAG 18.1.0.2: Let $X: \mathrm{CAlg}^{cn} \to \mathcal{S}$ be a functor. Then X is representable by a spectral DM stack if and only if it is nilcomplete, infinitesimally cohesive, admits a cotangent complex, and the functor $X|_{\mathrm{CAlg}^\heartsuit}$ is representable by a classical DM stack.

Heuristic principle:

Spectral/Derived algebraic geometry = classical algebraic geometry + deformation theory.

Idea of proof:

Step 1: We prove that X is a sheaf wrt the étale topology

Induction on $X|_{\mathrm{CAlg}^{\le n}}$.

Step 2: Approximation to étale morphisms:

Suppose we have a spectral DM stack $Y_0 = (\mathcal{Y}, \mathcal{O}_0)$ and a map $f_0: Y_0 \to X$ for which the relative cotangent complex $L_{Y_0/X}$ is 2-connective. Then f_0 factors as a composition

$$Y_0 \xrightarrow{g} Y \xrightarrow{f} X$$

where f is étale, $Y = (\mathcal{Y}, \mathcal{O})$ is a spectral DM stack, and g is induced by a 1-connective map $\mathcal{O} \to \mathcal{O}_0$.

Construct $\mathcal{O}$ as a limit of a tower of CAlg-valued sheaves on $\mathcal{Y}$

$$\cdots \longrightarrow \mathcal{O}_2 \longrightarrow \mathcal{O}_1 \longrightarrow \mathcal{O}_0$$

where each $Y_k := (\mathcal{Y}, \mathcal{O}_k)$ has a map $f_k: Y_k \to X$ such that the relative cotangent complex $L_{Y_i/X}$ is (2^k+1)-connective.

Step 3: Establish the assumption of Step 2 via a left Kan extension.