2.5 Pregeometries

Idea: generate geometries with simpler data.

Def: A pregeometry is an ∞-category $\mathcal{T}$ admitting $\underset{\text{finite}}{\wedge}$ products, equipped with an admissibility structure.

Def: $\mathcal{T}$ pregeometry, $\mathcal{X}$ ∞-topos. A $\mathcal{T}$-structure on $\mathcal{X}$ is a functor $\mathcal{O}: \mathcal{T} \to \mathcal{X}$ s.t.

1) $\mathcal{O}$ preserves finite products.
2) $\mathcal{O}$ preserves pullbacks along admissible morphisms
3) $\forall$ admissible covering $\{U_\alpha \to X\}$ in $\mathcal{T}$, the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \to \mathcal{O}(X)$ is an effective epimorphism in $\mathcal{X}$.

Denote $\mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \subset \mathrm{Fun}(\mathcal{T}, \mathcal{X})$, $\mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X}) \subset \mathrm{Str}_{\mathcal{T}}(\mathcal{X})$

A transformation of pregeometries $f: \mathcal{T} \to \mathcal{T}'$ is a functor which preserves finite products, adm morphisms, adm coverings, and pullbacks along adm morphisms.

A transformation of pregeometries $f: \mathcal{T} \to \mathcal{G}$ exhibits $\mathcal{G}$ as a geometric envelope of $\mathcal{T}$ if

1) $\mathcal{G}$ is a geometry with the coarsest structure s.t. f is a transformation of pregeometries.
2) $\forall$ ∞-category $\mathcal{C}$ idempotent complete and admitting finite limits,

Composition with f induces an equiv

$$\mathrm{Fun}^{lex}(\mathcal{G}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{ad}(\mathcal{T}, \mathcal{C})$$ preserves finite products, pullbacks along adm morphisms.

Proposition: Geometric envelope $f: \mathcal{T} \to \mathcal{G}$ exists. For any ∞-topos $\mathcal{X}$, Composition with f induces equivalence of ∞-categories

$$\mathrm{Str}_{\mathcal{G}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Str}_{\mathcal{T}}(\mathcal{X}) \qquad \mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Str}^{loc}_{\mathcal{T}}(\mathcal{X})$$

Def: $\mathcal{T}$ pregeometry. A $\mathcal{T}$-scheme is a $\mathcal{T}$-structured topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ equiv to a $\mathcal{G}$-scheme via the equivalence above.

Denote $\mathrm{Sch}(\mathcal{T}) \subset {}^{L}\mathrm{Top}(\mathcal{T})^{op}$ spanned by $\mathcal{T}$-schemes.

3. Main examples of derived geometry

3.1 Derived algebraic geometry (Zariski topology)

Let k be a commutative ring.

Define pregeometry $\mathcal{T}_{Zar}(k) := \{$ principle open subsets of some affine space $\mathbb{A}^n_k \}$

Admissible morphisms are open immersions

Admissible coverings are coverings by principle open subsets.

Define geometry $\mathcal{G}^{der}_{Zar}(k) :=$ opposite of the full subcategory of SCR_k (simplicial comm rings) spanned by compact objects.

Admissible morphisms are $\mathrm{Spec}\, B[\frac{1}{b}] \to \mathrm{Spec}\, B$ for some $b \in \pi_0 B$.

Admissible coverings are $\{\mathrm{Spec}\, B[\frac{1}{b_\alpha}] \to \mathrm{Spec}\, B\}$ s.t. b_α generate the unit ideal in $\pi_0 B$.

Proposition: $\mathcal{S}^{der}_{Zar}(k)$ is a geometric envelope of $\mathcal{T}_{Zar}(k)$

Def: A derived k-scheme is a $\mathcal{T}_{Zar}(k)$-scheme.

The theory of derived schemes are well-behaved when the underlying ∞-topoi are 0-localic. In the more general case, we must replace the Zariski topology by the étale topology.

3.2 Derived algebraic geometry (étale topology)

Let k commutative ring.

Define pregeometry $\mathcal{T}_{\acute{e}t}(k) := \{$affine schemes étale over some affine space $\mathbb{A}^n_k\}$

Admissible morphisms are étale morphisms. Admissible coverings are étale coverings.

Similar to the case of $\mathcal{T}_{Zar}(k)$, the geometric envelope of $\mathcal{T}_{\acute{e}t}(k)$ admits an explicit descriptions in terms of SCR.

Def: A derived Deligne-Mumford stack over k is a $\mathcal{T}_{\acute{e}t}(k)$-scheme.

3.3 Derived differential topology

Define pregeometry $\mathrm{Diff} := \{$smooth submanifolds of some $\mathbb{R}^n$, smooth maps$\}$

Admissible morphisms are open inclusions, adm coverings are coverings by open inclusion.

$\rightsquigarrow$ derived differential topology. not yet well-developed

3.4 Derived complex analytic geometry

Define pregeometry $\mathcal{T}_{an}(\mathbb{C}) := \{$ complex manifolds $\}$

Admissible morphisms are locally biholomorphic maps.

Adm coverings are coverings by admissible morphisms

$\rightsquigarrow$ Derived complex analytic geometry

3.5 Derived non-archimedean geometry

k non-archimedean field, e.g. $\mathbb{Q}_p$, $\mathbb{C}((t))$, $\cdots$

Define pregeometry $\mathcal{T}_{an}(k) := \{$ smooth (rigid) k-analytic spaces $\}$

Admissible morphisms are étale morphisms.

Admissible coverings are étale coverings.

$\rightsquigarrow$ Derived non-archimedean geometry. Developed by Porta and me in the last few years.

3.6 Spectral algebraic geometry

k $\mathbb{E}_\infty$-ring

Define geometry $\mathcal{G}^{nSp}_{Zar}(k) := \{$ compact objects of $\mathrm{CAlg}_k \}^{op}$

Adm morphisms are $\mathrm{Spec}\, B[\frac{1}{b}] \longrightarrow \mathrm{Spec}\, B$ for some $b \in \pi_0 B$

Adm coverings are $\{ \mathrm{Spec}\, B[\frac{1}{b_\alpha}] \longrightarrow \mathrm{Spec}\, B \}$ s.t. b_α generate the unit ideal in $\pi_0 B \}$

If k is connective, denote $\mathcal{G}^{Sp}_{Zar}(k) \subset \mathcal{G}^{nSp}_{Zar}(k)$ spanned by connective k-algebras

$\mathcal{G}^{Sp}_{Zar}(k)$ is a geometric envelope of a pregeometry defined in terms of localized polynomial $\mathbb{E}_\infty$-rings.

Def: A non-connective spectral k-scheme is a $\mathcal{G}^{nSp}_{Zar}(k)$-scheme.

A spectral k-scheme is a $\mathcal{G}^{Sp}_{Zar}(k)$-scheme.

Replacing the Zariski topology by the étale topology, we obtain the geometries $\mathcal{G}^{Sp}_{ét}(k) \subset \mathcal{G}^{nSp}_{ét}(k)$, and hence the notion of (non-connective) spectral Deligne-Mumford stacks.

SAG 1.6.6: The ∞-category of (non-connective) spectral schemes embedds fully faithfully into the ∞-category of (non-connective) spectral DM stacks. So we will focus on the study of spectral DM stacks.

Given $A \in CAlg_k$, we denote $\operatorname{Sp\acute{e}t} A := \operatorname{Spec}^{\mathcal{G}^{nSp}_{ét}(k)} A$

étale spectrum

4. Quasi-coherent sheaves on spectral DM stacks.

Quasi-coherent sheaves can be phrased in the language of geometries and structured topoi (see DAG VIII 2). Here we just spell out the definitions explicitely following SAG §2.

Def: $\mathcal{X}$ ∞-topos, $\mathcal{O} \in Shv_{CAlg}(\mathcal{X})$ sheaf of E_∞-rings.

By VII 1.18, $Shv_{CAlg}(\mathcal{X}) \simeq CAlg(Shv_{Sp}(\mathcal{X}))$.

Let $Mod_{\mathcal{O}}$ denote the ∞-category $Mod_{\mathcal{O}}(Shv_{Sp}(\mathcal{X}))$ of $\mathcal{O}$-module objects of $Shv_{Sp}(\mathcal{X})$. Its objects are called sheaves of $\mathcal{O}$-modules.

Def: Let $\mathrm{Mod} = \mathrm{Mod}(\mathrm{Sp})$, consisting of pairs (A, M), where A is an $\mathbb{E}_\infty$-ring, and M is an A-module.

Let $\infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{Mod}}$ denote the ∞-category of triples $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ where $\mathcal{X}$ is an ∞-topos, $\mathcal{O}$ is a strictly Henselian sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, i.e. a $\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$-structure on $\mathcal{X}$, and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules.

Example: A $\mathbb{E}_\infty$-ring, M A-module. Then the functor

$$\begin{aligned} \mathrm{CAlg}^{\text{ét}}_A &\longrightarrow \mathrm{Mod} \\ B &\longmapsto (B, B \otimes_A M) \end{aligned}$$

gives a Mod-valued sheaf $(\mathcal{O}, \mathcal{F})$ on the ∞-topos $\mathrm{Shv}^{\text{ét}}_A$, and $(\mathrm{Shv}^{\text{ét}}_A, \mathcal{O}, \mathcal{F}) \in \infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{Mod}}$.

We have a functor $\mathrm{Sp\acute{e}t}_{\mathrm{Mod}} : \mathrm{Mod}^{\mathrm{op}} \longrightarrow \infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{Mod}}$

which is right adjoint to the global sections functor.

Def: $X = (\mathcal{X}, \mathcal{O})$ non-conn spectral DM stack, $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules on X.

$\mathcal{F}$ is quasi-coherent if $\exists$ covering $\coprod_\alpha U_\alpha \to 1$ (i.e. effective epimorphism)

s.t. $\forall \alpha$, $\exists$ an $\mathbb{E}_\infty$-ring A_α, an A_α-module M_α, and an equivalence

$$(\mathcal{X}_{/U_\alpha}, \mathcal{O}|_{U_\alpha}, \mathcal{F}|_{U_\alpha}) \simeq \mathrm{Sp\acute{e}t}_{\mathrm{Mod}}(A_\alpha, M_\alpha) \text{ in } \infty\mathrm{Top}^{\mathrm{sHen}}_{\mathrm{Mod}}.$$

Denote $\mathrm{QCoh}(X) \subset \mathrm{Mod}_{\mathcal{O}}$.

Pullbacks and pushforwards of quasi-coherent sheaves:

$f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ map of spectrally ringed ∞-topoi:

Combining the pushforward functor $f_* : \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{Y}) \longrightarrow \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ with restriction of scalars along the map $\mathcal{O}_{\mathcal{X}} \longrightarrow f_* \mathcal{O}_{\mathcal{Y}}$, we obtain a pushforward functor $f_* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$. It admits a left adjoint $f^* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$.

Rem 1: f^* preserves quasi-coherence, f_* not always

2: The functor f^* and f_* are "already derived", no need to choose resolutions for their constructions.

3. functor of points