2.2 Geometries

Idea: In order to make sense of "Locally ringed", we need an extra structure on the ∞-category $\mathcal{C}$.

Def: $\mathcal{G}$ ∞-category, an admissibility structure on $\mathcal{G}$ consists of the following data:

(1) A subcategory $\mathcal{G}^{ad} \subset \mathcal{G}$ spanned by all the objects, and admissible morphisms satisfying the following conditions:

(i) Pullback of admissible morphism always exists, and is again admissible

(ii)
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$
g, h admissible $\Rightarrow$ f admissible

(iii) Retract of admissible morphism is admissible.

(2) A Grothendieck topology on $\mathcal{G}$, which is generated by admissible morphisms in the following sense: any covering sieve $\mathcal{G}^{(0)}_{/X} \subset \mathcal{G}_{/X}$ contains a covering sieve which is generated by a collection of admissible morphisms $\{U_\alpha \to X\}$.

Def: A geometry consists of the following data:

(1) An ∞-category $\mathcal{G}$ which admits finite limits and is idempotent complete.

(2) An admissibility structure on $\mathcal{G}$.

$\mathcal{G}, \mathcal{G}'$ geometries. A functor $f: \mathcal{G} \to \mathcal{G}'$ is a transformation of geometries if it preserves finite limits, admissible morphisms and admissible coverings.

Def: $\mathcal{G}$ geometry, $\mathcal{X}$ ∞-topos. A $\mathcal{G}$-structure on $\mathcal{X}$ is a left exact functor $\mathcal{O}: \mathcal{G} \to \mathcal{X}$ s.t. for every admissible covering $\{U_\alpha \to U\}$ in $\mathcal{G}$, the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \longrightarrow \mathcal{O}(X)$ is an effective epimorphism in $\mathcal{X}$.

the Čech nerve is a colimit diagram

We call the pair $(\mathcal{X}, \mathcal{O})$ a $\mathcal{G}$-structured topos.

Denote $\mathrm{Str}_{\mathcal{G}}(\mathcal{X}) \subset \mathrm{Fun}(\mathcal{G}, \mathcal{X})$ spanned by $\mathcal{G}$-structures on $\mathcal{X}$.

Given $\mathcal{O}, \mathcal{O}': \mathcal{G} \to \mathcal{X}$ $\mathcal{G}$-structures, a natural transformation $\alpha: \mathcal{O} \to \mathcal{O}'$ is a local transformation of $\mathcal{G}$-structures if for every admissible morphism $U \longrightarrow X$ in $\mathcal{G}$,

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{O}'(U) \arrow[d] \\ \mathcal{O}(X) \arrow[r] & \mathcal{O}'(X) \end{tikzcd}$$

is a pullback square in $\mathcal{X}$.

Denote $\mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X}) \subset \mathrm{Str}_{\mathcal{G}}(\mathcal{X})$ spanned by local transformations of $\mathcal{G}$-structures.

A morphism $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ of $\mathcal{G}$-structured topoi consists of a geometric morphism $f^*: \mathcal{Y} \to \mathcal{X}$ of ∞-topoi and a morphism $\alpha: f^*\mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}_{\mathcal{X}}$ in $\mathrm{Str}^{loc}_{\mathcal{G}}(\mathcal{X})$.

We have an ∞-category of $\mathcal{G}$-structured topoi, denoted by ${}^L\mathrm{Top}(\mathcal{G})^{op}$.

A geometry $\mathcal{G}$ is discrete if the admissible morphisms are precisely the equivalences, and the Grothendieck topology on $\mathcal{G}$ is trivial, i.e. a sieve $\mathcal{G}^{\circ}_{/X} \subset \mathcal{G}_{/X}$ on an object $X \in \mathcal{G}$ is a covering sieve if and only if $\mathcal{G}^{\circ}_{/X} = \mathcal{G}_{/X}$.

Example: Let $\mathcal{G}_{Zar} := \{\text{affine schemes of finite type over } \mathbb{Z}\}$

$= \{\text{finitely generated commutative rings}\}^{op}$

We endow it with a structure of geometry as follows:

(1) A morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is admissible if and only if it induces an isomorphism $B[\frac{1}{b}] \xrightarrow{\sim} A$ for some $b \in B$.

(2) A collection of admissible morphisms $\{\operatorname{Spec} A[\frac{1}{a_i}] \longrightarrow \operatorname{Spec} A\}$ is a covering if and only if $\{a_i\} \subset A$ generates the unit ideal in A.

X topological space, $\mathcal{X} := \mathrm{Shv}(X)$ ∞-category of sheaves of spaces on $\mathcal{X}$.

Then $\mathcal{G}_{Zar}$-structures on $\mathcal{X}$ are exactly sheaves of commutative rings $\mathcal{O}$ on the topological space X which are local.

2.3 Spectrum functor

Recall: A comm ring, the affine scheme $\operatorname{Spec} A$ is characterized by the following universal property: For every locally ringed space $(X, \mathcal{O}_X)$, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Locally\,Ringed\,Space}}((X, \mathcal{O}_X), \operatorname{Spec} A) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CommRing}}(A, \Gamma(X, \mathcal{O}_X))$$

$$\simeq \mathrm{Hom}_{\mathrm{Ringed\,Space}}((X, \mathcal{O}_X), (*, A))$$

Note that $\mathrm{Locally\,Ringed\,Space} \overset{\text{full}}{\subset} {}^{L}\mathrm{Top}(\mathcal{G}_{Zar})^{op}$

$\mathrm{Ringed\,Space} \overset{\text{full}}{\subset} {}^{L}\mathrm{Top}(\mathcal{G}_{Zar,0})^{op}$

the underlying discrete geometry

Theorem V2.1.1: Let $f: \mathcal{G} \to \mathcal{G}'$ be a transformation of geometries. Then the induced functor ${}^L\mathcal{T}\mathrm{op}(\mathcal{G}') \longrightarrow {}^L\mathcal{T}\mathrm{op}(\mathcal{G})$ admits a left adjoint.

We denote the left adjoint by $\mathrm{Spec}_{\mathcal{G}}^{\mathcal{G}'}$, and call it the relative spectrum functor associated to f.

Let $\mathcal{G}$ be a geometry, and $\mathcal{G}_0$ the associated discrete geometry

Let $\mathrm{Spec}^{\mathcal{G}}$ denote the composition

$$\mathrm{Ind}(\mathcal{G}^{op}) \simeq \mathrm{Str}_{\mathcal{G}_0}(\mathcal{S}) \longrightarrow {}^L\mathcal{T}\mathrm{op}(\mathcal{G}_0) \xrightarrow{\mathrm{Spec}_{\mathcal{G}_0}^{\mathcal{G}}} {}^L\mathcal{T}\mathrm{op}(\mathcal{G})$$

We call $\mathrm{Spec}^{\mathcal{G}}$ the absolute spectrum functor associated to $\mathcal{G}$.

Remark: 1) The analytication functor from algebraic geometry to analytic geometry is an example of the relative spectrum functor.

2) Consider the functor ${}^L\mathcal{T}\mathrm{op}(\mathcal{G}) \times \mathcal{G} \longrightarrow \mathcal{S}$

$$((\mathcal{X}, \mathcal{O}), \quad U) \longmapsto \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, \mathcal{O}(U))$$

It induces a functor ${}^L\mathcal{T}\mathrm{op}(\mathcal{G}) \longrightarrow \mathrm{Fun}(\mathcal{G}, \mathcal{S})$, which factors through $\mathrm{Fun}^{lex}(\mathcal{G}, \mathcal{S}) \simeq \mathrm{Ind}(\mathcal{G}^{op})$.

We let $\Gamma_{\mathcal{G}}: {}^L\mathcal{T}\mathrm{op}(\mathcal{G}) \longrightarrow \mathrm{Ind}(\mathcal{G}^{op})$, and call it the $\mathcal{G}$-structured global sections functor.

By construction, $\mathrm{Spec}^{\mathcal{G}}: \mathrm{Ind}(\mathcal{G}^{op}) \longrightarrow {}^L\mathcal{T}\mathrm{op}(\mathcal{G})$ is left adjoint to $\Gamma_{\mathcal{G}}$.

Rem: We also have an explicite description of $\mathrm{Spec}^{\mathcal{G}} X$ for any $X \in \mathrm{Pro}(\mathcal{G})$
$:= \mathrm{Ind}(\mathcal{G}^{op})^{op}$

2.4 Structured schemes

Def: $\mathcal{G}$ geometry. A morphism $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ in ${}^L\mathrm{Top}(\mathcal{G})$ is étale if

1) The underlying geometric morphism $f^*: \mathcal{X} \to \mathcal{Y}$ of ∞-topoi is étale

2) The induced map $f^*\mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{Y}$ is an equivalence in $\mathrm{Str}_\mathcal{G}(\mathcal{Y})$.

Example: If U is an object of $\mathcal{X}$, let $\mathcal{O}_\mathcal{X}|U$ denote the $\mathcal{G}$-structure on $\mathcal{X}_{/U}$ given by $\mathcal{G} \xrightarrow{\mathcal{O}_\mathcal{X}} \mathcal{X} \xrightarrow{\pi^*} \mathcal{X}_{/U}$.

Then $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \longrightarrow (\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X}|U)$ is an étale morphism in ${}^L\mathrm{Top}(\mathcal{G})$.

Def: $\mathcal{G}$ geometry. A $\mathcal{G}$-structured topos $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is an affine $\mathcal{G}$-scheme if it is equiv to $\mathrm{Spec}^\mathcal{G} A$ for some $A \in \mathrm{Pro}(\mathcal{G})$.

It is a $\mathcal{G}$-scheme if there is a collection of objects $\{U_\alpha\}$ of $\mathcal{X}$ st.

1) $\{U_\alpha\}$ covers $\mathcal{X}$, i.e. $\coprod_\alpha U_\alpha \longrightarrow 1_\mathcal{X}$ is an effective epimorphism.

2) $\forall \alpha$, $(\mathcal{X}_{/U_\alpha}, \mathcal{O}_\mathcal{X}|U_\alpha)$ is an affine $\mathcal{G}$-scheme.

Denote $\mathrm{Sch}(\mathcal{G}) \subset {}^L\mathrm{Top}(\mathcal{G})^{op}$ spanned by $\mathcal{G}$-schemes.

Proposition: 1) $\mathrm{Sch}(\mathcal{G})$ admits colimits along étale morphisms.

2) If the Grothendieck topology on $\mathrm{Pro}(\mathcal{G})$ is precanonical, then $\mathrm{Sch}(\mathcal{G})$ admits finite limits.

Example: 1) Geometry $\mathcal{G}_{Zar} := \{$affine schemes of finite type over $\mathbb{Z}\}$

Admissible morphisms are inclusions of principle open subsets.

Admissible coverings are coverings by principle open subsets.

Then we have a fully faithful embedding Scheme $\hookrightarrow Sch(\mathcal{G}_{Zar})$

The essential image consists of $\mathcal{G}_{Zar}$-schemes with 0-localic underlying ∞-topos.

2) Geometry $\mathcal{G}_{\acute{e}t} := \{$affine schemes of finite type over $\mathbb{Z}\}$

Admissible morphisms are étale morphisms, and admissible coverings are étale coverings.

Then we have a fully faithful embedding Deligne-Mumford stacks $\hookrightarrow Sch(\mathcal{G}_{\acute{e}t})$

The essential image consists of $\mathcal{G}_{\acute{e}t}$-schemes with 1-localic underlying ∞-topos.

2.5 Pregeometries

Idea: generate geometries with simpler data.

Def: A pregeometry is an ∞-category $\mathcal{T}$ admitting finite products, equipped with an admissibility structure.

Def: $\mathcal{T}$ pregeometry, $\mathcal{X}$ ∞-topos. A $\mathcal{T}$-structure on $\mathcal{X}$ is a functor $\mathcal{O}: \mathcal{T} \to \mathcal{X}$ s.t.

1) $\mathcal{O}$ preserves finite products.
2) $\mathcal{O}$ preserves pullbacks along admissible morphisms
3) $\forall$ admissible covering $\{U_\alpha \to X\}$ in $\mathcal{T}$, the induced map $\coprod_\alpha \mathcal{O}(U_\alpha) \longrightarrow \mathcal{O}(X)$ is an effective epimorphism in $\mathcal{X}$.

Denote $Str_{\mathcal{T}}(\mathcal{X}) \subset Fun(\mathcal{T}, \mathcal{X})$, $Str^{loc}_{\mathcal{T}}(\mathcal{X}) \subset Str_{\mathcal{T}}(\mathcal{X})$

A transformation of pregeometries $f: \mathcal{T} \to \mathcal{T}'$ is a functor which preserves finite products, adm morphisms, adm coverings, and pullbacks along adm morphisms.

A transformation of pregeometries $f: \mathcal{T} \to \mathcal{G}$ exhibits $\mathcal{G}$ as a geometric envelope of $\mathcal{T}$ if

1) $\mathcal{G}$ is a geometry with the coarsest structure s.t. f is a transformation of pregeometries.

2) $\forall$ ∞-category $\mathcal{C}$ idempotent complete and admitting finite limits, composition with f induces an equiv

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{G}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ad}}(\mathcal{T}, \mathcal{C})$$

preserves finite products, pullbacks along adm morphisms.