V. Structured spaces

Goal: generalize the notion of locally ringed space to the derived setting

1. ∞-Topoi

Idea: generalization of topological spaces

1.1 Giraud's axioms

Def: An ∞-category $\mathcal{X}$ is an ∞-topos it there exists a small ∞-category $\mathcal{C}$ and an accessible left exact localization functor $P(\mathcal{C}) \longrightarrow \mathcal{X}$.

More intrinsic characterization:

HTT 6.1.0.6: $\mathcal{X}$ ∞-category. $\mathcal{X}$ is an ∞-topos if and only if it satisfies the following analogue of Giraud's axioms:

(i) The ∞-category $\mathcal{X}$ is presentable

(ii) Colimits in $\mathcal{X}$ are universal:

For any morphism $f: T \to S$ in $\mathcal{X}$, the associated pullback functor $f^*: \mathcal{X}_{/S} \longrightarrow \mathcal{X}_{/T}$ preserves colimits

($\Leftrightarrow$ $\exists$ a right adjoint f_* by the adjoint functor theorem).

(iii) Coproducts in $\mathcal{X}$ are disjoint

Every cocartesian diagram

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \sqcup Y \end{tikzcd}$$

is also cartesian.

(iv) Every groupoid object of $\mathcal{X}$ is effective:

Rough idea: equivalence relations $\xleftrightarrow{1-1}$ quotients

Precise formulation:

Def: $\Delta = \{[n], n \geq 0\}$ the category of combinatorial simplices

$\Delta_+ = \Delta \cup \{[-1] := \phi\}$ $\qquad \Delta_+^{\leq n} \subset \Delta_+$ spanned by $\{[k]\}_{k \leq n}$.

$\mathcal{C}$ ∞-category

A simplicial object of $\mathcal{C}$ is a functor $U: \Delta^{op} \to \mathcal{C}$.

Denote $U_n := U([n])$.

An augmented simplicial object of $\mathcal{C}$ is functor $U^+: \Delta_+^{op} \to \mathcal{C}$.

A simplicial object of $\mathcal{C}$ is a groupoid object if for every $n \geq 0$, and every partition $[n] = S \cup S'$ s.t. $S \cap S' = \{s\}$ is a singleton, the diagram

$$\begin{tikzcd} U([n]) \arrow[r] \arrow[d] & U(S) \arrow[d] \\ U(S') \arrow[r] & U(\{s\}) \end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

HTT6.1.2.11: $\mathcal{C}$ ∞-category, $U^+: \Delta_+^{op} \to \mathcal{C}$ an augmented simplicial object of $\mathcal{C}$, TFAE:

(1) U^+ is a right Kan extension of $U^+|_{(\Delta_+^{\leq 0})^{op}}$.

(2) the underlying simplicial object U is a groupoid object of $\mathcal{C}$, and the diagram $U^+|_{(\Delta_+^{\leq 1})^{op}}$ is a pullback square
$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_0 \arrow[r] & U_{-1} \end{tikzcd}$$
in $\mathcal{C}$.

In this case, U^+ is called the Čech nerve of $u: U_0 \to U_{-1}$.

Def: A simplicial object U of an ∞-category $\mathcal{C}$ is an effective groupoid if it can be extended to a colimit diagram $U^+: \Delta_+^{op} \to \mathcal{C}$ such that U^+ is a Čech nerve.

1.2 Grothendieck topologies and sheaves

Def: $\mathcal{C}$ ∞-category. A sieve on $\mathcal{C}$ is a full subcategory $\mathcal{C}^{(0)} \subset \mathcal{C}$, s.t. for any $D \in \mathcal{C}^{(0)}$, $f: C \to D$ in $\mathcal{C}$, we have $C \in \mathcal{C}^{(0)}$.

For any object $C \in \mathcal{C}$, a sieve on C is a sieve on the ∞-category $\mathcal{C}_{/C}$.

A Grothendieck topology on $\mathcal{C}$ consists of, for every object C of $\mathcal{C}$, a collection of sieves on C, called covering sieves, satisfying the following properties:

(1) $\forall$ object $C \in \mathcal{C}$, the sieve $\mathcal{C}_{/C} \subset \mathcal{C}_{/C}$ on C is a covering sieve.

(2) $\forall$ morphism $f: C \to D$ in $\mathcal{C}$, f^* of a covering sieve is a covering sieve.

(3) $\forall$ object $C \in \mathcal{C}$, covering sieve $\mathcal{C}^{(0)}_{/C}$ on C, $\mathcal{C}^{(1)}{}_{/C}$ an arbitrary sieve on C. If for each $f: D \to C$ belonging to $\mathcal{C}^{(0)}{}_{/C}$, the pullback

$f^* \mathcal{C}^{(1)}_{/C}$ is a covering sieve on D, then $\mathcal{C}^{(1)}_{/C}$ is a covering sieve on C.

Rem: Grothendieck topology on $\mathcal{C}$ $\Longleftrightarrow$ Grothendieck topology on $h\mathcal{C}$.

Prop: $\mathcal{C}$ ∞-category, $C \in \mathcal{C}$, $j: \mathcal{C} \to \mathcal{P}(\mathcal{C})$ the Yoneda embedding.
We have a bijection $\{\text{subobjects of } j(C)\} \xrightarrow{\sim} \{\text{sieves on } C\}$

$$\text{monomorphism } i: U \to j(C) \longmapsto \mathcal{C}_{/C}(U)$$

$\mathcal{C}_{/C}(U) \subset \mathcal{C}_{/C}$ spanned by $f: D \to C$ s.t. $\exists$ a commutative triangle

$$\begin{tikzcd} j(D) \arrow[rr, "j(f)"] \arrow[dr] & & j(C) \\ & U \arrow[ur, "i"'] & \end{tikzcd}$$

Def: $\mathcal{C}$ ∞-category, S a collection of morphisms of $\mathcal{C}$. An object Z of $\mathcal{C}$ is S-local if for every morphism $s: X \to Y$ belonging to S, composition with s induces a homotopy equivalence $\mathrm{Map}_{\mathcal{C}}(Y, Z) \to \mathrm{Map}_{\mathcal{C}}(X, Z)$.

Def: $\mathcal{C}$ ∞-category equipped with a Grothendieck topology.
$j: \mathcal{C} \to \mathcal{P}(\mathcal{C})$ Yoneda embedding.
S the collection of all monomorphisms $U \to j(C)$ corresponding to the covering sieves $\mathcal{C}^{(0)}_{/C} \subset \mathcal{C}_{/C}$. A presheaf $\mathcal{F} \in \mathcal{P}(\mathcal{C})$ is a sheaf if it is S-local.
Denote $\mathrm{Shv}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$ full subcategory spanned by sheaves.

HTT 6.2.2.7: $\mathrm{Shv}(\mathcal{C})$ is an ∞-topos.

1.3 The ∞-category of ∞-topoi

Def: $\mathcal{X}$, $\mathcal{Y}$ ∞-topoi. A geometric morphism between $\mathcal{X}$ and $\mathcal{Y}$ is a pair of adjoint functors $\mathcal{X} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{Y}$ where f^* is left exact.

We usually only write one of f^* or f_*.

Define subcategories $\mathcal{L}Top$, $\mathcal{R}Top \subset Cat_\infty$ as follows:

(1) The objects of $\mathcal{L}Top$ and $\mathcal{R}Top$ are ∞-topoi

(2) A functor $f^*: \mathcal{X} \to \mathcal{Y}$ between ∞-topoi belongs to $\mathcal{L}Top$ if and only if it preserves small colimits and finite limits.

(3) A functor $f_*: \mathcal{X} \to \mathcal{Y}$ between ∞-topoi belongs to $\mathcal{R}Top$ if and only if f_* has a left adjoint which is left exact.

Rem: $\mathcal{L}Top \simeq \mathcal{R}Top^{op}$

HTT 6.3.2-4: $\mathcal{L}Top$ and $\mathcal{R}Top$ admit limits and colimits.

1.4 Étale morphisms of ∞-topoi

Proposition: $\mathcal{X}$ ∞-topos, $U \in \mathcal{X}$, then

(1) $\mathcal{X}_{/U}$ is an ∞-topos

(2) $\pi_!: \mathcal{X}_{/U} \to \mathcal{X}$ has a right adjoint π^* which commutes with colimits. Consequently, π^* has a right adjoint $\pi_*: \mathcal{X}_{/U} \to \mathcal{X}$, and (π^*, π_*) gives rise a geometric morphism of ∞-topoi.

Def: Such geometric morphisms are called **Etale morphisms**

HTT6.3.5.13: Let $RTop_{ét} \subset RTop_{ét}$ spanned by all ∞-topoi and étale geometric morphisms. Then $RTop_{ét}$ admits small colimits, i.e. we can glue ∞-topoi along étale open subsets.

2. Structured sheaves

2.1 Sheaves with values in a ∞-category

Recall: Goal: generalize the notion of locally ringed space to the derived setting.

Def: $\mathcal{C}$ arbitrary ∞-category. $\mathcal{X}$ ∞-topos. A **$\mathcal{C}$-valued sheaf** on $\mathcal{X}$ is a functor $\mathcal{X}^{op} \longrightarrow \mathcal{C}$ which preserves small limits.

Denote $Shv_{\mathcal{C}}(\mathcal{X}) \subset Fun(\mathcal{X}^{op}, \mathcal{C})$

Def: $\mathcal{T}$ ∞-category equipped with a Grothendieck topology. $\mathcal{C}$ ∞-category. A functor $\mathcal{O}: \mathcal{T}^{op} \to \mathcal{C}$ is a **$\mathcal{C}$-valued sheaf** on $\mathcal{T}$ if for every object $X \in \mathcal{T}$ and every covering sieve $\mathcal{T}^{o}_{/X} \subset \mathcal{T}_{/X}$, the composite map

$$(\mathcal{T}^{o}_{/X})^{\triangleleft} \subset (\mathcal{T}_{/X})^{\triangleleft} \longrightarrow \mathcal{T} \xrightarrow{\mathcal{O}^{op}} \mathcal{C}^{op}$$

is a colimit diagram in $\mathcal{C}^{op}$.

Denote $Shv_{\mathcal{C}}(\mathcal{T}) \subset Fun(\mathcal{T}^{op}, \mathcal{C})$

Example: $Shv(\mathcal{T}) \simeq Shv_{\mathcal{S}}(\mathcal{T})$

Proposition: $\mathcal{T}$ as above. $j: \mathcal{T} \to \mathcal{P}(\mathcal{T})$ Yoneda embedding

$L: \mathcal{P}(\mathcal{T}) \to \mathrm{Shv}(\mathcal{T})$ left adjoint to the inclusion

$\mathcal{C}$ arbitrary ∞-category admitting small limits.

Then the composition with $L \circ j$ induces an equiv of ∞-categories

$$\mathrm{Shv}_{\mathcal{C}}(\mathrm{Shv}(\mathcal{T})) \xrightarrow{\sim} \mathrm{Shv}_{\mathcal{C}}(\mathcal{T}).$$

2.2 Geometries

Idea: In order to make sense of "Locally ringed", we need an extra structure on the ∞-category $\mathcal{C}$.

Def: $\mathcal{G}$ ∞-category, an admissibility structure on $\mathcal{G}$ consists of the following data:

(1) A subcategory $\mathcal{G}^{ad} \subset \mathcal{G}$ spanned by all the objects, and admissible morphisms satisfying the following conditions:

(i) Pullback of admissible morphism always exists, and is again admissible

(ii)
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$
g, h admissible $\Rightarrow$ f admissible

(iii) Retract of admissible morphism is admissible.

(2) A Grothendieck topology on $\mathcal{G}$, which is generated by admissible morphisms in the following sense: any covering sieve $\mathcal{G}^{(0)}_{/X} \subset \mathcal{G}_{/X}$ contains a covering sieve which is generated by a collection of admissible morphisms $\{U_\alpha \to X\}$.

Def: A geometry consists of the following data:

(1) An ∞-category $\mathcal{G}$ which admits finite limits and is idempotent complete.

(2) An admissibility structure on $\mathcal{G}$.