Def: $\mathcal{C}$ presentable ∞-cat. For every derivation $\eta: A \to M$ in $\mathcal{C}$, form a pullback diagram

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & A \arrow[d, "\eta"] \\ 0 \arrow[r] & M \end{tikzcd}$$

in the ∞-category $\mathcal{M}^T(\mathcal{C})$.

A morphism $f: \tilde{A} \to A$ in $\mathcal{C}$ is a square-zero extension if $\exists$ a derivation $\eta: A \to M$ in $\mathcal{C}$ and an equivalence $\tilde{A} \simeq A^\eta$ in the ∞-cat $\mathcal{C}_{/A}$.

We say that $\tilde{A}$ is a square-zero extension of A by $M[-1]$.

Rem: By HTT4.3.1.9, the above pullback diagram in $\mathcal{M}^T(\mathcal{C})$ is equivalent to the following pullback diagram in $\mathcal{C}$

$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & A \arrow[d, "d_\eta"] \\ A \arrow[r, "d_0"'] & A \oplus M \end{tikzcd}$$

where $A \oplus M$ denotes the image of M under the functor $\Omega^\infty: Sp(\mathcal{C}_{/A}) \to \mathcal{C}$

d_0 is the section associated to the 0 derivation.

The main source of square-zero extensions in the setting of E_∞-rings is n-small extensions.

Def: For $n \geq 0$, a morphism $f: A \to B$ in CAlg is an n-small extension if $A \in \mathrm{CAlg}^{cn}$, $\mathrm{fib}(f) \in \mathrm{CAlg}_{[n, 2n]}$, and the multiplication map

$\mathrm{fib}(f) \underset{A}{\otimes} \mathrm{fib}(f) \longrightarrow \mathrm{fib}(f)$ is nullhomotopic.

Denote $\mathrm{Fun}_{n\text{-sm}}(\Delta^1, \mathrm{CAlg}) \subset \mathrm{Fun}(\Delta^1, \mathrm{CAlg})$

An object $(A, \eta: L_A \to M[1]) \in \mathrm{Der}(\mathrm{CAlg})$ is n-small if A is connective and $M \in \mathrm{Sp}_{[n,2n]}$.

Denote $\mathrm{Der}_{n\text{-sm}} \subset \mathrm{Der}(\mathrm{CAlg})$.

Theorem: The functor $\Phi: \mathrm{Der}(\mathrm{CAlg}) \longrightarrow \mathrm{Fun}(\Delta^1, \mathrm{CAlg})$

$$(A, \eta: L_A \to M[1]) \longmapsto (A^\eta \to A)$$

induces for each $n \geq 0$, an equivalence of ∞-cats

$$\mathrm{Der}_{n\text{-sm}} \xrightarrow{\sim} \mathrm{Fun}_{n\text{-sm}}(\Delta^1, \mathrm{CAlg}).$$

Corollary: 1) Every n-small extension in CAlg is a square-zero extension.

2) For any $A \in \mathrm{CAlg}^{cn}$, every map in the Postnikov tower

$$\cdots \longrightarrow \tau_{\leq 3} A \longrightarrow \tau_{\leq 2} A \longrightarrow \tau_{\leq 1} A \longrightarrow \tau_{\leq 0} A$$

is a square-zero extension.

Application: Given $A, B \in \mathrm{CAlg}^{cn}$, we can understand $\mathrm{Map}_{\mathrm{CAlg}}(A,B)$ as $\lim\limits_n \mathrm{Map}_{\mathrm{CAlg}}(A, \tau_{\leq n} B)$.

For $n=0$, we have $\mathrm{Map}_{\mathrm{CAlg}}(A, \tau_{\leq n} B) \simeq \mathrm{Hom}(\pi_0 A, \pi_0 B)$.

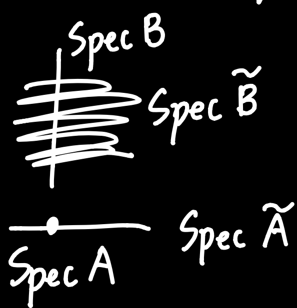
For $n>0$, we have a pullback square

$$\begin{array}{ccc} \tau_{\leq n} B & \longrightarrow & \tau_{\leq n-1} B \\ \downarrow & & \downarrow \\ \tau_{\leq n-1} B & \longrightarrow & \tau_{\leq n-1} B \oplus (\pi_n B)[n+1] \end{array}$$

This reduces us to the study of $\mathrm{Map}_{\mathrm{CAlg}}(A, \tau_{\leq n-1} B)$ and the linear problem of derivations from A to $(\pi_n B)[n+1]$, which is controlled by the cotangent complex.

2.2 Deformation theory of E_∞-rings

Def: Let A be an E_∞-ring, $\tilde{A}$ a square-zero extension of A by an A-module M, $B \in \mathrm{CAlg}_A$. Then a deformation of B to $\tilde{A}$ is a pair $(\tilde{B}, \alpha)$ where $\tilde{B} \in \mathrm{CAlg}_{\tilde{A}}$ and α is an equivalence $\tilde{B} \otimes_{\tilde{A}} A \simeq B$ in CAlg_A.



Rem: If A, M are connective, then $\tilde{B}$ is flat over $\tilde{A}$ if and only if B is flat over A.

Proof: $\Rightarrow$ ✓ $\Leftarrow$ let us show that for every discrete $\tilde{A}$-module N, $\tilde{B} \otimes_{\tilde{A}} N$ is discrete. Let $I := \ker(\pi_0 \tilde{A} \to \pi_0 A)$, we have a short exact seq

$$0 \to IN \to N \to N/IN \to 0 \qquad \text{modules over } \pi_0 \tilde{A}.$$

So it's enough to show that $\tilde{B} \otimes_{\tilde{A}} IN$ and $\tilde{B} \otimes_{\tilde{A}} N/IN$ are discrete.

Let $\mathrm{Der} := \mathrm{Der}(\mathrm{CAlg})$, the ∞-cat of derivations in CAlg.

Define a subcategory $\mathrm{Der}^+ \subset \mathrm{Der}$ as follows:

(i) objects: derivations $\eta: A \to M[1]$ where both A and M are connective.

(ii) morphism: $f: (\eta: A \to M[1]) \to (\eta': B \to N[1])$ s.t. $B \otimes_A M \xrightarrow{\sim} N$.

Proposition: A connective $\mathbb{E}_\infty$-ring, M connective A-module, $\eta : A \to M[1]$ a derivation. Then we have an equivalence of ∞-categories

$$\mathrm{Der}^{+}_{\eta/} \xrightarrow{\sim} \mathrm{CAlg}^{cn}_{A^\eta}$$

$$(\eta' : B \to N[1]) \longmapsto B^{\eta'} := \mathrm{fib}(\eta')$$

Idea: Giving a deformation $\tilde{B}$ of B over $\tilde{A}$ is equivalent to providing a factorization of $\eta_B : B \otimes_A L_A \longrightarrow B \otimes_A M[1]$

as a composition $B \otimes_A L_A \longrightarrow L_B \xrightarrow{\eta'} B \otimes_A M[1]$

and the corresponding extension is given by $\tilde{B} = B^{\eta'}$.

In particular, B admits a deformation over $\tilde{A}$ if and only if the composite map $L_{B/A}[-1] \to B \otimes_A L_A \xrightarrow{\eta_B} B \otimes_A M[1]$ vanishes.

2.3 Connectivity of the cotangent complex

Theorem: $f : A \to B$ map of connective $\mathbb{E}_\infty$-rings. If $\mathrm{cofib}(f)$ is n-connective for $n \geq 0$ then there is a canonical $(2n)$-connective map of B-modules

$$\varepsilon_f : B \otimes_A \mathrm{cofib}(f) \longrightarrow L_{B/A}.$$

Construction of the map ε_f: We have $\eta : L_B \to L_{B/A}$ map of B-modules $\rightsquigarrow$ B^η the associated square-zero extension of B by $L_{B/A}[-1]$.

Since the restriction of η to L_A is nullhomotopic, the map f factors as a composition $A \xrightarrow{f'} B^\eta \xrightarrow{f''} B$

So we obtain a map of A-modules $\mathrm{cofib}(f) \to \mathrm{cofib}(f'')$, hence a map of B-modules $\varepsilon_f : B \otimes_A \mathrm{cofib}(f) \longrightarrow \mathrm{cofib}(f'') \simeq L_{B/A}$.

Corollary 1: $f: A \to B$ map of connective E_∞-rings. If $\mathrm{cofib}(f)$ is n-connective for $n \geq 0$, then the relative cotangent complex $L_{B/A}$ is n-connective. The converse holds provided that f induces an isomorphism $\pi_0 A \xrightarrow{\sim} \pi_0 B$.

Proof: fiber sequence of B-modules

$$\mathrm{fib}(\varepsilon_f) \longrightarrow B \otimes_A \mathrm{cofib}(f) \longrightarrow L_{B/A}$$

Corollary 2: A connective E_∞-ring. Then the absolute cotangent complex L_A is connective.

Proof: Apply Cor 1 to the unit map $S \to A$ in the case $n=0$.

Corollary 2′: $f: A \to B$ map of connective E_∞-rings. Then the relative cotangent complex $L_{B/A}$ is connective.

Corollary 3: $f: A \to B$ map of connective E_∞-rings. Then

$$f \text{ is an equivalence} \iff \begin{cases} f \text{ induces an isomorphism } \pi_0 A \xrightarrow{\sim} \pi_0 B. \\ \text{the relative cotangent complex } L_{B/A} \simeq 0. \end{cases}$$

Corollary 4: $f: A \to B$ map of connective E_∞-rings s.t. $\mathrm{cofib}(f)$ is n-connective for $n \geq 0$. Then the induced map $L_f : L_A \to L_B$ has n-connective cofiber. In particular, the canonical map $\pi_0 L_A \to \pi_0 L_{\pi_0 A}$ is an isomorphism.

Corollary 5: $f: A \to B$ map of connective $\mathbb{E}_\infty$-rings s.t. $\mathrm{cofib}(f)$ is n-connective for $n \geq 0$. Then there exists a canonical $(2n-1)$-connective map of A-module $\mathrm{cofib}(f) \longrightarrow L_{B/A}$.

Proposition: $f: A \to B$ map of connective $\mathbb{E}_\infty$-rings. Then $\pi_0 L_{B/A} \simeq \Omega_{\pi_0 B/\pi_0 A}$ as $\pi_0 B$-modules.

Proof: By fiber sequence for $L_{B/A}$, and short exact sequence $\Omega_{\pi_0 B/\pi_0 A}$,

$\rightsquigarrow$ Lemma: A discrete $\mathbb{E}_\infty$-ring, $\pi_0 L_A \simeq \Omega_A$ as discrete A-modules.

We show that $\pi_0 L_A$ and Ω_A corepresent the same functor on the category of discrete A-modules: let M be a discrete A-module, we have

$$\mathrm{Map}_{\mathrm{Mod}_A}(\pi_0 L_A, M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, M) \simeq \mathrm{Map}_{\mathrm{CAlg}/A}(A, A \oplus M)$$
$$\simeq \mathrm{Map}_{\mathrm{Rings}/A}(A, A \oplus M).$$

2.4 Finiteness of the cotangent complex

Theorem: A connective $\mathbb{E}_\infty$-ring, B connective $\mathbb{E}_\infty$-algebra over A.

1) If B is locally of finite presentation over A, then $L_{B/A}$ is perfect as B-module.

The converse holds provided that $\pi_0 B$ is finitely presented over $\pi_0 A$.

2) If B is almost of finite presentation over A, then $L_{B/A}$ is almost perfect as B-module.

The converse holds provided that $\pi_0 B$ is finitely presented over $\pi_0 A$.

3. Étale morphisms

Def: A map $\phi: A \to B$ of E_∞-rings is étale if $\pi_0 A \to \pi_0 B$ is étale and B is flat as A-module.

Theorem: A E_∞-ring. Every étale map of discrete commutative rings $\pi_0 A \to \pi_0 B$ can be lifted (essentially unique) to a étale map $\phi: A \to B$ of E_∞-rings.

Corollary: The relative cotangent complex of an étale morphism of E_∞-rings vanishes.

DAG VII. 8.9: $A \to B$ map of connective E_∞-rings s.t. $L_{B/A}$ vanishes.

TFAE:
(1) $\pi_0 B$ is finitely presented over $\pi_0 A$
(2) B is finitely presented over A
(3) B is almost finitely presented over A
(4) $A \to B$ is étale.