IV. Cotangent complexes

1. The cotangent complex formalism

Goal: derive Kähler differentials

Recall: A commutative ring, M A-module

A derivation from A to M is a map $d: A \longrightarrow M$ satisfying

$$d(x+y) = dx + dy \qquad d(xy) = x\,dy + y\,dx$$

Let $\mathrm{Der}(A, M)$ be the abelian group of derivations from A to M.

Fix A, the functor $M \longmapsto \mathrm{Der}(A, M)$ is corepresented by an A-module Ω_A, called the A-module of absolute Kähler differentials.

Explicitly, Ω_A = free module generated by the symbols $\{dx\}_{x \in A}$

$/$ relations $d(x+y) = dx + dy$, $d(xy) = x\,dy + y\,dx$, $x, y \in A$.

Reformulation:

Let $B = A \oplus M$, equipped with the ring structure given by

$$(a, m)(a', m') = (aa', am' + a'm)$$

called a trivial square-zero extension.

Then $\mathrm{Der}(A, M)$ = sections of the projection map $A \oplus M \longrightarrow A$.

Let $\mathrm{Ring} = \{\text{commutative rings}\}$

$\mathrm{Ring}^{+} = \{(A, M) \mid A \text{ is a commutative ring, } M \text{ is an } A\text{-module}\}$

Mor: $(f, f'): (A, M) \to (B, N)$

$f: A \to B$ ring morphism $f': M \to N$ map of A-modules.

$G: \mathrm{Ring}^+ \to \mathrm{Ring}$ $(A, M) \mapsto A \oplus M$ trivial square-zero extension.

G admits a left adjoint $F: A \mapsto (A, \Omega_A)$.

Steps for generalizing the above construction to derived geometry:

1) Generalize trivial square-zero extension
2) Generalize Ring^+ to $\mathcal{C}^+$ for any presentable ∞-category $\mathcal{C}$.
   $\llcorner$: $T_{\mathcal{C}}$ called the tangent bundle to $\mathcal{C}$.
3) Define the cotangent complex functor $L: \mathcal{C} \to T_{\mathcal{C}}$ via adjunction.
4) Define derivation via the tangent correspondence to $\mathcal{C}$.

1.1 Trivial square-zero extension

Goal: Given A $\mathbb{E}_\infty$-ring, $M \in \mathrm{Mod}_A$, construct the trivial square-zero extension $A \oplus M$. We want a functorial construction, i.e. a trivial square-zero extension functor

$$G: \mathrm{Mod}_A \longrightarrow \mathrm{CAlg}_{/A}.$$

Construction: Let X be an object of $\mathrm{Sp}(\mathrm{CAlg}_{/A})$.
Then the 0^{th}-space $\Omega^\infty(X)$ is a pointed object of $\mathrm{CAlg}_{/A}$, i.e. an $\mathbb{E}_\infty$-ring B which fits into a commutative diagram

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ A \arrow[ur] \arrow[rr, "id"] & & A \end{tikzcd}$$

Note that the fiber of f inherits the structure of an A-module

$\rightsquigarrow$ functor $F' : Sp(CAlg_{/A}) \longrightarrow Mod_A$

HA 7.3.4.14: F' is an equivalence of ∞-categories.

Define the trivial square-zero extension functor G to be the composition

$$Mod_A \xrightarrow{\sim} Sp(CAlg_{/A}) \xrightarrow{\Omega^\infty} CAlg_{/A}$$

Denote $A \oplus M := G(M)$.

HA 7.3.4.15: Forgetting the algebra structure, $A \oplus M$ is canonically identified with the coproduct of A and M.

HA 7.3.4.17: The multiplication on $\pi_*(A \oplus M)$ is given on homogeneous elements by the formula $(a, m)(a', m') = (aa', am' + (-1)^{|a'||m|} a'm)$.

In particular, if A and M are discrete, then $A \oplus M$ is identified with the classical trivial square-zero extension.

1.2 Stable envelopes and tangent bundles

Idea: make the above construction in families, i.e. fiberwise stabilization.

Let's first give a characterization of the stabilization $Sp(\mathcal{C})$:

Def: $\mathcal{C}$ presentable ∞-category. A stable envelope of $\mathcal{C}$ is a functor $u: \mathcal{C}' \to \mathcal{C}$ s.t.

(i) $\mathcal{C}'$ is a presentable stable ∞-category.

(ii) u admits a left adjoint

(iii) $\forall$ presentable stable ∞-category $\mathcal{E}$, composition with u induces an equiv of ∞-cats.

$$\mathrm{RFun}(\mathcal{E}, \mathcal{C}') \longrightarrow \mathrm{RFun}(\mathcal{E}, \mathcal{C})$$

functors which are right adjoints

Example: $\Omega^\infty_{\mathcal{C}}: \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ exhibits $\mathrm{Sp}(\mathcal{C})$ as a stable envelope of $\mathcal{C}$.

Def: A functor $p: X \to S$ of ∞-categories is a presentable fibration if it is both cartesian and cocartesian, and every fiber $X_s = X \times_S \{s\}$ is a presentable ∞-category.

Def: A stable envelope of a presentable fibration $p: \mathcal{C} \to \mathcal{D}$ is a functor $u: \mathcal{C}' \to \mathcal{C}$ s.t.

(i) $p \circ u$ is a presentable fibration.

(ii) u carries $(p\circ u)$-cartesian morphisms of $\mathcal{C}'$ to p-cartesian morphisms of $\mathcal{C}$

(iii) $\forall D \in \mathcal{D}$, the induced map $\mathcal{C}'_D \to \mathcal{C}_D$ is a stable envelope of $\mathcal{C}'_D$.

Def: $\mathcal{C}$ presentable ∞-category. A tangent bundle to $\mathcal{C}$ is a functor

$$T_{\mathcal{C}} \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$$

which exhibits $T_{\mathcal{C}}$ as the stable envelope of the presentable fibration

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}.$$

Idea: Objects of $T_{\mathcal{C}}$ are pairs (A, M), where $A \in \mathcal{C}$, $M \in \mathrm{Sp}(\mathcal{C}_{/A})$.

For $\mathcal{C} = \mathrm{CAlg}$, $M \in \mathrm{Sp}(\mathrm{CAlg}_{/A}) \simeq \mathrm{Mod}_A$.

The functor $T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ sends (A, M) to the projection

$A \oplus M \to A$.

Explicit construction of tangent bundle $T_{\mathcal{C}}$:

$$\mathrm{Exc}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$$

$$(X : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}) \longmapsto (X(S^0) \to X(*))$$

Def: $\mathcal{C}$ presentable ∞-category. The absolute cotangent complex functor $L : \mathcal{C} \to T_{\mathcal{C}}$ is a left adjoint to the composition

$$T_{\mathcal{C}} \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}.$$

Rem: relative adjunction
$$\begin{tikzcd} T_{\mathcal{C}} \arrow[rr, "G"] \arrow[dr] & & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

Rem: For $A \in \mathcal{C}$, the object $L_A \in \mathrm{Sp}(\mathcal{C}_{/A}) \simeq (T_{\mathcal{C}})_A$ corresponds to the image of $\mathrm{id}_A \in \mathcal{C}_{/A}$ under the suspension spectrum functor $\Sigma^\infty_+ : \mathcal{C}_{/A} \to \mathrm{Sp}(\mathcal{C}_{/A})$.

1.3 The relative cotangent complex

$\mathcal{C}$ presentable ∞-category, $A \in \mathcal{C}$ $\rightsquigarrow$ absolute cotangent complex $L_A \in Sp(\mathcal{C}_{/A})$.

Goal: define a relative cotangent complex $L_{B/A}$ associated to a morphism $f: A \to B$ in $\mathcal{C}$.

Idea: Recall that for Kähler differentials, we have an exact sequence

$$\Omega_A \underset{A}{\otimes} B \longrightarrow \Omega_B \longrightarrow \Omega_{B/A} \longrightarrow 0$$ for a homomorphism of rings.

So we want to define $L_{B/A}$ via some cofiber sequence.

Def: $\mathcal{C}$ presentable ∞-category, $p: T_{\mathcal{C}} \to \mathcal{C}$ tangent bundle.

A relative cofiber sequence in $T_{\mathcal{C}}$ is a pushout square in $T_{\mathcal{C}}$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Sp(\mathcal{C}_{/p(X)}) \ni 0 & \longrightarrow & Z \end{array}$$

s.t. each column lies in a fiber of p.

Let $\mathcal{E} \subset Fun(\Delta^1 \times \Delta^1, T_{\mathcal{C}}) \underset{Fun(\Delta^1 \times \Delta^1, \mathcal{C})}{\times} Fun(\Delta^1, \mathcal{C})$

spanned by relative cofiber sequences.

The relative cotangent complex functor is the composition

$$\mathrm{Fun}(\Delta^1,\mathcal{C}) \xrightarrow{L} \mathrm{Fun}(\Delta^1, T_{\mathcal{C}}) \longrightarrow \mathcal{E} \longrightarrow T_{\mathcal{C}}$$

make relative cofiber sequence

take lower right corner

$$(f: A\to B) \longmapsto L_{B/A} \in (T_{\mathcal{C}})_B \simeq Sp(\mathcal{C}_{/B})$$

Rem: By definition, we have a relative cofiber sequence in $T_{\mathcal{C}}$

$$\begin{tikzcd} L_A \arrow[r] \arrow[d] & L_B \arrow[d] \\ 0 \arrow[r] & L_{B/A} \end{tikzcd}$$

$\overset{\text{HTT 4.3.1.9}}{\rightsquigarrow}$ Cofiber sequence $f_! L_A \to L_B \to L_{B/A}$ in $(T_{\mathcal{C}})_B \simeq Sp(\mathcal{C}_{/B})$

where $f_!: Sp(\mathcal{C}_{/A}) \to Sp(\mathcal{C}_{/B})$ denotes the functor induced by the cocartesian fibration p.

Example: • For $f: A\to B$, A an initial object of $\mathcal{C}$, we have

$$L_B \xrightarrow{\sim} L_{B/A} \in Sp(\mathcal{C}_{/B}).$$

• For $f: A \xrightarrow{\sim} B$ an equivalence, we have $L_{B/A} \simeq 0 \in Sp(\mathcal{C}_{/B})$

Proposition: $\mathcal{C}$ presentable ∞-category, $T_{\mathcal{C}}$ tangent bundle

$$\begin{tikzcd} & B \arrow[dr] & \\ A \arrow[ur] \arrow[rr] & & C \end{tikzcd}$$

commutative diagram in $\mathcal{C}$

$\Rightarrow$ pushout diagram

$$\begin{tikzcd} L_{B/A} \arrow[r] \arrow[d] & L_{C/A} \arrow[d] \\ 0 \simeq L_{B/B} \arrow[r] & L_{C/B} \end{tikzcd}$$

in $T_{\mathcal{C}}$ (also a relative cofiber sequence.)

$\Rightarrow$ cofiber sequence $f_! L_{B/A} \to L_{C/A} \to L_{C/B}$ in $Sp(\mathcal{C}_{/C})$.

Proposition: Given a pushout diagram
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ A' \arrow[r] & B' \end{tikzcd}$$
in $\mathcal{C}$,

we have an equivalence $f_! L_{B/A} \xrightarrow{\sim} L_{B'/A'}$, i.e. $L_{B/A} \to L_{B'/A'}$ is a p-cocartesian morphism in $T_{\mathcal{C}}$.

2. Deformation theory

2.1 Square-zero extensions

Recall: R commutative ring. A square-zero extension of R is a comm ring $\tilde{R}$ with a surjection $\phi: \tilde{R} \to R$ s.t. $(\ker\phi)^2 = 0$. In this case, $M := \ker\phi$ inherits an R-module structure.

Def: $\mathcal{C}$ presentable ∞-category. $L: \mathcal{C} \to T_{\mathcal{C}}$ a cotangent complex functor.

$\xrightarrow{\text{unstraightening}}$ cocartesian fibration
$$\begin{array}{ccc} M^T(\mathcal{C}) & = & \{\mathcal{C} \underset{L}{\overset{p}{\rightleftarrows}} T_{\mathcal{C}}\} \\ \downarrow & & \downarrow \\ \Delta^1 & = & \{0 \longrightarrow 1\} \end{array}$$

We call $M^T(\mathcal{C})$ the tangent correspondence to $\mathcal{C}$.

It has a projection map $p: M^T(\mathcal{C}) \to \Delta^1 \times \mathcal{C}$

Def: A derivation in $\mathcal{C}$ is a morphism $\eta: A \to M$ in $M^T(\mathcal{C})$ where

$A \in \mathcal{C}$, $M \in (T_{\mathcal{C}})_A$. By the cocartesian property, it can also be identified with a map $d: L_A \to M$ in the fiber $(T_{\mathcal{C}})_A$.

Let $\mathrm{Der}(\mathcal{C})$ be the ∞-category of derivations in $\mathcal{C}$.

Def: $\mathcal{C}$ presentable ∞-category. For every derivation $\eta: A \to M$ in $\mathcal{C}$, form a pullback diagram
$$\begin{tikzcd} A^\eta \arrow[r] \arrow[d] & A \arrow[d, "\eta"] \\ 0 \arrow[r] & M \end{tikzcd}$$
in the ∞-cat $\mathcal{M}^T(\mathcal{C})$.

A morphism $f: \tilde{A} \to A$ in $\mathcal{C}$ is a square-zero extension if there exists a derivation $\eta: A \to M$ in $\mathcal{C}$ and an equiv $\tilde{A} \simeq A^\eta$ in $\mathcal{C}_{/A}$.

We also call $\tilde{A}$ a square-zero extension of A by $M[-1]$.