HA 7.2.1.19: R E_1-ring, $M \in \mathrm{RMod}_R$, $N \in \mathrm{LMod}_R$. There exists a spectral sequence $\{E_r^{p,q}, d_r\}_{r\geq 2}$ with E_2-page $E_2^{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q$ which converges to $\pi_{p+q}(M \otimes_R N)$.

Applications of the spectral sequence:

1) If R, M, N are all discrete, then $\pi_n(M \otimes_R N) \simeq \mathrm{Tor}_n^R(M,N)$.

2) If R, M, N are all connective, then $M \otimes_R N$ is connective and

$$\pi_0(M \otimes_R N) \simeq \pi_0 M \otimes_{\pi_0 R} \pi_0 N.$$

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Flat and projective modules over a connective E_1-ring.

Def: R E_1-ring. A left R-module M is *free* if it is equivalent to a coproduct of copies of R. A free left module M is *finitely generated* if it is a finite coproduct of copies of R.

Def: Let $\mathcal{C}$ be an ∞-category which admits geometric realizations of simplicial objects. An object $X \in \mathcal{C}$ is *projective* if the functor $\mathrm{Map}_{\mathcal{C}}(X, \cdot): \mathcal{C} \to \mathcal{S}$ corepresented by X commutes with geometric realizations.

Def: R connective $\mathbb{E}_1$-ring. A left R-module is projective if it is a projective object of the ∞-category LMod_R^{cn} of connective left R-modules

Def: $\mathcal{C}$ ∞-category, $X, Y \in \mathcal{C}$. Y is called a retract of X if there exists a 2-simplex $\Delta^2 \to \mathcal{C}$ corresponding to a diagram

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"'] & & Y \end{tikzcd}$$

HA 7.2.2.7-8: R connective $\mathbb{E}_1$-ring. P connective left R-module. Then

P is projective $\Leftrightarrow$ P is a retract of a free R-module M

P is projective and $\pi_0 P$ is finitely generated over $\pi_0 R$

$\Leftrightarrow$ P is a compact projective object of LMod_R^{cn}

$\Leftrightarrow$ P is a retract of a finitely generated free R-module M.

Def: R $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R$ is flat if $\pi_0 M$ is flat over $\pi_0 R$,

$\forall n \in \mathbb{Z}$, $\pi_n R \underset{\pi_0 R}{\otimes} \pi_0 M \xrightarrow{\sim} \pi_n M$.

Easy consequences:

1) R $\mathbb{E}_1$-ring, $f: M \to N$ map of flat R-modules

Then f is an equiv $\Leftrightarrow$ it induces an isomorphism $\pi_0 M \xrightarrow{\sim} \pi_0 N$.

2) R connective $\mathbb{E}_1$-ring. A flat left R-module M is projective

$\Leftrightarrow$ $\pi_0 M$ is projective over $\pi_0 R$.

3) R $\mathbb{E}_1$-ring, $M \in \mathrm{RMod}_R$, $N \in \mathrm{LMod}_R$, N flat, $\forall n \in \mathbb{Z}$, we have

$$\mathrm{Tor}_0^{\pi_0 R}(\pi_n M, \pi_0 N) \xrightarrow{\sim} \pi_n(M \otimes_R N).$$

Lazard's theorem: R connective $\mathbb{E}_1$-ring, N connective left R-module.

TFAE: 1) N is a filtered colimit of finitely generated free modules

2) N is a filtered colimit of projective left R-modules

3) N is flat

4) The functor $M \mapsto M \otimes_R N$ is left t-exact, i.e. it carries $(\mathrm{RMod}_R)_{\leq 0}$ into $\mathrm{Sp}_{\leq 0}$.

5) If M is discrete, then $M \otimes_R N$ is discrete.

Finiteness properties of rings and modules (Ref HA 7.2.4)

Def: R $\mathbb{E}_1$-ring. A left R-module M is perfect if it belongs to $\mathrm{LMod}_R^{\mathrm{perf}}$, the smallest stable subcat of LMod_R which contains R and is closed under retracts.

Similarly we define $\mathrm{RMod}_R^{\mathrm{perf}}$, and perfect right R-module.

Idea: M is perfect if it can be built from finitely many copies of R by forming shifts, extensions, and retracts.

Recall: $\mathcal{C}$ ∞-category admitting filtered colimits. An object $X \in \mathcal{C}$ is called compact if the corepresentable functor $\mathrm{Map}_{\mathcal{C}}(X, \cdot)$ commutes with filtered colimits.

Proposition: R $\mathbb{E}_1$-ring. An object of LMod_R is compact if and only if it is perfect.

Cor: R connective $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R^{\mathrm{perf}}$, then

(1) $\pi_m M = 0$ for $m << 0$

(2) If $\pi_m M = 0$ for all $m < k$, then $\pi_k M$ is finitely presented over $\pi_0 R$.

Proposition: Duality between left and right modules:

R $\mathbb{E}_1$-ring. The relative tensor product functor

$$\underset{R}{\otimes} : \mathrm{RMod}_R \times \mathrm{LMod}_R \longrightarrow \mathrm{Sp}$$

induces fully faithful embeddings

$$\theta : \mathrm{RMod}_R \longrightarrow \mathrm{Fun}(\mathrm{LMod}_R, \mathrm{Sp}) \qquad \theta' : \mathrm{LMod}_R \longrightarrow \mathrm{Fun}(\mathrm{RMod}_R, \mathrm{Sp}).$$

Essential image = functors preserving small colimits.

Proposition: R $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R$. Then M is perfect if and only if $\exists\, M^\vee \in \mathrm{RMod}_R$ s.t. the composition $\mathrm{LMod}_R \xrightarrow{M^\vee \otimes_R \cdot} \mathrm{Sp} \xrightarrow{\Omega^\infty} \mathcal{S}$ is equiv to the functor corepresented by M. In this case, $M^\vee$ is also perfect.

called dual of M

Def-Lem: $\mathcal{C}, \mathcal{D}$ ∞-cats. A functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{S}$ is a perfect pairing if it satisfies the following two equiv conditions:

(1) The induces map $f: \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{D}, \mathcal{S}) = \mathcal{P}(\mathcal{D}^{op})$ is fully faithful and the essential image of f coincides with the essential image of the

Yoneda embedding $\mathcal{D}^{op} \to \mathcal{P}(\mathcal{D}^{op})$.

(2) .... $f': \mathcal{D} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}) = \mathcal{P}(\mathcal{C}^{op})$ ....

.... $\mathcal{C}^{op} \to \mathcal{P}(\mathcal{C}^{op})$.

A perfect pairing $\rightsquigarrow$ an equiv $\mathcal{C}$ and $\mathcal{D}^{op}$.

Proposition: R $\mathbb{E}_1$-ring. The bifunctor

$$\mathrm{RMod}_R^{perf} \times \mathrm{LMod}_R^{perf} \xrightarrow{\otimes_R} Sp \xrightarrow{\Omega^\infty} \mathcal{S}$$

is a perfect pairing.

Warning: R commutative noetherian ring, M discrete finitely generated R-module. Then M is not generally perfect.

Def: $\mathcal{C}$ compactly generated ∞-cat (i.e. presentable and ω-accessible). An object $C \in \mathcal{C}$ is almost compact if $\tau_{\leq n} C$ is a compact object of $\tau_{\leq n} C$ is a compact object of $\tau_{\leq n}\mathcal{C}$ for all $n > 0$.

Def: R connective $\mathbb{E}_1$-ring. A left R-module M is almost perfect if there exists an integer k such that $M \in (\mathrm{LMod}_R)_{\geq k}$ and is almost compact as an object of $(\mathrm{LMod}_R)_{\geq k}$.

Denote $\mathrm{LMod}_R^{aperf} \subset \mathrm{LMod}_R$.

Prop: $M \in (\mathrm{LMod}_R^{aperf})_{\geq 0}$, then M is the geometric realization of a simplicial left R-module $P_\bullet$ where each P_n is free and finitely generated.

Recall: An associative ring R is left coherent if every finitely generated left ideal of R is finitely presented (as a left R-module).

Def: R $\mathbb{E}_1$-ring. R is left coherent if the following conditions are satisfied:

(1) R is connective

(2) $\pi_0 R$ is left coherent.

(3) For each $n \geqslant 0$, $\pi_n R$ is finitely presented as a left module over $\pi_0 R$.

Proposition: R left coherent $\mathbb{E}_1$-ring, $M \in \mathrm{LMod}_R$. Then M is almost perfect if and only if (i) $\pi_m M = 0$ $\forall m \ll 0$

(ii) $\pi_m M$ is finitely presented over $\pi_0 R$, $\forall m \in \mathbb{Z}$.

Proposition: R left coherent $\mathbb{E}_1$-ring. Then t-structure on LMod_R

$\rightsquigarrow$ t-structure on $\mathrm{LMod}_R^{\mathrm{aperf}}$.

Proposition: R connective $\mathbb{E}_1$-ring, M connective left R-module. TFAE

(1) M is a retract of a finitely generated free R-module

(In particular, M is perfect.)

(2) M is flat and almost perfect.

Def: R connective $\mathbb{E}_1$-ring. A left R-module M has Tor-amplitude $\leq n$ if for every discrete R-module N, $\pi_i(N \otimes_R M)$ vanishes for $i > n$.

M is of finite Tor-amplitude if it has Tor-amplitude $\leq n$ for some n.

Remark: A connective left R-module M has Tor-amplitude ≤ 0 if and only if M is flat.

Proposition: R connective $\mathbb{E}_1$-ring. $M \in \mathrm{LMod}_R$

Assume M almost perfect. Then M is perfect $\Longleftrightarrow$ M has finite Tor-amplitude.

Def: R connective $\mathbb{E}_\infty$-ring. $\mathrm{Free}: \mathrm{LMod}_R \longrightarrow \mathrm{CAlg}_R$

A connective $\mathbb{E}_\infty$-algebra over R. We say A is

- finitely generated and free if $A \simeq \mathrm{Free}(M)$ for some finitely generated and free $M \in \mathrm{LMod}_R$.
- of finite presentation if A is a finite colimit of finitely generated and free algebras.
- locally of finite presentation if A is a compact object of CAlg_R.
- almost of finite presentation if A is an almost compact object of CAlg_R