Example: the smash product symmetric monoidal structure on Sp:

the ∞-category Sp of spectra admits a symmetric monoidal structure, which is uniquely determined by the following properties:

(a) The bifunctor $\otimes : Sp \times Sp \longrightarrow Sp$ preserves small colimits separately in each variable.

(b) The unit object of Sp is the sphere spectrum $\mathbb{S}$.

Lurie's construction: Consider the cartesian symmetric monoidal structure on $\mathcal{P}r^{St} \subset \mathcal{C}at_\infty$, spanned by presentable stable ∞-categories and colimit-preserving functors. Realize Sp as the unit object of $\mathcal{P}r^{St}$.

3. Algebra in the stable homotopy category

3.1 Rings and modules

Sequences of ∞-operads:
$$\begin{array}{ccccccccc} \mathbb{E}_0^{\otimes} & \hookrightarrow & \mathbb{E}_1^{\otimes} & \longrightarrow & \mathbb{E}_2^{\otimes} & \longrightarrow & \cdots & \longrightarrow & \mathbb{E}_\infty^{\otimes} \\ & & \wr\| & & & & & & \wr\| \\ & & \mathrm{Assoc}^{\otimes} & & & & & & \mathrm{Comm}^{\otimes} = \mathcal{F}in_* \end{array}$$

$\mathbb{E}_k^{\otimes}$: ∞-operad of little k-cubes

Def: An $\mathbb{E}_k$-ring is an $\mathbb{E}_k$-algebra object of Sp. Let $\mathrm{Alg}^{(k)} := \mathrm{Alg}_{\mathbb{E}_k}(Sp)$

Let $\mathrm{Alg} := \mathrm{Alg}^{(1)}$, $\mathrm{CAlg} := \mathrm{Alg}^{(\infty)}$.

Def: R $\mathbb{E}_1$-ring. $\mathrm{LMod}_R := \mathrm{LMod}_R(Sp)$ the ∞-cat of left R-modules.

HA 7.1.1.5: R $\mathbb{E}_1$-ring. The ∞-cat LMod_R and RMod_R are stable.

R $\mathbb{E}_1$-ring. $\forall n \in \mathbb{Z}$, $\pi_n R :=$ n-th homotopy group of the underlying spectrum.

Recall adjunction: $\Sigma^\infty_+ : \mathcal{S} \rightleftarrows \mathrm{Sp}(\mathcal{S}) : \Omega^\infty$ we have $\pi_n R \simeq \pi_0 \mathrm{Map}_{\mathrm{Sp}}(S[n], R)$,

$\otimes$ exact in each variable $\Rightarrow$ $S[n] \otimes S[m] \simeq S[n+m]$ $\forall n, m \in \mathbb{Z}$.

$$\mathrm{Map}_{\mathrm{Sp}}(S[n], R) \times \mathrm{Map}_{\mathrm{Sp}}(S[m], R) \longrightarrow \mathrm{Map}_{\mathrm{Sp}}(S[n] \otimes S[m], R \otimes R) \longrightarrow \mathrm{Map}_{\mathrm{Sp}}(S[n+m], R)$$

$\rightsquigarrow$ bilinear map $\pi_n R \times \pi_m R \longrightarrow \pi_{n+m} R$

$\rightsquigarrow$ a graded associative ring structure on $\pi_* R := \bigoplus_n \pi_n R$.

If R is an $\mathbb{E}_k$-ring for $k \geq 2$, then the multiplication on $\pi_* R$ is graded commutative, i.e. $\forall x \in \pi_n R$, $y \in \pi_m R$, we have $xy = (-1)^{nm} yx$.

In particular, $\pi_0 R$ is a commutative ring. $\forall n \in \mathbb{Z}$, $\pi_n R$ is a module over $\pi_0 R$.

R $\mathbb{E}_1$-ring, M left R-module. The action map $R \otimes M \longrightarrow M$

$\rightsquigarrow$ bilinear maps $\pi_n R \times \pi_m M \longrightarrow \pi_{n+m} M$

$\rightsquigarrow$ $\pi_* M := \bigoplus_n \pi_n M$ has the structure of a graded left module over $\pi_* R$.

Def: A spectrum X is connective if $\pi_n X \simeq 0$ for $n < 0$.

$\mathrm{Sp}^{\mathrm{cn}} \subset \mathrm{Sp}$ full subcat spanned by connective spectra.

An $\mathbb{E}_k$-ring is connective if its underlying spectrum is connective.

$\mathrm{Alg}^{\mathrm{cn}} \subset \mathrm{Alg}$, $\mathrm{CAlg}^{\mathrm{cn}} \subset \mathrm{CAlg}$, Notation: R $\mathbb{E}_1$-ring. $\mathrm{LMod}_R^{\geq 0} \subset \mathrm{LMod}_R$ spanned by R-modules M with $\pi_n M = 0$ $\forall n < 0$.

$\mathrm{LMod}_R^{\leq 0}$

$M, N \in \mathrm{LMod}_R$, $\mathrm{Ext}^i_R(M, N) := \pi_0 \mathrm{Map}_{\mathrm{LMod}_R}(M, N[i])$.

HA7.1.1.13: R connective $\mathbb{E}_1$-ring.

Then $(\mathrm{LMod}_R^{\geq 0}, \mathrm{LMod}_R^{\leq 0}) \rightsquigarrow$ accessible t-structure on LMod_R.

π_0 determines an equivalence of the heart $\mathrm{LMod}_R^{\heartsuit}$ with the abelian category $\mathcal{A}$ of left $\pi_0 R$-modules. $\rightsquigarrow$ right t-exact functor $\theta : \mathcal{D}^-(\mathcal{A}) \to \mathrm{LMod}_R$.

HA7.1.1.15: If R is discrete, i.e. $\pi_i(R) = 0$ $\forall i = 0$, then θ induces an equiv of $\mathcal{D}^-(\mathcal{A})$ with the ∞-cat of right bounded objects of LMod_R.

$\rightsquigarrow$ equiv of ∞-cats $\mathcal{D}(\mathcal{A}) \simeq \mathrm{LMod}_R$.

$\rightsquigarrow$ it can be promoted to an equiv of symmetric monoidal ∞-cats.

Recognition principles: when is a stable ∞-category of the form LMod_R or RMod_R?

Schwede-Shipley theorem: $\mathcal{C}$ stable ∞-cat. Then $\mathcal{C}$ is equiv to RMod_R for some $\mathbb{E}_1$-ring R, if and only if $\mathcal{C}$ is presentable and $\exists$ a compact object $C \in \mathcal{C}$ which generates $\mathcal{C}$ in the following sense: if $D \in \mathcal{C}$ is an object having the property that $\mathrm{Ext}^n_{\mathcal{C}}(C, D) \simeq 0$ for $\forall n \in \mathbb{Z}$, then $D \simeq 0$.

HA7.1.2.7: $\mathcal{C}$ symmetric monoidal ∞-cat. Then $\mathcal{C}$ is equiv to $\mathrm{Mod}_R^{\otimes}$ for some $\mathbb{E}_\infty$-ring R if and only if the following conditions are satisfied:

(1) $\mathcal{C}$ is stable and presentable, $\otimes$ preserve small colimits

(2) The unit object $\mathbb{1} \in \mathcal{C}$ is compact.

(3) The object $\mathbb{1}$ generates $\mathcal{C}$ as above.

3.2 Explicit models for algebras over discrete commutative rings:

Def: R comm ring. A differential graded algebra over R is a graded associative algebra A_* over R equipped with a differential $d: A_* \to A_{*-1}$ satisfying the following conditions: • $d^2=0$

• d is a (graded) derivation, i.e. we have the Leibniz rule

$$d(xy)=(dx)y + (-1)^{|x|} x\,dy.$$

Morphism $\phi: A_* \to B_*$ is a homomorphism of graded R-algebra st. $\phi(dx) = d\phi(x)$.

$Alg^{dg}(R)$: category of dg algebras over R.

A map $\phi: A_* \to B_*$ of dg-algebras is a quasi-isomorphism if it induces a quasi-isom of chain complexes over R.

HA7.1.4.6: R comm ring. We have an equiv of ∞-cats:

$$Alg^{dg}(R)[\text{quasi-isomorphisms}^{-1}] \simeq Alg_R \; (:= Alg_{\mathbb{E}_1}(LMod_R))$$

Def: A dg-algebra A_* over a comm ring R is a commutative differential graded algebra (cdga) if $\forall\, x \in A_m,\ y \in A_n$, we have $xy = (-1)^{mn} yx$.

$CAlg^{dR}(R) \subset Alg^{dg}(R)$ full subcat.

HA 7.1.4.11. R comm ring of char 0 i.e. $R \supset \mathbb{Q}$ (otherwise we have trouble with model structures for cdgas)

We have an equiv of ∞-cats:

$$CAlg^{dg}(R)[\text{quasi-isom}^{-1}] \simeq CAlg_R \; (:= CAlg(LMod_R(Sp)) \simeq CAlg_{R/})$$

HA 7.1.4.18: R comm ring, Alg_R^{disc} category of discrete associative R-algebras, Alg_R^{Simp} category of simplicial objects of Alg_R^{disc}. Then we have an equiv of ∞-cats

$$Alg_R^{Simp}[\text{homotopy equiv of underlying simplicial sets}^{-1}] \simeq Alg_R^{cn}.$$

HA 7.1.4.20: R comm ring. $CAlg^{disc}$ category of discrete commutative R-algebras. $CAlg_R^{Simp}$ cat of simplicial objects of $CAlg_R^{disc}$. We have a functor

$$CAlg_R^{Simp}[\text{———}^{-1}] \longrightarrow CAlg_R^{cn}.$$

It is an equiv if $R \supset \mathbb{Q}$.

3.3 Properties of rings and modules

3.3.1: Free resolutions and spectral sequences.

Def: $\mathcal{C}$ presentable ∞-cat, S a collection of objects of $\mathcal{C}$, A simplicial object $X_\bullet$ of $\mathcal{C}$ is S-free if $\forall n$, $\exists$ a coproduct F of objects of S and a map $F \to X_n$ in $\mathcal{C}$ which induces an equiv

$$L_n(X) \sqcup F \xrightarrow{\sim} X_n.$$

$\llcorner$ n^{th} latching object,
consisting of all "degenerate simplices".

Let $C \in \mathcal{C}$, $X_\bullet$ a simplicial object of $\mathcal{C}_{/C}$, $X_\bullet$ is called an S-hypercovering of C if for every object $Y \in S$ corepresenting a functor $\chi : \mathcal{C} \to \mathcal{S}$, the simplicial object $\chi(X_\bullet)$ is a hypercovering in the ∞-topos $\mathcal{S}_{/\chi(C)}$.

Example: R associative ring, $\mathcal{A}$ category of left R-modules, $S = \{R\}$.

$M_\bullet$ simplicial object $\xrightarrow{\text{Dold-Kan correspondence}}$ P_* corresponding chain complex

Then $M_\bullet$ is S-free $\Leftrightarrow$ each P_n is a free left R-module.

$M_\bullet$ is an S-hypercovering of a left R-module M

$\Leftrightarrow$ the associated chain complex $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ is exact.

HA 7.2.1.4-9: $\mathcal{C}$ presentable ∞-cat, S a set of objects of $\mathcal{C}$. Then for every object $C \in \mathcal{C}$, there exists an S-free S-hypercovering $X_\bullet : \Delta^{op} \to \mathcal{C}_{/C}$, unique up to simplicial homotopy.

Def: R $\mathbb{E}_1$-ring, $N \in \mathrm{LMod}_R$. N is quasi-free if $N \simeq \bigoplus_{\alpha \in A} R[n_\alpha]$ $\in \mathbb{Z}$

For $M \in \mathrm{RMod}_R$, $N \in \mathrm{LMod}_R$, we can study the homotopy groups of $M \otimes_R N$ by resolving N by quasi-free R-modules.

$P_\bullet$ S-free S-hypercovering of N where $S := \{R[n], n \in \mathbb{Z}\}$.

$\rightsquigarrow$ spectral sequence with E_2-page $E_2^{p,*} \simeq \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)$

which converges to $\pi_{p+q}(M \underset{R}{\otimes} N)$.