4. Examples

∞-categories arise naturally by inverting a collection of morphisms in an ordinary category.

Given an ∞-category $\mathcal{C}$ and a collection of morphisms W in $\mathcal{C}$, we can construct an ∞-category $\mathcal{C}[W^{-1}]$ and a functor $\alpha: \mathcal{C} \to \mathcal{C}[W^{-1}]$ s.t. for every ∞-category $\mathcal{D}$, composition with α induces a fully faithful embedding

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

whose essential image consists of those functors which carry every morphism in W to an equivalence in $\mathcal{D}$.

Example 1: $\mathcal{K}an$: category of Kan complexes

W : collection of homotopy equivalences

$\mathcal{S} := \mathcal{K}an[W^{-1}]$ called the ∞-category of spaces

2. $\mathcal{W}\mathcal{K}an$: category of weak Kan complexes (model for our ∞-categories)

W : collection of equivalences of ∞-categories

$\mathcal{C}at_\infty := \mathcal{W}\mathcal{K}an[W^{-1}]$ called the ∞-category of (small) ∞-categories

5. Fibrations of ∞-categories

Idea: Want a family of ∞-categories parametrized by an ∞-category

$$\begin{array}{ccc} \mathcal{C} & \quad & x \xrightarrow{\tilde{f}} y \\ p\downarrow & & \\ \mathcal{D} & & d \xrightarrow{f} d' \end{array}$$

$\forall d \in \mathcal{D}$, we want the fiber $\mathcal{C}_d := \mathcal{C} \times_{\mathcal{D}} \{d\}$ to be an ∞-category.

$\forall$ mor $d \xrightarrow{f} d'$ in $\mathcal{D}$, we want a functor $f^*: \mathcal{C}_{d'} \to \mathcal{C}_d$.

So for each $y \in \mathcal{C}_{d'}$, we need to choose a mor $x \xrightarrow{\tilde{f}} y$ in $\mathcal{C}$ lifting f, and set $f^* y = x$.

Def: $p: \mathcal{C} \to \mathcal{D}$ ∞-cats. A mor $f: x \to y$ in $\mathcal{C}$ is called p-cartesian if it is a final object in $\mathcal{C}_{/y} \times_{\mathcal{D}_{/\bar{y}}} \mathcal{D}_{\bar{f}}$, where the bars denote the images by p.

If f is p-cartesian, we call it a p-cartesian lift of $\bar{f}$ at y.

The functor p is called a cartesian fibration if for all $y \in \mathcal{C}$ and $\bar{f}: \bar{x} \to p(y)$, there is a p-cartesian lift of $\bar{f}$ at y.

Dually, f is called p-cocartesian if it is cartesian wrt $p^{op}: X^{op} \to S^{op}$.

p is called cocartesian if $p^{op}: X^{op} \to S^{op}$ is cartesian.

HTT2.4.1.10: The mor f is p-cartesian if and only if for any $z \in \mathcal{C}$, we have a pullback diagram of mapping spaces

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(z, x) & \xrightarrow{f \circ -} & \mathrm{Map}_{\mathcal{C}}(z, y) \\ \downarrow & & \downarrow \end{array}$$

$$\mathrm{Map}_{\mathcal{D}}(\bar{z}, \bar{x}) \xrightarrow{f \circ -} \mathrm{Map}_{\mathcal{D}}(\bar{z}, \bar{y})$$

Families of initial objects in a cartesian fibration

HTT 2.4.4.9: $p: \mathcal{C} \to \mathcal{D}$ cartesian fibration of ∞-cats. Assume $\forall d \in \mathcal{D}$, the ∞-cat $\mathcal{C}_d := \mathcal{C} \times_{\mathcal{D}} \{d\}$ has an initial object. Then $\exists$ a functor $q: \mathcal{D} \to \mathcal{C}$ which is a section of p, s.t. $q(d)$ is an initial object of $\mathcal{C}_d$ for $\forall d \in \mathcal{D}$.

Examples of cartesian fibrations:

HTT 2.4.7.12: $f: \mathcal{C} \to \mathcal{D}$ functor between ∞-cats. The projection

$$p: \mathrm{Fun}(\Delta^1, \mathcal{D}) \times_{\mathrm{Fun}(\{1\}, \mathcal{D})} \mathcal{C} \longrightarrow \mathrm{Fun}(\{0\}, \mathcal{D})$$

is a cartesian fibration. Moreover, a morphism of is p-cartesian if and only if its image in $\mathcal{C}$ is an equivalence.

Example: $\mathcal{C}$ ∞-category. Then $ev_0: \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a cartesian fibration, and $ev_1: \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a cocartesian fibration. Moreover, if $\mathcal{C}$ has pushouts, then ev_0 is also a cocartesian fibration, if $\mathcal{C}$ has pullbacks, then ev_1 is also a cartesian fibration.

Straightening and unstraightening

HTT 3.2.0.1: $\mathcal{C}$ ∞-cat, we have an equiv of ∞-cats

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty) \underset{St}{\overset{Un}{\rightleftarrows}} (\mathrm{Cat}_\infty)^{cart}_{/\mathcal{C}}$$

where $(\mathrm{Cat}_\infty)^{cart}_{/\mathcal{C}}$ is the subcategory of $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ spanned by cartesian fibrations, and functors preserving cartesian morphisms.

Examples: Applying straightening to the previous example, we obtain funtors

$$\mathcal{C} \to \mathrm{Cat}_\infty, \quad x \mapsto \mathcal{C}_{/x}, \quad (x \xrightarrow{f} y) \mapsto (\mathcal{C}_{/x} \xrightarrow{f\circ -} \mathcal{C}_{/y})$$

$$\mathcal{C} \to \mathrm{Cat}_\infty, \quad x \mapsto \mathcal{C}_{x/}, \quad (x \xrightarrow{f} y) \mapsto (\mathcal{C}_{y/} \xrightarrow{-\circ f} \mathcal{C}_{x/})$$

If $\mathcal{C}$ has pushouts, we have

$$\mathcal{C} \to \mathrm{Cat}_\infty, \quad x \mapsto \mathcal{C}_{x/}, \quad (x \xrightarrow{f} y) \mapsto (\mathcal{C}_{x/} \xrightarrow{-\sqcup_x y} \mathcal{C}_{y/})$$

If $\mathcal{C}$ has pullbacks, we have

$$\mathcal{C} \to \mathrm{Cat}_\infty, \quad x \mapsto \mathcal{C}_{/x}, \quad (x \xrightarrow{f} y) \mapsto (\mathcal{C}_{/y} \xrightarrow{-\times_y x} \mathcal{C}_{/x})$$

Unstraightening allows us explicit constructions of limits and colimits in Cat_∞

HTT3.3.3.2: Diagram $F: I \to \mathrm{Cat}_\infty \xrightarrow{Un}$ cartesian fibration $p: X \to I^{op}$.

Then $$\lim_I F \simeq \mathrm{Fun}^{cart}_{/I^{op}}(I^{op}, X)$$

where RHS denotes the full subcat of $\mathrm{Fun}_{/I^{op}}(I^{op}, X)$ spanned by functors that send every mor of I^{op} to a p-cartesian mor in X.

Example: Consider the diagram of ∞-cats

$$\begin{tikzcd} & \mathcal{E} \arrow[d, "g"] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

$\xrightarrow{\text{Unstraightening}}$ cartesian fibration

$$\begin{tikzcd} \mathcal{X} := \{\mathcal{C} & \mathcal{D} \arrow[d, "p"] & \mathcal{E}\} \\ \{*_1 & *_2 \arrow[l, "\alpha"'] \arrow[r, "\beta"] & *_3\} \end{tikzcd}$$

By HTT 2.4.1.10, for any $c \in \mathcal{C}$, $d \in \mathcal{D}$, $e \in \mathcal{E}$, we have

$$\mathrm{Map}_{\mathcal{X}}(d, c) \simeq \mathrm{Map}_{\mathcal{D}}(d, f(c))$$

$$\mathrm{Map}_{\mathcal{X}}(d, e) \simeq \mathrm{Map}_{\mathcal{D}}(d, g(e))$$

$$\mathrm{Map}_{\mathcal{X}}(c, d) \simeq \mathrm{Map}_{\mathcal{X}}(e, d) \simeq \mathrm{Map}_{\mathcal{X}}(c, e) \simeq \phi$$

A section s of p consists of the following data:

- $c := s(*_1) \in \mathcal{C}$, $d := s(*_2) \in \mathcal{D}$ and $e := s(*_3) \in \mathcal{E}$
- $s(\alpha)$: $d \to c$ in $\mathcal{X}$ i.e. a mor $d \to f(c)$ in $\mathcal{D}$
- $s(\beta)$: $d \to e$ in $\mathcal{X}$ i.e. a mor $d \to g(e)$ in $\mathcal{D}$

s is cartesian : $f(c) \simeq d \simeq g(e)$

So cartesian sections s of p $\rightsquigarrow$ an obj in $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$.

For colimits, we have

HTT 3.3.4.3: Diagram $F: I \to \mathrm{Cat}_\infty \overset{Un}{\rightsquigarrow}$ cocartesian fibration $p: X \to I$

Let W be the collection of all p-cocartesian morphisms in X.

$$\operatorname*{colim}_{I} F \simeq X[W^{-1}]$$

Fibration of spaces

Def: A right (left) fibration is a (co)cartesian fibration $p: \mathcal{C} \to \mathcal{D}$ whose fibers are spaces (i.e. $\mathcal{S} \subset \mathrm{Cat}_\infty$).

Straightening and Unstraightening

HTT 2.2.1.2: $\mathcal{C}$ ∞-cat. We have an equiv of ∞-cats

$$\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) \underset{St}{\overset{Un}{\rightleftarrows}} (\mathrm{Cat}_\infty)^{rfib}_{/\mathcal{C}}$$

where RHS is the subcat of $(\mathrm{Cat}_\infty)_{/\mathcal{C}}$ spanned by right fibrations.

6. Kan extensions

Def: $f: \mathcal{C} \to \mathcal{D}$ ∞-cats, $\bar{p}: K^{\triangleright} \to \mathcal{C}$ diagram, $p = \bar{p}|_K$

$\bar{p}$ is called an f-colimit of p if $\mathcal{C}_{\bar{p}/} \xrightarrow{\sim} \mathcal{C}_{p} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\bar{p}/}$

Example: For $f: \mathcal{C} \to *$, $\bar{p}: K^{\triangleright} \to \mathcal{C}$ is an f-colimit iff it is a colimit

Def: Given comm diag of ∞-cats

$$\begin{tikzcd} \mathcal{C}^\circ \arrow[r, "F_0"] \arrow[d, hook, "\text{full}"'] & \mathcal{D} \arrow[d, "p"] \\ \mathcal{C} \arrow[r] \arrow[ur, "F"] & \mathcal{D}' \end{tikzcd}$$

F is called a p-left Kan extension of F_0 at $C \in \mathcal{C}$ if the induced diagram

$$\begin{tikzcd} (\mathcal{C}^\circ_{/C}) \arrow[r, "F_C"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ (\mathcal{C}^\circ_{/C})^{\triangleright} \arrow[r] \arrow[ur] & \mathcal{D}' \end{tikzcd}$$

exhibits $F(C)$ as a p-colimit of F_C.

F is called a p-left Kan extension of F_0 if it is a p-left Kan extension of F_0 at C for every $C \in \mathcal{C}$.

HTT 4.3.2.15: Given ∞-cats

$$\begin{tikzcd} \mathcal{C}^\circ \subset_{\text{full}} \mathcal{C} \arrow[dr] & & \mathcal{D} \arrow[dl, "p"] \\ & \mathcal{D}' & \end{tikzcd}$$

Let $K \subset \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ be the full subcat spanned by functors that are left Kan extensions of their restrictions to $\mathcal{C}^\circ$.

Let $K' \subset \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^\circ, \mathcal{D})$ be the full subcat spanned by functors $F_0 : \mathcal{C}^\circ \to \mathcal{D}$ s.t. $\forall C \in \mathcal{C}$, the induced diagram $\mathcal{C}^\circ_{/C} \to \mathcal{D}$ has a p-colimit. Then the restriction functor $K \to K'$ is an equiv of ∞-cats.

HTT 4.3.2.16: Assume $\forall F_0 \in \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^\circ, \mathcal{D})$, $\exists F \in \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D})$ which is a p-left Kan extension of F_0. Then the restriction map $i^* : \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^\circ, \mathcal{D})$ admits a section $i_!$ whose essential image consists of precisely those functors F which are p-left extensions of $F|_{\mathcal{C}^\circ}$.

left Kan extension functor

Left Kan extensions $\xleftrightarrow{\text{dual}}$ Right Kan extensions.

7. Presentable ∞-categories

Def: $\mathcal{C}$ ∞-cat, $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ ∞-cat of presheaves on $\mathcal{C}$.

$\mathcal{D}$ ∞-cat, $\mathcal{P}_{\mathcal{D}}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$ $\mathcal{D}$-valued presheaves

HTT 5.1.2.3: $\mathcal{P}(\mathcal{C})$ admits all small limits and colimits, and they can be computed pointwise.

Construction of the Yoneda embedding:

Category of twisted arrows: I, J linearly ordered sets, let $I \star J$ denote the coproduct $I \sqcup J$, equipped with the unique linear ordering which restricts to the given linear orderings on I and J, and satisfies $i \leq j$ for $i \in I$ and $j \in J$.

Δ : cat of combinatorial simplices

Functor $Q: \Delta \to \Delta$

$$I \mapsto I \star I^{op}$$
$$[n] \mapsto [2n+1]$$

For $\mathcal{C}$ ∞-cat, we define $\mathrm{TwArr}(\mathcal{C})$ to be the simplicial set

$$[n] \mapsto \mathcal{C}(Q[n]) = \mathcal{C}([2n+1])$$

Informally, objects of $\mathrm{TwArr}(\mathcal{C})$ are morphisms $f: C \to D$ in $\mathcal{C}$, and morphisms in $\mathrm{TwArr}(\mathcal{C})$ are given by comm diag

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & D \\ C' \arrow[r, "f'"] & D' \arrow[u] \end{tikzcd}$$

We have canonical inclusions $I \hookrightarrow I \star I^{op} \hookleftarrow I^{op}$

$\rightsquigarrow$ maps of simplicial sets $\mathcal{C} \leftarrow \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C}^{op}$

HA 5.2.1.3: The canonical map $\lambda: \mathrm{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$ is a right fibration.

$\xrightarrow{\text{Straightening}}$ functor $\mathcal{C} \times \mathcal{C}^{op} \to \mathcal{S}$

$\rightsquigarrow$ Yoneda embedding $j: \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) = \mathcal{P}(\mathcal{C})$

HTT 5.1.3.1–2: The Yoneda embedding is fully faithful. It preserves all small limits which exist in $\mathcal{C}$.

HTT 5.1.5.8: The Yoneda embedding generates $\mathcal{P}(\mathcal{C})$ under small colimits.