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A complex semidefinite programming rounding approximation algorithm for the balanced Max-3-Uncut problem

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Outline

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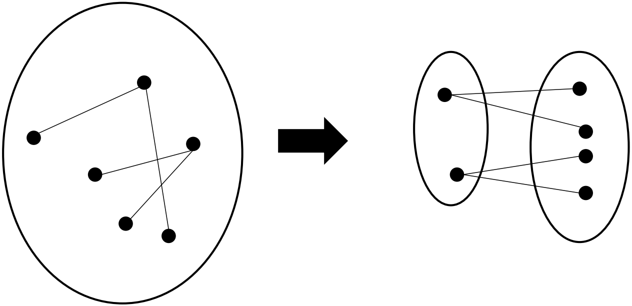
Introduction

- Graph Partition Problem
  - The most famous problem is Max Cut.



Max Cut

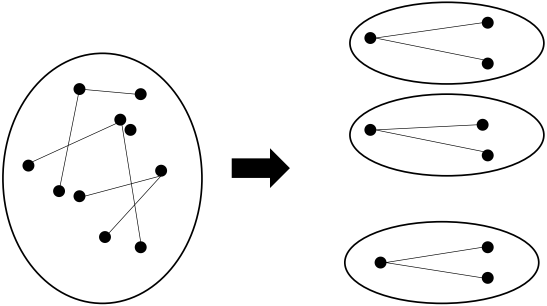
Introduction

- Graph Partition Problem
  - The most famous problem is Max Cut.
  - There are also some other variant problems of Max Cut problem:
    - Max Bisection(balanced version of Max-Cut): adding equal cardinality constraint
    - Max-$\frac{n}{2}$-Uncut: balanced version and calculating the weight not in the cut.
    - $\cdots$

Introduction

We study the balanced Max-3-Uncut.

- Problem description
  Given a weighted graph $G = (V, E)$(Assume $|V|$ is a multiple of 3)
  - weight function $w : E \to R_+$
  - Goal: partition V into three subsets S_1, S_2, and S_3 with equal cardinality such that the total weight of the edges from the same subsets is maximized



Introduction

- Methods
  - Approximation algorithm(attractive algorithm with bounded solution)
  - linear program $\rightarrow$ Semidefinite program
    The real space $\mathbb{R}$ $\rightarrow$ The complex space $\mathbb{C}$
- Results
  - Based on the complex semdefinite programming rounding technique, we proposed a 0.3456-approximation algorithm for the balanced Max-3-Uncut.

Introduction

- Literature review
  - Based on semidefinite programming in $\mathbb{R}$
    - Goemans and Williamson(J. ACM, 1995) for the Max-Cut: 0.87856, semidefinite programming rounding using randomly hyperplane;
    - Frieze and Jerrum(Algorithmica, 2006) for the Max-Bisection: 0.6514, semidefinite programming rounding + greedy swapping;
    - Austin et al. (SODA, 2013) for the Max-Bisection: 0.8776, semidefinite programming hierarchies rounding.(Best until now)
    - Halperin and Zwick(Random Structures and Algorithms, 2002) for the Balanced Max-2-Uncut: 0.6436, semidefinite programming rounding using randomly hyperplane.
    - Wu et al.(J Combin Opt, 2013) for the Balanced Max-2-Uncut: 0.8776, semidefinite programming rounding using randomly hyperplane.

Introduction

- Literature review
  - Based on semidefinite programming in $\mathbb{C}$
    - Goemans and Williamson(J. Comput. Syst. Sci., 2004) for the Max-3-Cut: $\left(\frac{7}{12}\frac{3}{4\pi^2}\arccos^2(-1/4)-\epsilon\right) \approx (0.8360-\epsilon)$, for any given $\epsilon > 0$, complex semidefinite programming rounding.
    - Ling (COCOA, 2009) for Max-3-Section: 0.6733, complex semidefinite programming rounding + greedy swapping.

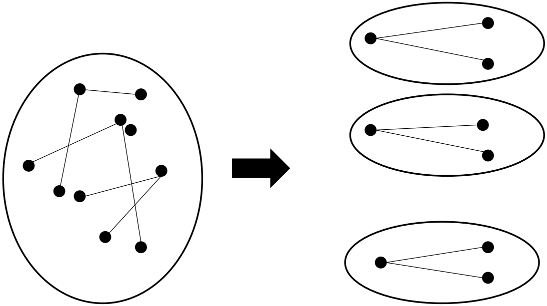
Formulation

The Balanced Max-3-Uncut can be described by

$$\max_{\substack{(S_1,S_2,S_3)\in\mathcal{P}(V)\\ |S_1|=|S_2|=|S_3|}} \sum_{i,j\in S_1} w_{ij} + \sum_{i,j\in S_2} w_{ij} + \sum_{i,j\in S_3} w_{ij},$$

where

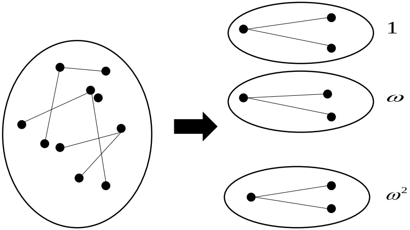
$$\mathcal{P}(V) := \{(S_1,S_2,S_3) : S_1\cup S_2\cup S_3 = V, \text{ and } S_k\cap S_l=\emptyset \text{ for all } k\neq l\}$$



Formulation

$$\text{Max} - \text{Cut} \qquad x^2 = 1(-1, 1)$$
$$\text{Balaced Max} - 3 - \text{Uncut} \qquad x^3 = 1(1, \omega = e^{-\frac{2}{3}\pi i}, \omega^2)$$

$$\max \quad \frac{1}{3}\sum_{i<j} w_{ij}(1 + y_i \cdot y_j + y_j \cdot y_i)$$
$$\text{s. t.} \quad \sum_{i \in V} y_i = 0,$$
$$y_i \in \{1, \omega, \omega^2\}, \qquad \forall i \in V.$$



Introduction the variable $y_i \in \{1, \omega, \omega^2\}$ for each $i \in V$.

Formulation

- Relaxation: $y_i \in \mathbb{C} \to v_i \in \mathbb{C}^n, \| v_i \| = 1$
- Tighter relaxation: Since $y_i \in \{1, \omega, \omega^2\}$ for all $i \in V$, we must have

$$\begin{aligned} y_i \cdot y_j + y_j \cdot y_i &\geq -1, \qquad \forall i,j \in V, \\ \omega \cdot (y_i \cdot y_j) + \omega^2 \cdot (y_j \cdot y_i) &\geq -1, \qquad \forall i,j \in V, \\ \omega^2 \cdot (y_i \cdot y_j) + \omega \cdot (y_j \cdot y_i) &\geq -1, \qquad \forall i,j \in V, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \mathrm{Re}(y_i \cdot y_j) &\geq -\frac{1}{2}, \qquad \forall i,j \in V, \\ \mathrm{Re}(\omega \cdot (y_i \cdot y_j)) &\geq -\frac{1}{2}, \qquad \forall i,j \in V, \\ \mathrm{Re}(\omega^2 \cdot (y_i \cdot y_j)) &\geq -\frac{1}{2}, \qquad \forall i,j \in V. \end{aligned}$$

Formulation

By adding the above extra inequalities into the above program, we get the complex semidefinite programming relaxations as follows.

$$
\begin{aligned}
\max \quad & \frac{1}{3}\sum_{i<j} w_{ij}(1 + v_i \cdot v_j + v_j \cdot v_i) \\
\text{s. t.} \quad & \mathrm{Re}(v_i \cdot v_j) \geq -\frac{1}{2}, \qquad \forall i, j \in V, \\
& \mathrm{Re}(\omega \cdot (v_i \cdot v_j)) \geq -\frac{1}{2}, \qquad \forall i, j \in V, \\
& \mathrm{Re}(\omega^2 \cdot (v_i \cdot v_j)) \geq -\frac{1}{2}, \qquad \forall i, j \in V, \\
& \sum_{i,j} v_i \cdot v_j = 0, \\
& \| v_i \| = 1, \qquad \forall i \in V, \\
& v_i \in \mathbb{C}^n, \qquad \forall i \in V.
\end{aligned} \tag{2.1}
$$

Algorithm

Step 1 Solve complex semidefinite programming

Solve the (2.1) to obtain an optimal solution $\{v_i\}$, leading to a complex semidefinite matrix $V := (v_i \cdot v_j)$.

Step 2 Generate random complex variable

For a given parameter $\theta \in [0, 1]$, choose a random vector $\xi \sim N(0, \theta V + (1-\theta)I)$, where I is the $n \times n$ identity matrix.

Step 3 Obtain solution for the Max-3-Uncut

$$\hat{y}_i = \begin{cases} 1, & \mathrm{Arg}(\xi_i) \in [0, \frac{2}{3}\pi); \\ \omega, & \mathrm{Arg}(\xi_i) \in [\frac{2}{3}\pi, \frac{4}{3}\pi); \\ \omega^2, & \mathrm{Arg}(\xi_i) \in [\frac{4}{3}\pi, 2\pi). \end{cases}$$

Let $S_1 := \{i : \hat{y}_i = 1\}$, $S_2 := \{i : \hat{y}_i = \omega\}$, and $S_3 := \{i : \hat{y}_i = \omega^2\}$.

Algorithm

Step 4 Swap greedy to obtain solution for the balanced Max-3-Uncut

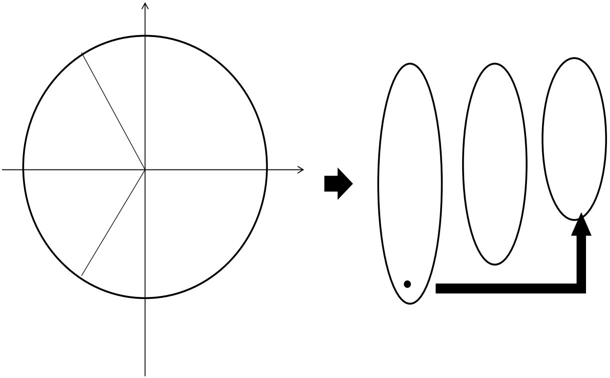
Assume, without loss of generality, $|S_1| \geq |S_2| \geq |S_3|$. Initialize $\hat{S}_\ell = S_\ell$ $(\ell = 1, 2, 3)$. Denote the final partition with equal cardinality as $\tilde{S}_1$, $\tilde{S}_2$, and $\tilde{S}_3$.

Case 4.1. If $|S_1| \geq |S_2| \geq \frac{n}{3} \geq |S_3|$, then iteratively, perform the following operations (i)-(ii) until $|\hat{S}_\ell| = \frac{n}{3}$ for each $\ell = 1, 2$:

- (i) Sort the vertices in $\hat{S}_\ell$ such that $\delta(i_1) \geq \ldots \geq \delta(i_{|\hat{S}_\ell|})$ where $\delta(i) = \sum_{i' \in \hat{S}_\ell} w_{i'i}$ $(i \in \hat{S}_\ell)$.
- (ii) Move the point $i_{|\hat{S}_\ell|}$ from $\hat{S}_\ell$ to $\hat{S}_3$; namely, $\hat{S}_\ell = \hat{S}_\ell \backslash \{i_{|\hat{S}_1|}\}$, and $\hat{S}_3 = \hat{S}_3 \cup \left\{i_{|\hat{S}_\ell|}\right\}$.

Case 4.2. Operate similarly for the case of $|S_1| \geq \frac{n}{3} \geq |S_2| \geq |S_3|$.

Algorithm



Analysis

- First, we define

$$z(\gamma) := \frac{W(S_1, S_2, S_3)}{W^*} + \gamma \frac{C}{C^*},$$

where

- $W(S_1, S_2, S_3)$ is the weight of the partition is S_1, S_2, S_3.
- W^* is the optimal solution of the complex semidefinite relaxation of the Max-3-Uncut.
- $C = |S_1||S_2| + |S_1||S_3| + |S_2||S_3|$.
- $C^* = \frac{n^2}{3}$.
- $\mu(x) = \frac{C}{C^*}$, where $x = \left(\frac{S_1}{n}, \frac{S_2}{n}, \frac{S_3}{n}\right)$.

If the following hold, we can estimate the approximation ratio.

- $E\left[\frac{W(S_1,S_2,S_3)}{W^*}\right] \geq \alpha(\theta)$, which estimates the ratio of weight between the solution before swapping process and the optimal solution.
- $E\left[\frac{C}{C^*}\right] \geq \beta(\theta)$, which estimate the ratio of cardinality between the solution before swapping process and the optimal solution.
- $\frac{W(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3)}{W^*} \geq r(x)\frac{W(S_1,S_2,S_3)}{W^*}$ which estimate the ratio between the weight of the solution before the swapping process and the solution after the swapping process.

Thus $z(\gamma)$ is a balance between the weight and the cardinality.

Analysis

Denote $\mu(x) = \frac{C}{C^*}$, where $x = \left(\frac{S_1}{n}, \frac{S_2}{n}, \frac{S_3}{n}\right)$.

$$z(\gamma) := \frac{W(S_1, S_2, S_3)}{W^*} + \gamma \frac{C}{C^*}$$

$$\begin{aligned} \Rightarrow \quad \frac{W(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3)}{W^*} &\geq r(x) \frac{W(S_1, S_2, S_3)}{W^*} \\ &\geq r(x) \left(\alpha(\theta) + \gamma\beta(\theta) - \gamma \frac{C}{C^*} \right) \\ &= r(x) \left(\alpha(\theta) + \gamma\beta(\theta) - \gamma\mu(x) \right). \end{aligned}$$

Analysis

Denote $R(x;\theta,\gamma) = r(x)(\alpha(\theta) + \gamma\beta(\theta) - \gamma\mu(x))$. Thus, the approximation ratio is

$$\max_{\gamma,\theta} \min_{x\in\Delta} R(x;\theta,\gamma),$$

where $\Delta = \left\{ x = (x_1, x_2, x_3) \;\middle|\; \begin{array}{l} \sum_{i=1}^{3} x_i = 1 \\ x_1 \geq x_2 \geq x_3 \geq 0 \end{array} \right\}.$

Analysis

Next, we need to analyze the three inequalities, give the close form of $\alpha(\theta)$, $\beta(\theta)$ and $r(x)$.

$\alpha(\theta)$

Note that $\alpha(\theta) = \frac{\mathrm{E}[W(S_1,S_2,S_3)]}{W^*}$.

$$\begin{aligned}
& \mathrm{E}\left[W(S_1, S_2, S_3)\right] \\
= \quad & \frac{1}{3}\sum_{i<j} w_{ij}\left(1 + 2\mathrm{Re}\mathrm{E}[\hat{y}_i \cdot \hat{y}_j]\right) \\
\geq \quad & \frac{1}{3}\sum_{i<j} w_{ij}\left(1 + 2\mathrm{Re}(v_i \cdot v_j)\right) \frac{1 + 2\mathrm{Re}\mathrm{E}[\hat{y}_i \cdot \hat{y}_j]}{1 + 2\mathrm{Re}(v_i \cdot v_j)} \\
= \quad & W^*\left(\min_{\{v_i\} \text{ satisfies the constraints of SDP relaxation}} \frac{1 + 2\mathrm{Re}\mathrm{E}[\hat{y}_i \cdot \hat{y}_j]}{1 + 2\mathrm{Re}(v_i \cdot v_j)}\right),
\end{aligned}$$

then, we need to give the expected value of $\hat{y}_i \cdot \hat{y}_j$ if we need to estimate $\alpha(\theta)$.

$\alpha(\theta)$

By Goemans and Williamson (J. Comput. Syst. Sci. 2004) and Zhang and Huang (SIAM J. Optim. 2006), we have the following lemma which estimates the expected value of the real part of the feasible solution obtained by the algorithm.

Lemma

The real part of the expected value of $\hat{y}_i \cdot \hat{y}_j$ *is*

$$\frac{9}{8\pi^2}\left[\arccos^2\left(-\mathrm{Re}(\theta v_i \cdot v_j)\right) - \frac{1}{2}\arccos^2(-\mathrm{Re}(\omega \cdot (\theta v_i \cdot v_j))) - \frac{1}{2}\arccos^2(-\mathrm{Re}(\omega^2 \cdot (\theta v_i \cdot v_j)))\right].$$

$\alpha(\theta)$

Thus, we can obtain the value of $\alpha(\theta)$.

Lemma

For a given $\theta \in [0,1]$, the ratio of the expected weight of (S_1, S_2, S_3) and W^* is no less than $\alpha(\theta)$, where $\alpha(\theta)$ is

$$
\begin{array}{ll}
\min & g(\theta, z_1, z_2) \\
\text{s. t.} & -\frac{1}{2} \le z_1 \le 1, \\
& -\frac{1}{2} \le -\frac{1}{2} z_1 + \frac{\sqrt{3}}{2} z_2 \le 1, \\
& -\frac{1}{2} \le -\frac{1}{2} z_1 - \frac{\sqrt{3}}{2} z_2 \le 1, \\
& z_1^2 + z_2^2 \le 1.
\end{array}
$$

In the above,

$$
\begin{aligned}
g(\theta, z_1, z_2) \quad := \quad & \frac{1}{1+2z_1}\left\{1 + \frac{9}{4\pi^2}\left[\arccos^2(-\theta z_1) - \frac{1}{2}\arccos^2\left(\frac{1}{2}\theta z_1 - \frac{\sqrt{3}}{2}\theta z_2\right)\right.\right. \\
& \left.\left. -\frac{1}{2}\arccos^2\left(\frac{1}{2}\theta z_1 + \frac{\sqrt{3}}{2}\theta z_2\right)\right]\right\}.
\end{aligned}
$$

$\beta(\theta)$

Note that $\beta(\theta) = \frac{|S_1||S_2|+|S_2||S_3|+|S_1||S_3|}{n^2/3}$.

By Ling(COCOCA'09), we can obtain that

Lemma

- $f(x) := \frac{9}{8\pi^2}\left(\arccos^2(-x) - \arccos^2\left(\frac{1}{2}x\right)\right)$.
- $c(\theta) := \min\limits_{-\frac{1}{2}\leq x\leq 1} \frac{f(\theta)-f(\theta x)}{1-x}$.
- $\beta(\theta) := \left(1-\frac{1}{n}\right)\left(1-f(\theta)+c(\theta)\right)$.

$r(x)$

Note $\frac{W(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3)}{W^*} \geq r(x)\frac{W(S_1,S_2,S_3)}{W^*}$.

By the swapping process, we need to consider two cases of $|S_2| \geq \frac{n}{3}$ and $|S_2| < \frac{n}{3}$.

Lemma

When $|S_1| \geq |S_2| \geq \frac{n}{3} \geq |S_3|$, *we have*

$$\frac{W\left(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3\right)}{W(S_1, S_2, S_3)} \geq \frac{1}{81x_1^2x_2^2},$$

where $x = (x_1, x_2, x_3) = \left(\frac{|S_1|}{n}, \frac{|S_2|}{n}, \frac{|S_3|}{n}\right)$.

$r(x)$

Note $\frac{W(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3)}{W^*} \geq r(x)\frac{W(S_1,S_2,S_3)}{W^*}$.

By the swapping process, we need to consider two cases of $|S_2| \geq \frac{n}{3}$ and $|S_2| < \frac{n}{3}$.

Lemma

When $|S_1| \geq \frac{n}{3} \geq |S_2| \geq |S_3|$, *we have*

$$\frac{W\left(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3\right)}{W(S_1, S_2, S_3)} \geq \frac{1}{9x_1^2},$$

where $x = (x_1, x_2, x_3) = \left(\frac{|S_1|}{n}, \frac{|S_2|}{n}, \frac{|S_3|}{n}\right)$.

$r(x)$

Then, we have

$$r(x) := \begin{cases} \dfrac{1}{81x_1^2x_2^2}, & \text{if } x_2 \geq \dfrac{1}{3}; \\ \dfrac{1}{9x_1^2}, & \text{if } x_2 < \dfrac{1}{3}. \end{cases}$$

Analysis

Note that $R(x;\theta,\gamma) := r(x)(\alpha(\theta)+\gamma\beta(\theta)-\gamma\mu(x))$.

- $$R_1(\theta,\gamma) := \frac{27}{256}\gamma^4\left(\frac{1+\sqrt{1-\frac{8(\alpha(\theta)+\gamma\beta(\theta))}{9\gamma}}}{\alpha(\theta)+\gamma\beta(\theta)}\right)^3\left(1-3\sqrt{1-\frac{8(\alpha(\theta)+\gamma\beta(\theta))}{9\gamma}}\right).$$

- $$R_2(\theta,\gamma) := \gamma\frac{\alpha(\theta)+\gamma\beta(\theta)-\gamma}{4(\alpha(\theta)+\gamma\beta(\theta))-3\gamma}.$$

Setting $\theta := 0.3115$ and $\gamma = 12.1855$, then, we obtain the approximation ratio is 0.3456. In this case, $\alpha(\theta) = 0.4521$ and $\beta(\theta) = 0.9952$.

Future work

- A tighter bound for $r(x)$.
- Complex semidefinite hierarchies relaxation and its corresponding rounding technique.

Thank you!