ERRATUM ON "GROWTH TIGHTNESS FOR GROUPS WITH CONTRACTING ELEMENTS"

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Let's first recall some defintions from [1]. Assume that a group G acts properly on a geodesic metric space (Y, d). A subset $X \subset Y$ has *contracting* property if any geodesic far from X has an uniform bounded projection to X. An infinite subgroup H in G is called *contracting* if for some (hence any) $o \in Y$, the subset Ho is contracting in Y. Moreover, H is called *strongly contracting* if for some (hence any) $o \in Y$, the collection $\{gHo : g \in G\}$ is a contracting system with bounded intersection in Y.

An element $h \in G$ is called (resp. *strongly*) *contracting* if the subgroup $\langle h \rangle$ is (resp. strongly) contracting. The following theorem is the main result in [1].

Theorem 0.1. *If a non-elementary group G admits a geometric action on a metric space (Y, d) with a contracting element, then G is growth tight.*

If h is a contracting element such that the orbital map $n \mapsto h^n o$ is a quasi-isometric embedding map, then h is called a *quasi-isometrically embedded* contracting element. This is the definition of a contracting element used in [2].

The first objective of this Erratum is to explain some mistakes in [1] which leads that the above theorem was proved under the existence **quasi-isometrically embedded** contracting element. The second obejctive is to give a proof with technics therein of the following result.

Lemma 0.2. *Suppose G acts properly on a proper geodesic metric space (Y, d) with a contracting element. Then G contains a quasi-isometrically embeded contracting element.*

As a consequence, this proves Theorem 0.1 as it is without the quasi-isometrical embeding assumption.

1. Corrections

1.1. **Quasi-geodesicity of admissibe paths.** The admissible path γ in [1, Corollary 3.2] was originally claimed to be a $(\Lambda, 0)$-quasi-geodesic, i.e. a bi-Lipschitz path. This is certainly wrong when the concatenated admissible path is not simple. However the quasi-geodesicity does follow from Proposition 3.1 there, which says that the endpoints of each p_i stay uniformly close to the geodesic with same endpoints of γ. Thus, [1, Corollary 3.2] should be corrected as follows.

Corollary 1.1. *There are constants $D = D(\lambda, c, \nu, \tau) > 0, \Lambda = \Lambda(\lambda, c, \nu, \tau) > 1$ such that given $D_0 > D$ any $(D_0, \lambda, c, \nu, \tau)$-admissible path is a (Λ, Λ)-quasigeodesic.*

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1.2. **Corrections with quasi-isometrically embedded contracting elements.** Let H be a contracting subgroup. On page 308 in Section 4, the following set

$$E(H) := \{g \in G : \exists r > 0, diam(N_r(gHo) \cap N_r(Ho)) = \infty\}$$

was claimed to be a group. This may not be true but we could see that $E(H)$ is a finite union of double H-cosets.

Lemma 1.2. *The set $E(H)$ is a finite union of double H-cosets.*

Proof. Denote by $C > 0$ the contraction and quasiconvexity constants of Ho. We claim that if $diam(N_r(gHo) \cap N_r(Ho)) = \infty$ for some $r > 0$, then $diam(N_C(gHo) \cap N_C(Ho)) = \infty$. Indeed, choose a geodesic γ of length $\ell(\gamma) > 2r + C$ with two endpoints in $N_r(gHo) \cap N_r(Ho)$. By the contracting property, γ has to intersect $N_C(Ho)$: if not, we have $\Pi_{Ho}(\gamma) \le C$ and thus $\ell(\gamma) \le \Pi_{Ho}(\gamma) + d(\gamma_-, Ho) + d(\gamma_+, Ho) \le C + 2r$, giving a contradiction. The claim follows.

If $N_C(gHo) \cap N_C(Ho) \ne \emptyset$, there exist $h_1, h_2 \in H$ such that $h_1 g h_2 \in B(o, 2C)$. The proper action implies the finiteness of $B(o, 2C)$, and consequently, $E(H)$ is a finite union of double H-cosets. □

Accordingly, the statements of Lemmas 4.2 and 4.3 with quasi-isometrically embedding assumption will become true:

Lemma 1.3. [2, Lemma 2.11] *Let h be a quasi-isometrically embedded contracting element. Then $E(h) = E(\langle h \rangle)$ is a group and contains $\langle h \rangle$ as a finite index subgroup. Moreover, $E(h)$ is strongly contracting.*

Proof. Since $\langle h \rangle o$ is a quasiconvex by the contracting property and in this case, it is a quasi-geodesic, we have that an infinite intersection of $N_r(g\langle h \rangle o)$ and $N_r(\langle h \rangle o)$ implies a finite Hausdorff distance of $g\langle h \rangle o$ and $\langle h \rangle o$. So $E(h)$ is a group. □

The sentence "a contracting subgroup always produces a strongly contracting subgroup" on page 308 is confusing. First, a contracting subgroup might not be contained in a strongly contracting one. For instance, a finite index subgroup is contracting, but never strongly contracting. A less trivial example can be given in free group generated by a, b. A subgroup $H = \langle b, a^{-1}ba \rangle$ has the property that $\{b^n a : n \in \mathbb{Z}\} \subset aH \cap Ha \subset aH \cap N_1(H)$, where $N_1(H)$ denotes the 1-neighborhood of H.

If the subgroup H is replaced with $\langle h \rangle$ for a quasi-isometrically embedded contracting element, then the proof of Lemma 4.4, Corollary 4.5, Lemma 4.6 in Section 4 remains true. Then Theorem 0.1 is proved by the same argments, provided that a quasi-isometrically embedded contracting element exists.

In the Lemma 1 of [1, appendix], the strongly contracting assumption is also necessary.

Lemma 1.4. *Suppose G acts properly on a proper geodesic metric space (Y, d) with a strongly contracting subgroup H. Then H is a hyperbolically embedded subgroup.*

2. Existence of quasi-isometrically embedded contracting elements

We now explain how to prove Lemma 0.2. The following statement corrects [1, Lemma 4.4] with "$F \subset E(H)$"there replaced by "$F \subset H$" below.

Lemma 2.1. *Suppose G acts properly on (Y, d) with a contracting subgroup H. For any $k \in G \setminus E(H)$ and $o \in Y$, there exists $D = D(k, o) > 0$ with the following property. Choose a finite set $F \subset H$ such that $d(o, fo) > D$ for all $f \in F$. Then for any $W \in \mathcal{W}(F, k)$, the set $o(W)$ is contracting with contracting constant depending on F.*

Proof. The first sentence of proof of Lemma 4.4 is false: "Since $E(H)$ is strongly contracting, $\mathbb{X} = \{gE(H)o : g \in G\}$ is (μ, ϵ)-contracting with ν-bounded intersection for some fixed μ, ϵ, ν." We now work with the constracting system $\mathbb{X} = \{gHo : g \in G\}$, which may not necessarily have bounded intersection. For given $k \notin E(H)$, we have $diam(N_r(kHo) \cap N_r(Ho)) < \infty$ and thus $N_r(kHo)$ and $N_r(Ho)$ have bounded projection by τ for a constant $\tau > 0$. In particular, $[1, k]$ has τ-bounded projection to both kHo and Ho.

Assume that $W = h_0kh_1k \cdots h_nk \in \mathcal{W}(F, k)$ for $n > 0, h_i \in F$. Consider the normal path $\gamma = p_0q_1p_1 \ldots p_{n-1}q_n$, where p_i are geodesics corresponding to h_i and q_i geodesics corresponding to k. Each p_i has the two endpoints in $X_i = h_0kh_1k \cdots h_{i-1}kH \in \mathbb{X}$. We can always translate X_i, q_i, X_{i+1} to a standard position Ho, k, kHo, so that q_i and q_{i+1} have τ-bounded projection to X_i. Hence, γ is a $(D, 1, 0, \nu, \tau)$ admissible path.

The remaining proof runs exactly as that of Lemma 4.4. □

With the assumption $h \in E(H)$ replaced with $h \in H$, the following corollary is the same as [1, Corollay 4.5] with the stronger conclusion of quasi-isometrically contracting elements. This then proves Lemma 0.2.

Corollary 2.2. *Suppose G acts properly on (Y, d) with a contracting subgroup H. Consider $k \in G \setminus E(H)$ and $o \in Y$. Then there exists $D = D(k, o) > 0$ such that for any $h \in H$ with $d(o, ho) > D$ the element hk is quasi-isometrically contracting in G.*

Similarly, the proof of [1, Lemma 4.6] using Corollary 2.2 can show the following.

Lemma 2.3. *Suppose a non-elementary group G acts properly on (X, d) with a contracting element. Let Γ be an infinite normal subgroup. Then Γ contains infinitely many quasi-isometrically contracting elements.*

Therefore, with the above corrections, Theorem 0.1 is true with existence of a contracting element.

References

1. W. Yang, *Growth tightness for groups with contracting elements*, Math. Proc. Cambridge Philos. Soc **157** (2014), 297 – 319.
2. ______, *Statistically convex-cocompact actions of groups with contracting elements*, International Mathematics Research Notices, No. 23, 2019, 7259–7323.

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