教学提纲

课程介绍

几何群论是通过构造在几何空间上的作用来研究无限的离散群的研究领域，其中具有负曲率特征的空间和群的研究占有特别重要的位置。本课程主要学习双曲群与相对双曲群等具有非正曲率特征群的基本理论，及在低维拓扑等领域中的应用。

第一章：双曲群 - Hyperbolic groups

内容框架

A. Definitions:
Thin-triangle property
Four points inequality
Isoperimetric inequality
Uniform convergence action

B. Local to global principle
Finite tree approximations
Local geodesics are global

C. Automatic structures
Growth series
Geodesic regular languages

D. Gromov and Busemann boundary:
visual metrics
quasi-isometry induces homeomorphisms
Cannon conjectures

E. Prototype examples:
free groups
one relator groups
small cancellation groups
negatively curved manifold

F. Quasi-convex subgroups

Finite subgroups
Virtually cyclic subgroups
Free subgroups by Ping-Pong
General theory

G. Solving word, conjugacy, and isomorphism problems

Word problem: Dehn Algorithm
Conjugacy problem

参考资料

1. 本课程讲义：http://bicmr.pku.edu.cn/~wyang/ggt/GGTnotes.pdf （主要给出上课内容的证明，没有太多引导，仅供参考。建议阅读下面的书籍了解背景知识）

双曲群的简洁介绍：

2. Bowditch, **A course on geometric group theory**, Mathematical Society of Japan, Tokyo, 2006

非常全面的介绍：

3. M. Bridson and A. Haefliger, **Metric spaces of non-positive curvature**, vol. 319, Grundlehren der Mathematischen Wissenschaften, SpringerVerlag, Berlin, 1999.

4. Cornelia Druţu and Michael Kapovich.**Geometric group theory**, volume 63 ofAmerican MathematicalSociety Colloquium Publications. American Mathematical Society, Providence, RI, 2018

组合群论介绍：

5. CF Miller III, **Combinatorial group theory**, http://www.ms.unimelb.edu.au/˜cfm/notes/cgt-notes.pdf.

面向本科生读物，介绍了许多具体的群（尽管没有介绍双曲群）：

6. J.Meier, **Groups, graphs and trees: An introduction to the geometry of infinite groups**, Cambridge University Press, Cambridge, 2008.

Automatic的介绍：

7. B. Farb, **Automatic groups: A guided tour**. L'Enseignement Math., Vol. 38 (1992), 291-313. http://www.math.uchicago.edu/~farb/papers/auto.ps

8.（原始文献）Epstein， Cannon，Holt，Levy, Paterson, Thurston: **Word processing in groups**. Jones and Bartlett Publishers, Boston, MA, 1992. xii+330 pp. ISBN: 0-86720-244-0

收敛群作用：

9. Bowditch， **Convergence groups and configuration spaces**. Geometric group theory down under (Canberra, 1996), 23–54, de Gruyter, Berlin, 1999.

相关论文

1. 双曲群原始论文：

M. Gromov, **Hyperbolic groups**, in “Essays in Group Theory” (G. M. Gersten, ed.), MSRI Publ. 8, 1987, pp. 75-263

2. 解释Gromov上述文章的三篇（早期）文献：

H. Short (editor), **Notes on hyperbolic groups**, Group theory from a geometrical viewpoint, World Scientific Publishing Co., Inc., 1991. https://www.i2m.univ-amu.fr/~short/Papers/MSRInotes2004.pdf

E. Ghys and P. de la Harpe (editors), **Sur les groupes hyperboliques d'aprs Mikhael Gromov.** Progress in Mathematics, 83. Birkhuser Boston, Inc., Boston, MA, 1990. xii+285 pp. ISBN 0-8176-3508-4

M. Coornaert, T. Delzant and A. Papadopoulos, **Geometrie et theorie des groupes: les groupes hyperboliques de Gromov**, Lecture Notes in Mathematics, vol. 1441, Springer-Verlag, Berlin, 1990, x+165 pp.

3. 双曲群边界理论的非常全面的综述（五星推荐）：
I. Kapovich and N. Benakli, **Boundaries of hyperbolic groups**, Combinatorial and geometric group theory (New York, 2000/Hoboken, NJ, 2001), Contemp. Math., vol. 296, Amer. Math. Soc., Providence, RI, 2002, pp. 39–93.

4. 双曲群的组合定理：M. Bestvina and M. Feighn, **A combination theorem for negatively curved groups,** J. Differential Geom. 35 (1992), no. 1, 85–101.

5. 双曲群的动力学刻画和spliting（边界拓扑的重要应用）：

B. Bowditch: **Cut points and canonical splittings of hyperbolic groups**, Acta Math. 180 (1998), no. 2, 145–186.
B. Bowditch: **A topological characterisation of hyperbolic groups**, J. Amer. Math. Soc., 11, no. 3, (1998) 643–667

6. 双曲群边界的Patterson-Sullivan测度：

M. Coornaert. **Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov.** *Pacific J. Math.* **159**(2) (1993), 241–270.

（原始文献）S. Patterson. **The limit set of a Fuchsian group**. *Acta Math.* **136**(1) (1976), 241–273.

（原始文献）D. Sullivan. **The density at infinity of a discrete group of hyperbolic motions**. *Publ. Math. Inst. Hautes Études Sci.* **50**(1979), 171–202.

7. 收敛群作用：

P. Tukia. **Conical limit points and uniform convergence groups**. *J. Reine Angew. Math.* **501** (1998), 71–98.

（原始文献）F. W. Gehring and G. J. Martin, **Discrete quasiconformal groups I,** Proc. London Math. Soc. (3) **55** (1987), 331–358.

P. Tukia, **Convergence groups and Gromov's metric hyperbolic spaces**, New Zealand J. Math. **23** (1994), 157–187.

第二章：Bass-Serre 理论

结构框架

A. Groups acting on trees:

Amalgamated product and HNN extension

Graph of groups

Property (FA)

B. Groups with infinitely ends:

Stallings ends theorem
Dunwoody's accessibility theorem

C. Examples:

examples of Caylay graphs
JSJ decompositions
graph manifolds

参考资料

Scott and Wall: **Topological methods in group theory**: http://bicmr.pku.edu.cn/~wyang/ggt/scott_wall.pdf 偏重拓扑的角度，非常直观和富有启发性

J.-P Serre, **Trees**, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation 偏重代数的方式，清晰明快。

课程讲义：讲义.

相关论文

第三章：相对双曲群 - Relatively Hyperbolic Groups

内容框架

A. Definitions

1. Gromov-Bowditch Definition
2. Farb Definition
3. Osin Definition
4. Dynamic approach:
   Bowditch, Yaman
   Gerasimov
5. Intrinsic characterization:
   relative triangle property (Drutu - Sapir)
   Transition points (Hruska)
   Floyd metrics (Gerasimov - Potyagailo)

B. Boundary Theory

1. Bowditch, Floyd and ends boundaries
2. Relations between boundaries and peripheral structures
3. Metric measure structures

C. Prototype examples

1. Free products
2. Infinitely ended groups
3. hyperbolic 3-manifolds with finite volume
4. mixed 3-manifolds.

D. Relatively quasi-convex subgroups

1. Maximal parabolic subgroups
2. Fully quasi-convex subgroups

E. Applications:

1. Group theoretical Dehn filling
2. Small cancellation theory

参考论文

比较全面的综述（推荐）：

G. C. Hruska, Relative hyperbolicity and relative quasiconvexity for countable groups, Algebr. Geom. Topol. 10 (2010), no. 3, 1807–1856

Farb定义：

B. Farb, Relatively hyperbolic groups, Geom. Funct. Anal. 8 (1998) 810 – 840

Bowditch的定义：

B. Bowditch, Relatively hyperbolic groups, Internat. J. Algebra Comput. 22 (2012), no. 3, 1250016, 66

Osin的定义：

Denis Osin. Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algo rithmic problems. Memoirs of the American Mathematical Society, 20, 2006

Dynamical定义：

B. Bowditch, Convergence groups and configuration spaces, in: Geometric group theory down under, Walter de Gruyter, Berlin (1999), 23–54.A.
Yaman, A topological characterisation of relatively hyperbolic groups, J. reine angew. Math. 566 (2004), 41–89.
P. Tukia, Conical limit points and uniform convergence groups, J. reine. angew. Math. 501 (1998), 71–98.

渐近锥定义和内蕴的定义

C. Drutu and M. Sapir, Tree-graded spaces and asymptotic cones of groups, Topology 44 (2005), no. 5, 959–1058.

Floyd边界

William Floyd, Group completions and limit sets of Kleinian groups. Inventiones Mathematicae, 57:205–2018, 1980
A. Karlsson, Free subgroups of groups with non-trivial floyd boundary, Comm. Algebra 31 (2003), 5361–5376.

Bowditch边界与Floyd边界的关系：

Victor Gerasimov. Expansive convergence groups are relatively hyperbolic. Geom. Funct. Anal., 19(1):137–169, 2009.
Victor Gerasimov. Floyd maps for relatively hyperbolic groups.Geometric and Functional Analysis, 22:1361–1399, 2012

Dehn filling和small cancellation groups（重要应用）

D. Osin, Small cancellations over relatively hyperbolic groups and embedding theorems, Ann. Math. 172 (2010), no. 1, 1-39.
D. Osin, Peripheral fillings of relatively hyperbolic groups, Invent. Math. 167 (2007), no. 2, 295-326.

进一步发展

相对双曲群的理论大部分结果被推广到更一般地称为Acylindrically hyperbolic groups上，使得可以用来研究曲面的映射类群。如下是几篇原始的相关论文：

D. Osin. Acylindrically hyperbolic groups.Trans. Amer. Math. Soc., 368(2):851–888, 2016

F. Dahmani, V. Guirardel, D. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces, Memoirs AMS 245 (2017), no. 1156.

Mladen Bestvina, Ken Bromberg, and Koji Fujiwara. Constructing group actions on quasi-trees andapplications to mapping class groups.Publ. Math. Inst. Hautes ´Etudes Sci., 122:1–64, 2015.