Lecture: Fast Proximal Gradient Methods

http://bicmr.pku.edu.cn/~wenzw/opt-2018-fall.html

Acknowledgement: this slides is based on Prof. Lieven Vandenberghe's lecture notes

Outline

1. fast proximal gradient method (FISTA)
2. FISTA with line search
3. FISTA as descent method
4. Nesterov's second method
5. Proof by estimating sequence

Fast (proximal) gradient methods

- Nesterov (1983, 1988, 2005): three projection methods with $1/k^2$ convergence rate
- Beck & Teboulle (2008): FISTA, a proximal gradient version of Nesterov's 1983 method
- Nesterov (2004 book), Tseng (2008): overview and unified analysis of fast gradient methods
- several recent variations and extensions

this lecture

FISTA and Nesterov's 2nd method (1988) as presented by Tseng

FISTA (basic version)

$$\text{minimize} \quad f(x) = g(x) + h(x)$$

- g convex, differentiable, with $\textbf{dom}\, g = \mathbb{R}^n$
- h closed, convex, with inexpensive prox_{th} oprator

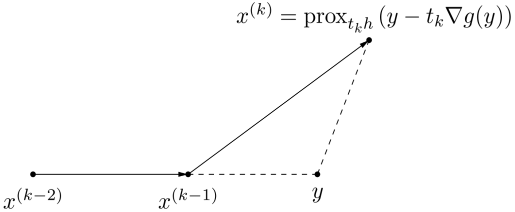
algorithm: choose any $x^{(0)} = x^{(-1)}$; for $k \geq 1$, repeat the steps

$$y = x^{(k-1)} + \frac{k-2}{k+1}(x^{(k-1)} - x^{(k-2)})$$
$$x^{(k)} = \text{prox}_{t_k h}(y - t_k \nabla g(y))$$

- step size t_k fixed or determined by line search
- acronym stands for 'Fast Iterative Shrinkage-Thresholding Algorithm'

Interpretation

- first iteration ($k = 1$) is a proximal gradient step at $y = x^{(0)}$
- next iterations are proximal gradient steps at extrapolated points y

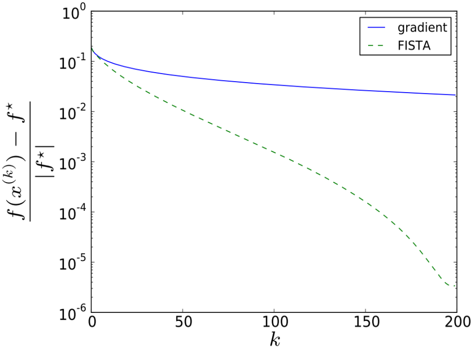


note: $x^{(k)}$ is feasible (in $\mathbf{dom}\ h$); y may be outside $\mathbf{dom}\ h$

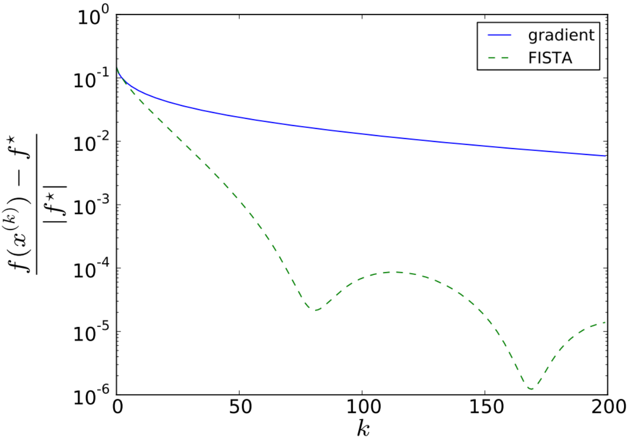
Example

$$\text{minmize} \quad \log \sum_{i=1}^{m} \exp(a_i^T x + b_i)$$

randomly generated data with $m = 2000$, $n = 1000$, same fixed step size



another instance



FISTA is not a descent method

Convergence of FISTA

assumptions

- g convex with $\mathbf{dom}\, g = \mathbb{R}^n$; ∇g Lipschitz continuous with constant L:

$$\|\nabla g(x) - \nabla g(y)\|_2 \leq L\|x - y\|_2 \qquad \forall x, y$$

- h is closed and convex (so that $\mathrm{prox}_{th}(u)$ is well defined)
- optimal value f^* is finite and attained at x^* (not necessarily unique)

convergence result: $f(x^{(k)}) - f^*$ decreases at least as fast as $1/k^2$

- with fixed step size $t_k = 1/L$
- with suitable line search

Reformulation of FISTA

define $\theta_k = 2/(k+1)$ and introduce an intermediate variable $v^{(k)}$

algorithm: choose $x^{(0)} = v^{(0)}$; for $k \geq 1$, repeat the steps

$$
\begin{aligned}
y &= (1-\theta_k)x^{(k-1)} + \theta_k v^{(k-1)} \\
x^{(k)} &= \mathrm{prox}_{t_k h}(y - t_k \nabla g(y)) \\
v^{(k)} &= x^{(k-1)} + \frac{1}{\theta_k}(x^{(k)} - x^{(k-1)})
\end{aligned}
$$

substituting expression for $v^{(k)}$ in formula for y gives FISTA of page 4

Important inequalities

choice of θ_k: the sequence $\theta_k = 2/(k+1)$ satisfies $\theta_1 = 1$ and

$$\frac{1-\theta_k}{\theta_k^2} \leq \frac{1}{\theta_{k-1}^2}, \qquad k \geq 2$$

upper bound on g from Lipschitz property

$$g(u) \leq g(z) + \nabla g(z)^T(u-z) + \frac{L}{2}\|u-z\|_2^2 \qquad \forall u, z$$

upper bound on h from definition of prox-operator

$$h(u) \leq h(z) + \frac{1}{t}(w-u)^T(u-z) \qquad \forall w,\ u = \mathrm{prox}_{th}(w),\ z$$

Note $\min_u th(u) + \frac{1}{2}\|u-w\|_2^2$ gives $0 \in t\partial h(u) + (u-w)$ gives $0 \in t\partial h(u) + (u-w)$. Hence, $\frac{1}{t}(w-u) \in \partial h(u)$.

Progress in one iteration

define $x = x^{(i-1)}, x^+ = x^{(i)}, v = v^{(i-1)}, v^+ = v^{(i)}, t = t_i, \theta = \theta_i$

- upper bound from Lipschitz property: if $0 < t \leq 1/L$

$$g(x^+) \leq g(y) + \nabla g(y)^T (x^+ - y) + \frac{1}{2t} \|x^+ - y\|_2^2 \tag{1}$$

- upper bound from definition of prox-operator:

$$h(x^+) \leq h(z) + \nabla g(y)^T (z - x^+) + \frac{1}{t} (x^+ - y)^T (z - x^+) \quad \forall z$$

- add the upper bounds and use convexity of g

$$f(x^+) \leq f(z) + \frac{1}{t} (x^+ - y)^T (z - x^+) + \frac{1}{2t} \|x^+ - y\|_2^2 \quad \forall z$$

- make convex combination of upper bounds for $z = x$ and $z = x^*$

$$\begin{aligned}
&f(x^+) - f^* - (1-\theta)(f(x) - f^*) \\
&\quad = f(x^+) - \theta f^* - (1-\theta) f(x) \\
&\quad \le \frac{1}{t}(x^+ - y)^T(\theta x^* + (1-\theta)x - x^+) + \frac{1}{2t}\|x^+ - y\|_2^2 \\
&\quad = \frac{1}{2t}\left(\|y - (1-\theta)x - \theta x^*\|_2^2 - \|x^+ - (1-\theta)x - \theta x^*\|_2^2\right) \\
&\quad = \frac{\theta^2}{2t}\left(\|v - x^*\|_2^2 - \|v^+ - x^*\|_2^2\right)
\end{aligned}$$

conclusion: if the inequality (1) holds at iteration i, then

$$\begin{aligned}
&\frac{t_i}{\theta_i^2}\left(f(x^{(i)}) - f^*\right) + \frac{1}{2}\|v^{(i)} - x^*\|_2^2 \\
&\quad \le \frac{(1-\theta_i)t_i}{\theta_i^2}\left(f(x^{(i-1)}) - f^*\right) + \frac{1}{2}\|v^{(i-1)} - x^*\|_2^2
\end{aligned} \tag{2}$$

Analysis for fixed step size

take $t_i = t = 1/L$ and apply (2) recursively, using $(1-\theta_i)/\theta_i^2 \leq 1/\theta_{i-1}^2$;

$$\begin{aligned}
&\frac{t}{\theta_k^2}\left(f(x^{(k)}) - f^*\right) + \frac{1}{2}\|v^{(k)} - x^*\|_2^2 \\
&\quad \leq \frac{(1-\theta_1)t}{\theta_1^2}\left(f(x^{(0)}) - f^*\right) + \frac{1}{2}\|v^{(0)} - x^*\|_2^2 \\
&\quad = \frac{1}{2}\|x^{(0)} - x^*\|_2^2
\end{aligned}$$

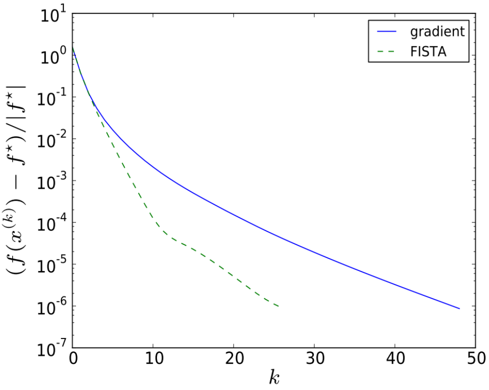
therefore

$$f(x^{(k)}) - f^* \leq \frac{\theta_k^2}{2t}\|x^{(0)} - x^*\|_2^2 = \frac{2L}{(k+1)^2}\|x^{(0)} - x^*\|_2^2$$

conclusion: reaches $f(x^{(k)}) - f^* \leq \epsilon$ after $\mathcal{O}(1/\sqrt{\epsilon})$ iterations

Example: quadratic program with box constraints

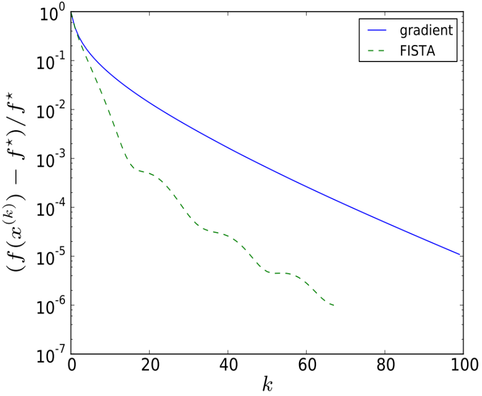
$$\begin{array}{ll} \text{minimize} & (1/2)x^T A x + b^T x \\ \text{subject to} & 0 \preceq x \preceq \mathbf{1} \end{array}$$



$n = 3000$; fixed step size $t = 1/\lambda_{\max}(A)$

1-norm regularized least-squares

$$\text{minimize} \quad \frac{1}{2}\|Ax - b\|_2^2 + \|x\|_1$$



randomly generated $A \in \mathbb{R}^{2000\times 1000}$; step $t_k = 1/L$ with $L = \lambda_{\max}(A^TA)$

Outline

Key steps in the analysis of FISTA

- the starting point (page 11) is the inequality

$$g(x^+) \le g(y) + \nabla g(y)^T(x^+ - y) + \frac{1}{2t}\|x^+ - y\|_2^2 \qquad (1)$$

this inequality is known to hold for $0 < t \le 1/L$

- if (1) holds, then the progress made in iteration i is bounded by

$$\begin{aligned} &\frac{t_i}{\theta_i^2}\left(f(x^{(i)}) - f^*\right) + \frac{1}{2}\|v^{(i)} - x^*\|_2^2 \\ &\quad \le \frac{(1-\theta_i)t_i}{\theta_i^2}\left(f(x^{(i-1)} - f*\right) + \frac{1}{2}\|v^{(i-1)}) - x^*\|_2^2 \end{aligned} \qquad (2)$$

- to combine these inequalities recursively, we need

$$\frac{(1-\theta_i)t_i}{\theta_i^2} \le \frac{t_{i-1}}{\theta_{i-1}^2} \qquad (i \ge 2) \qquad (3)$$

- if $\theta_1 = 1$, combing the inequalities (2) from $i = 1$ to k gives the bound

$$f(x^{(k)}) - f^* \leq \frac{\theta_k^2}{2t_k} \|x^{(0)} - x^*\|_2^2$$

conclusion: rate $1/k^2$ convergence if (1) and (3) hold with

$$\frac{\theta_k^2}{t_k} = \mathcal{O}(\frac{1}{k^2})$$

FISTA with fixed step size

$$t_k = \frac{1}{L}, \qquad \theta_k = \frac{2}{k+1}$$

these values satisfies (1) and (3) with

$$\frac{\theta_k^2}{t_k} = \frac{4L}{(k+1)^2}$$

FISTA with line search (method 1)

replace update of x in iteration k (page 9) with

$$
\begin{aligned}
& t := t_{k-1} \qquad (\text{define } t_0 = \hat{t} > 0) \\
& x := \mathrm{prox}_{th}(y - t\nabla g(y)) \\
& \text{while } g(x) > g(y) + \nabla g(y)^T(x-y) + \frac{1}{2t}\|x-y\|_2^2 \\
& \qquad t := \beta t \\
& \qquad x := \mathrm{prox}_{th}(y - t\nabla g(y)) \\
& \text{end}
\end{aligned}
$$

- inequality (1) holds trivially, by the backtracking exit condition
- inequality (3) holds with $\theta_k = 2/(k+1)$ because $t_k \le t_{k-1}$
- Lipschitz continuity of ∇g guarantees $t_k \ge t_{\min} = \min\{\hat{t}, \beta/L\}$
- preserves $1/k^2$ convergence rate because $\theta_k^2/t_k = \mathcal{O}(1/k^2)$:

$$
\frac{\theta_k^2}{t_k} \le \frac{4}{(k+1)^2 t_{\min}}
$$

FISTA with line search (method 2)

replace update of y and x in iteration k (page 9) with

$$
\begin{aligned}
& t := \hat{t} > 0 \\
& \theta := \text{positive root of } t_{k-1}\theta^2 = t\theta_{k-1}^2(1-\theta) \\
& y := (1-\theta)x^{(k-1)} + \theta v^{(k-1)} \\
& x := \text{prox}_{th}(y - t\nabla g(y)) \\
& \text{while } g(x) > g(y) + \nabla g(y)^T(x-y) + \frac{1}{2t}\|x-y\|_2^2 \\
& \qquad t := \beta t \\
& \qquad \theta := \text{positive root of } t_{k-1}\theta^2 = t\theta_{k-1}^2(1-\theta) \\
& \qquad y := (1-\theta)x^{(k-1)} + \theta v^{(k-1)} \\
& \qquad x := \text{prox}_{th}(y - t\nabla g(y)) \\
& \text{end}
\end{aligned}
$$

assume $t_0 = 0$ in the first iteration $(k = 1)$, *i.e.*, take $\theta_1 = 1, y = x^{(0)}$

discussion

- inequality (1) holds trivially, by the backtracking exit condition
- inequality (3) holds trivially, bu construction of θ_k
- Lipschitz contimuity of ∇g guarantees $t_k \geq t_{\min} = \min\{\hat{t}, \beta/L\}$
- θ_i is defined as the positive root of $\theta_i^2/t_i = (1-\theta_i)\theta_{i-1}^2/t_{i-1}$; hence

$$\frac{\sqrt{t_{i-1}}}{\theta_{i-1}} = \frac{\sqrt{(1-\theta_i)t_i}}{\theta_i} \leq \frac{\sqrt{t_i}}{\theta_i} - \frac{\sqrt{t_i}}{2}$$

  combine inequalities from $i = 2$ to k to get $\sqrt{t_i} \leq \frac{\sqrt{t_k}}{\theta_k} - \frac{1}{2}\sum_{i=2}^{k}\sqrt{t_i}$
- rearranging shows that $\theta_k^2/t_k = \mathcal{O}(1/k^2)$:

$$\frac{\theta_k^2}{t_k} \leq \frac{1}{(\sqrt{t_1} + \frac{1}{2}\sum_{i=2}^{k}\sqrt{t_i})^2} \leq \frac{4}{(k+1)^2 t_{\min}}$$

Comparison of line search methods

method 1

- uses nonincreasing stepsizes (enforces $t_k \leq t_{k-1}$)
- one evaluation of $g(x)$, one prox_{th} evaluation per line search iteration

method 2

- allows non-monotonic step sizes
- one evaluation of $g(x)$, one evaluation of $g(y)$, $\nabla g(y)$, one evaluation of prox_{th} per line search iteration

the two strategies cann be combined and extended in various ways

Outline

Descent version of FISTA

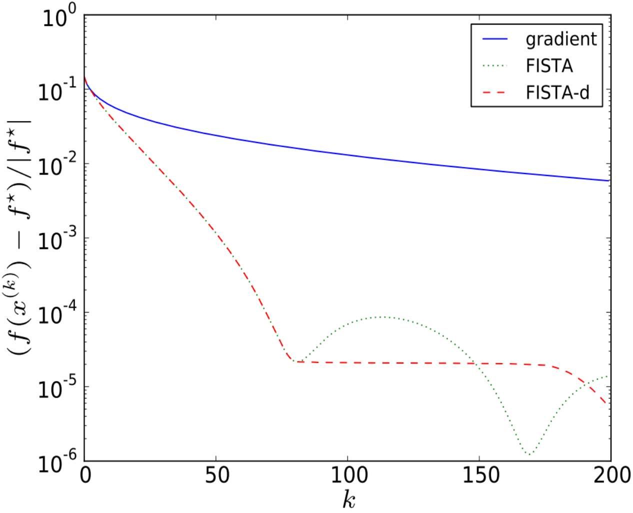
choose $x^{(0)} = v^{(0)}$; for $k \geq 1$, repeat the steps

$$
\begin{aligned}
y &= (1-\theta_k)x^{(k-1)} + \theta_k v^{(k-1)} \\
u &= \mathrm{prox}_{t_k h}(y - t_k \nabla g(y)) \\
x^{(k)} &= \begin{cases} u & f(u) \leq f(x^{(k-1)}) \\ x^{(k-1)} & \text{otherwise} \end{cases} \\
v^{(k)} &= x^{(k-1)} + \frac{1}{\theta_k}(u - x^{(k-1)})
\end{aligned}
$$

- step 3 implies $f(x^{(k)}) \leq f(x^{(k-1)})$
- use $\theta_k = 2/(k+1)$ and $t_k = 1/L$, or one of the line search methods
- same iteration complexity as original FISTA
- changes on page 11: replace x^+ with u and use $f(x^+) \leq f(u)$

Example

(from page 7)



Outline

Nesterov's second method

algorithm: choose $x^{(0)} = v^{(0)}$; for $k \geq 1$, repeat the steps

$$\begin{aligned} y &= (1-\theta_k)x^{(k-1)} + \theta_k v^{(k-1)} \\ v^{(k)} &= \mathrm{prox}_{(t_k/\theta_k)h}\left(v^{(k-1)} - \frac{t_k}{\theta_k}\nabla g(y)\right) \\ x^{(k)} &= (1-\theta_k)x^{(k-1)} + \theta_k v^{(k)} \end{aligned}$$

- use $\theta_k = 2/(k+1)$ and $t_k = 1/L$, or one of the line search methods
- identical to FISTA if $h(x) = 0$
- unlike in FISTA, y is feasible (in $\mathbf{dom}\ h$) if we take $x^{(0)} \in \mathbf{dom}\ h$

Convergence of Nesterov's second method

assumptions

- g convex; ∇g is Lipschitz continuous on $\mathbf{dom}\ h \subseteq \mathbf{dom}\ g$

$$\nabla g(x) - \nabla g(y)\|_2 \leq L\|x - y\|_2 \qquad \forall x, y \in \mathbf{dom}\ h$$

- h is closed and convex (so that $\mathrm{prox}_{th}(u)$ is well defined)
- optimal value f^* is finite and attained at x^* (not necessarily unique)

convergence result: $f(x^{(k)}) - f^*$ decrease at least as fast as $1/k^2$

- with fixed step size $t_k = 1/L$
- with suitable line search

Analysis of one iteration

define $x = x^{(i-1)}, x^+ = x^{(i)}, v = v^{(i-1)}, v^+ = v^{(i)}, t = t_i, \theta = \theta_i$

- from Lipschitz property if $0 < t \leq 1/L$

$$g(x^+) \leq g(y) + \nabla g(y)^T (x^+ - y) + \frac{1}{2t} \|x^+ - y\|_2^2$$

- plug in $x^+ = (1-\theta)x + \theta v^+$ and $x^+ - y = \theta(v^+ - v)$

$$g(x^+) \leq g(y) + \nabla g(y)^T ((1-\theta)x + \theta v^+ - y) + \frac{\theta^2}{2t} \|v^+ - v\|_2^2$$

- from convexity of g, h

$$\begin{aligned} g(x^+) &\leq (1-\theta) g(x) + \theta(g(y) + \nabla g(y)^T (v^+ - y)) + \frac{\theta^2}{2t} \|v^+ - v\|_2^2 \\ h(x^+) &\leq (1-\theta) h(x) + \theta h(v^+) \end{aligned}$$

- upper bound on h from page 10 (with $u = v^+,\ w = v - (t/\theta)\nabla(y)$)

$$h(v^+) \le h(z) + \nabla g(y)^T (z - v^+) - \frac{\theta}{t}(v^+ - v)^T (v^+ - z) \quad \forall z$$

- combine the upper bounds on $g(x^+), h(x^+), h(v^+)$ with $z = x^*$

$$\begin{aligned} f(x^+) \ &\le \ (1-\theta) f(x) + \theta f^* - \frac{\theta^2}{t}(v^+ - v)^T (v^+ - x^*) + \frac{\theta^2}{2t}\|v^+ - v\|_2^2 \\ &= \ (1-\theta) f(x) + \theta f^* + \frac{\theta^2}{2t}(\|v - x^*\|_2^2 - \|v^+ - x^*\|_2^2) \end{aligned}$$

this is identical to final inequality (2) in the analysis of FISTA on page 12

$$\begin{aligned} &\frac{t_i}{\theta_i^2}\left(f(x^{(i)}) - f^*\right) + \frac{1}{2}\|v^{(i)} - x^*\|_2^2 \\ &\quad \le \frac{(1-\theta_i)t_i}{\theta_i^2}\left(f(x^{(i-1)}) - f^*\right) + \frac{1}{2}\|v^{(i-1)} - x^*\|_2^2 \end{aligned}$$

References

surveys of fast gradient methods

- Yu. Nesterov, *Introductory Lectures on Convex Optimization. A Basic Course* (2004)
- P. Tseng, *On accelerated proximal gradient methods for convex-concave optimization* (2008)

FISTA

- A. Beck and M. Teboulle, *A fast iterative shrinkage-thresholding algorithm for linear inverse problems*, SIAM J. on Imaging Sciences (2009)
- A. Beck and M. Teboulle, *Gradient-based algorithms with applications to signal recovery*, in: Y. Eldar and D. Palomar (Eds.), *Convex Optimization in Signal Processing and Communications* (2009)

line search strategies

- FISTA papers by Beck and Teboulle
- D. Goldfarb and K. Scheinberg, *Fast first-order methods for composite convex optimization with line search* (2011)
- Yu. Nesterov, *Gradient methods for minimizing composite objective function* (2007)
- O. Güler, *New proximal point algorithms for convex minimization*, SIOPT (1992)

Nesterov's third method (not covered in this lecture)

- Yu. Nesterov, *Smooth minimization of non-smooth functions*, Mathematical Programming (2005)
- S. Becker, J. Bobin, E.J. Candès, *NESTA: a fast and accurate first-order method for sparse recovery*, SIAM J. Imaging Sciences (2011)

Outline

FOM Framework: $f^* = \min_x \{f(x),\ x \in X\}$

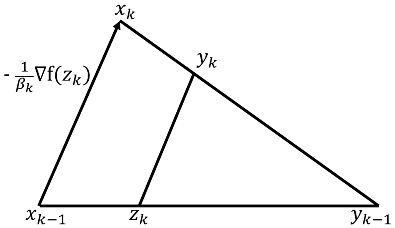
$f(x) \in C_L^{1,1}(X)$ convex. $X \subseteq \mathbb{R}^n$ closed convex. Find $\bar{x} \in X$: $f(\bar{x}) - f^* \leq \epsilon$

FOM Framework

Input: $x_0 = y_0$, choose $L\gamma_k \leq \beta_k$, $\gamma_1 = 1$. for $k = 1, 2, ..., N$ do

1. $z_k = (1 - \gamma_k)y_{k-1} + \gamma_k x_{k-1}$
2. $x_k = \text{argmin}_{x \in X} \left\{ \langle \nabla f(z_k), x \rangle + \frac{\beta_k}{2} \|x - x_{k-1}\|_2^2 \right\}$
3. $y_k = (1 - \gamma_k)y_{k-1} + \gamma_k x_k$

- Sequences: $\{x_k\}$, $\{y_k\}$, $\{z_k\}$. Parameters: $\{\gamma_k\}$, $\{\beta_k\}$.



Lemma 1.(Estimating sequence)

Let $\gamma_t \in (0,1]$, $t = 1, 2, \ldots$, denote $\Gamma_t = \begin{cases} 1 & t = 1 \\ (1-\gamma_t)\Gamma_{t-1} & t \geq 2 \end{cases}$. If the sequences $\{\Delta_t\}_{t\geq 0}$ satisfies $\Delta_t \leq (1-\gamma_t)\Delta_{t-1} + B_t \quad t = 1, 2, \ldots$, then we have $\Delta_k \leq \Gamma_k(1-\gamma_1)\Delta_0 + \Gamma_k \sum_{i=1}^{k} \frac{B_i}{\Gamma_i}$

Remark:

1. Let $\Delta_k = f(x_k) - f(x^*)$ or $\Delta_k = \|x_k - x^*\|_2^2$
2. Estimate $\{x_k\}$, let $\underbrace{f(x_k) - f(x^*)}_{\Delta_k} \leq (1-\gamma_k)\underbrace{(f(x_{k-1}) - f(x^*))}_{\Delta_{k-1}} + B_k$
3. Note $\Gamma_k = (1-\gamma_k)(1-\gamma_{k-1})\ldots(1-\gamma_2)$; If $\gamma_k = \frac{1}{k} \Rightarrow \Gamma_k = \frac{1}{k}$;

   If $\gamma_k = \frac{2}{k+1} \Rightarrow \Gamma_k = \frac{2}{k(k+1)}$; If $\gamma_k = \frac{3}{k+2} \Rightarrow \Gamma_k = \frac{6}{k(k+1)(k+2)}$

FOM Framework: Convergence

Main Goal: $\underbrace{f(y_k) - f(x^*)}_{\Delta_k} \leq (1-\gamma_k)\underbrace{(f(y_{k-1}) - f(x^*))}_{\Delta_{k-1}} + B_k.$

We have: $f(x) \in C_L^{1,1}(X)$; convexity; optimality condition of subproblem.

$$\begin{aligned} f(y_k) &\leq f(z_k) + \langle \nabla f(z_k), y_k - z_k \rangle + \frac{L}{2}\|y_k - z_k\|^2 \\ &= (1-\gamma_k)[f(z_k) + \langle \nabla f(z_k), y_{k-1} - z_k \rangle] + \gamma_k[f(z_k) + \langle \nabla f(z_k), x_k - z_k \rangle] + \frac{L\gamma_k^2}{2}\|x_k - x_{k-1}\|^2 \\ &\leq (1-\gamma_k) f(y_{k-1}) + \gamma_k[f(z_k) + \langle \nabla f(z_k), x_k - z_k \rangle] + \frac{L\gamma_k^2}{2}\|x_k - x_{k-1}\|^2 \end{aligned}$$

Since $x_k = \operatorname{argmin}_{x \in X}\left\{\langle \nabla f(z_k), x \rangle + \frac{\beta_k}{2}\|x - x_{k-1}\|_2^2\right\}$, by the optimal condition

$$\Rightarrow \langle \nabla f(z_k) + \beta_k(x_k - x_{k-1}), x_k - x \rangle \leq 0, \quad \forall\, x \in X$$

$$\Rightarrow \langle x_{k-1} - x_k, x_k - x \rangle \leq \frac{1}{\beta_k}\langle \nabla f(x_k), x - x_k \rangle$$

$$\begin{aligned} \frac{1}{2}\|x_k - x_{k-1}\|^2 &= \frac{1}{2}\|x_{k-1} - x\|^2 - \langle x_{k-1} - x_k, x_k - x \rangle - \frac{1}{2}\|x_k - x\|^2 \\ &\leq \frac{1}{2}\|x_{k-1} - x\|^2 + \frac{1}{\beta_k}\langle \nabla f(z_k), x - x_k \rangle - \frac{1}{2}\|x_k - x\|^2 \end{aligned}$$

Note $L\gamma_k \leq \beta_k$

FOM Framework: Convergence

Main inequality:

$$f(y_k) - f(x) \leq (1 - \gamma_k)[f(y_{k-1} - f(x))] + \frac{\beta_k \gamma_k}{2}(\|x_{k-1} - x\|^2 - \|x_k - x\|^2)$$

Main estimation:

$$f(y_k) - f(x) \leq \frac{\Gamma_k(1-\gamma_1)}{\Gamma_1}(f(y_0) - f(x)) + \frac{\Gamma_k}{2}\underbrace{\sum_{i=1}^{k} \frac{\beta_i \gamma_i}{\Gamma_i}\left(\|x_{i-1} - x\|^2 - \|x_i - x\|^2\right)}_{(*)}$$

$$\begin{aligned}(*) &= \frac{\beta_1 \gamma_1}{\Gamma_1}\|x_0 - x\|^2 + \sum_{i=2}^{k}\left(\frac{\beta_i \gamma_i}{\Gamma_i} - \frac{\beta_{i-1}\gamma_{i-1}}{\Gamma_{i-1}}\right)\|x_{i-1} - x\|^2 - \beta_k \gamma_k \Gamma_k \|x_k - x\|^2 \\ &\leq \frac{\beta_1 \gamma_1}{\Gamma_1}\|x_0 - x\|^2 + \sum_{i=2}^{k}\left(\frac{\beta_i \gamma_i}{\Gamma_i} - \frac{\beta_{i-1}\gamma_{i-1}}{\Gamma_{i-1}}\right) \cdot D_X^2 \qquad (\text{here } D_X = \sup_{x,y \in X}\|x - y\|)\end{aligned}$$

Observation:

If $\frac{\beta_k \gamma_k}{\Gamma_k} \geq \frac{\beta_{k-1}\gamma_{k-1}}{\Gamma_{k-1}} \Rightarrow (*) \leq \frac{\beta_k \gamma_k}{\Gamma_k} D_X^2 \Rightarrow f(y_k) - f(x) \leq \frac{\beta_k \gamma_k}{2} D_X^2$

If $\frac{\beta_k \gamma_k}{\Gamma_k} \leq \frac{\beta_{k-1}\gamma_{k-1}}{\Gamma_{k-1}} \Rightarrow (*) \leq \frac{\beta_1 \gamma_1}{\Gamma_1}\|x_0 - x\|^2 \Rightarrow f(y_k) - f(x) \leq \Gamma_k \frac{\beta_1 \gamma_1}{2}\|x_0 - x\|^2$

Main results:

1. Let $\beta_k = L$, $\gamma_k = \frac{1}{k} \Rightarrow \Gamma_k = \frac{1}{k}$, $\frac{\beta_k \gamma_k}{\Gamma_k} = L$. We have

$$f(y_k) - f(x^*) \le \frac{L}{2k} D_X^2, \quad f(y_k) - f(x^*) \le \frac{L}{2k} \|x_0 - x^*\|^2$$

2. Let $\beta_k = \frac{2L}{k}$, $\gamma_k = \frac{2}{k+1} \Rightarrow \Gamma_k = \frac{2}{k(k+1)}$, $\frac{\beta_k \gamma_k}{\Gamma_k} = 2L$. We have

$$f(y_k) - f(x^*) \le \frac{2L}{k(k+1)} D_X^2, \quad f(y_k) - f(x^*) \le \frac{4L}{k(k+1)} \|x_0 - x^*\|^2$$

3. Let $\beta_k = \frac{3L}{k+1}$, $\gamma_k = \frac{3}{k+2} \Rightarrow \Gamma_k = \frac{6}{k(k+1)(k+2)}$, $\frac{\beta_k \gamma_k}{\Gamma_k} = \frac{3Lk}{2} \ge \frac{\beta_{k-1}\gamma_{k-1}}{\Gamma_{k-1}}$.
We have

$$f(y_k) - f(x^*) \le \frac{9L}{2(k+1)(k+2)} D_X^2$$