

# Subgradient

Acknowledgement: this slides is based on Prof. Lieven Vandenberghe's lecture notes

- definition
- subgradient calculus
- duality and optimality conditions
- directional derivative

# Basic inequality

recall basic inequality for convex differentiable  $f$ :

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

- the first-order approximation of  $f$  at  $x$  is a global lower bound
- $\nabla f(x)$  defines non-vertical supporting hyperplane to **epi**  $f$  at  $(x, f(x))$

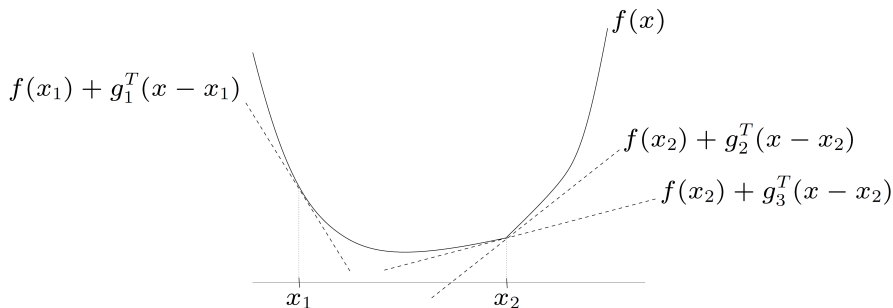
$$\begin{bmatrix} \nabla f(x) \\ -1 \end{bmatrix} \left( \begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0 \quad \forall (y, t) \in \mathbf{epi} f$$

what if  $f$  is not differentiable?

# Subgradient

$g$  is a **subgradient** of a convex function  $f$  at  $x \in \mathbf{dom} f$  if

$$f(y) \geq f(x) + g^T(y - x) \quad \forall y \in \mathbf{dom} f$$



$g_2, g_3$  are subgradients at  $x_2$ ;  $g_1$  is a subgradient at  $x_1$

## properties

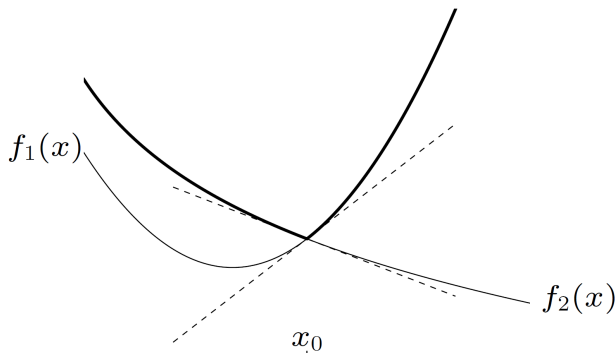
- $f(x) + g^\top(y - x)$  is a global lower bound on  $f(y)$
- $g$  defines non-vertical supporting hyperplane to  $\mathbf{epi} f$  at  $(x, f(x))$

$$\begin{bmatrix} g \\ -1 \end{bmatrix} \left( \begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0 \quad \forall (y, t) \in \mathbf{epi} f$$

- if  $f$  is convex and differentiable, then  $\nabla f(x)$  is a subgradient of  $f$  at  $x$
- algorithms for nondifferentiable convex optimization
- unconstrained optimality:  $x$  minimizes  $f(x)$  if and only if  $0 \in \partial f(x)$
- KKT conditions with nondifferentiable functions

## Example

$$f(x) = \max\{f_1(x), f_2(x)\} \quad f_1, f_2 \text{ convex and differentiable}$$



- subgradients at  $x_0$  form line segment  $[\nabla f_1(x_0), \nabla f_2(x_0)]$
- if  $f_1(\hat{x}) > f_2(\hat{x})$ , subgradient of  $f$  at  $\hat{x}$  is  $\nabla f_1(\hat{x})$
- if  $f_1(\hat{x}) < f_2(\hat{x})$ , subgradient of  $f$  at  $\hat{x}$  is  $\nabla f_2(\hat{x})$

# Subdifferential

the **subdifferential**  $\partial f(x)$  of  $f$  at  $x$  is the set of all subgradients:

$$\partial f(x) = \{g \mid g^\top (y - x) \leq f(y) - f(x)\} \quad \forall y \in \mathbf{dom} f$$

- $\partial f(x)$  is a closed convex set (possibly empty) (follows from the definition:  $\partial f(x)$  is an intersection of halfspaces)
- if  $x \in \mathbf{int} \mathbf{dom} f$  then  $\partial f(x)$  is nonempty and bounded (proof on next two pages)

*proof:* we show that  $\partial f(x)$  is nonempty when  $x \in \mathbf{int\,dom\,}f$

- $(x, f(x))$  is in the boundary of the convex set  $\mathbf{epi\,}f$
- therefore there exists a supporting hyperplane to  $\mathbf{epi\,}f$  at  $(x, f(x))$ :

$$\exists (a, b) \neq 0, \quad \begin{bmatrix} a \\ b \end{bmatrix}^\top \left( \begin{bmatrix} y \\ t \end{bmatrix} - \begin{bmatrix} x \\ f(x) \end{bmatrix} \right) \leq 0 \quad \forall (y, t) \in \mathbf{epi\,}f$$

- $b > 0$  gives a contradiction as  $t \rightarrow \infty$
- $b = 0$  gives a contradiction for  $y = x + \epsilon a$  with small  $\epsilon > 0$
- therefore  $b < 0$  and  $g = a/|b|$  is a subgradient of  $f$  at  $x$

*proof:* we show that  $\partial f(x)$  is empty when  $x \in \mathbf{int\, dom\,} f$

- for small  $r > 0$ , define a set of  $2n$  points

$$B = \{x \pm re_k | k = 1, \dots, n\} \subset \mathbf{dom\,} f$$

and define  $M = \max_{y \in B} f(y) < \infty$

- for every nonzero  $g \in \partial f(x)$ , there is a point  $y \in B$  with

$$f(y) \geq f(x) + g^\top (y - x) = f(x) + r \|g\|_\infty$$

(choose an index  $k$  with  $|g_k| = \|g\|_\infty$ , and take  $y = x + r \mathbf{sign}(g_k) e_k$ )

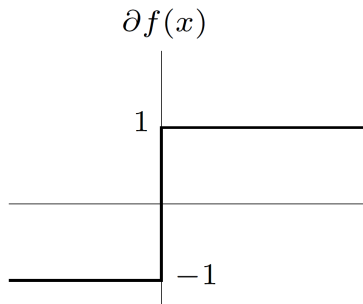
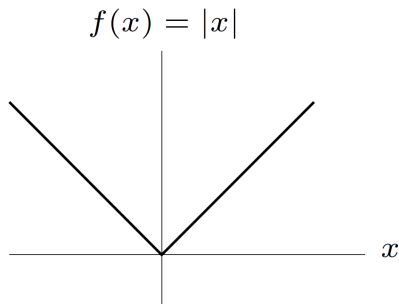
- therefore  $\partial f(x)$  is bounded:

$$\sup_{g \in \partial f(x)} \|g\|_\infty \leq \frac{M - f(x)}{r}$$



# Examples

**absolute value**  $f(x) = |x|$



**Euclidean norm**  $f(x) = \|x\|_2$

$$\partial f(x) = \frac{1}{\|x\|_2}x \text{ if } x \neq 0, \quad \partial f(x) = \{g \mid \|g\|_2 \leq 1\} \text{ if } x = 0\}$$

# Monotonicity

subdifferential of a convex function is a **monotone operator**:

$$(u - v)^\top (x - y) \geq 0 \quad \forall x, y, u \in \partial f(x), v \in \partial f(y)$$

*proof*: by definition

$$f(y) \geq f(x) + u^\top (y - x), f(x) \geq f(y) + v^\top (x - y)$$

combining the two inequalities shows monotonicity

# 连续性

设 $f(x)$ 是闭凸函数且 $\partial f$ 在点 $\bar{x}$ 附近存在且非空. 若序列 $x^k \rightarrow \bar{x}$ ,  $g^k \in \partial f(x^k)$ 为 $f(x)$ 在点 $x^k$ 处的次梯度, 且 $g^k \rightarrow \bar{g}$ , 则 $\bar{g} \in \partial f(\bar{x})$ .

- 对任意 $y \in \text{dom}f$ , 根据次梯度的定义,

$$f(y) \geq f(x^k) + \langle g^k, y - x^k \rangle.$$

对上述不等式两边取下极限, 我们有

$$\begin{aligned} f(y) &\geq \liminf_{k \rightarrow \infty} [f(x^k) + \langle g^k, y - x^k \rangle] \\ &\geq f(\bar{x}) + \langle \bar{g}, y - \bar{x} \rangle, \end{aligned}$$

其中第二个不等式利用了 $f(x)$ 的下半连续性以及 $g^k \rightarrow \bar{g}$ , 由此可推出 $\bar{g} \in \partial f(\bar{x})$ .

# Examples of non-subdifferentiable functions

the following functions are not subdifferentiable at  $x = 0$

- $f : \mathbf{R} \rightarrow \mathbf{R}, \text{dom } f = \mathbf{R}_+$

$$f(x) = 1 \quad \text{if } x = 0, \quad f(x) = 0 \quad \text{if } x > 0$$

- $f : \mathbf{R} \rightarrow \mathbf{R}, \text{dom } f = \mathbf{R}_+$

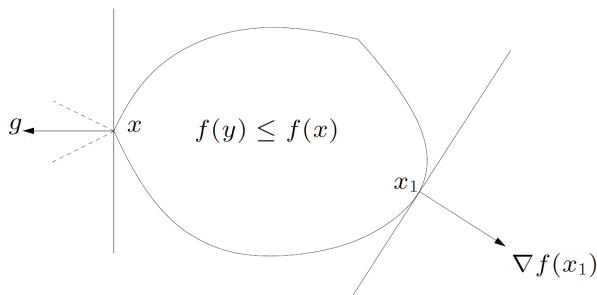
$$f(x) = -\sqrt{x}$$

the only supporting hyperplane to  $\text{epi } f$  at  $(0, f(0))$  is vertical

# Subgradients and sublevel sets

if  $g$  is a subgradient of  $f$  at  $x$ , then

$$f(y) \leq f(x) \implies g^\top (y - x) \leq 0$$



nonzero subgradients at  $x$  define supporting hyperplanes to sublevel set

$$\{y \mid f(y) \leq f(x)\}$$

- definition
- **subgradient calculus**
- duality and optimality conditions
- directional derivative

# Subgradient calculus

**weak subgradient calculus:** rules for finding *one* subgradient

- sufficient for most nondifferentiable convex optimization algorithms
- if you can evaluate  $f(x)$ , you can usually compute a subgradient

**strong subgradient calculus:** rules for finding  $\partial f(x)$  (*all subgradients*)

- some algorithms, optimality conditions, etc., need entire subdifferential
- can be quite complicated

we will assume that  $x \in \mathbf{int\,dom\,}f$

# Basic rules

**differentiable functions:**  $\partial f(x) = \{\nabla f(x)\}$  if  $f$  is differentiable at  $x$

**nonnegative combination**

if  $h(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x)$  with  $\alpha_1, \alpha_2 \geq 0$ , then

$$\partial h(x) = \alpha_1 \partial f_1(x) + \alpha_2 \partial f_2(x)$$

(r.h.s. is addition of sets)

**affine transformation of variables:** if  $h(x) = f(Ax + b)$ , then

$$\partial h(x) = A^\top \partial f(Ax + b)$$



# Pointwise maximum

$$f(x) = \max\{f_1(x), \dots, f_m(x)\}$$

define  $I(x) = \{i \mid f_i(x) = f(x)\}$ , the 'active' functions at  $x$

**weak result:** to compute a subgradient at  $x$ ,  
choose any  $k \in I(x)$ , and any subgradient of  $f_k$  at  $x$

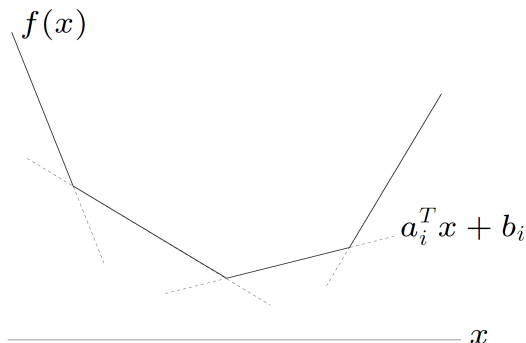
**strong result**

$$\partial f(x) = \mathbf{conv} \bigcup_{i \in I(x)} \partial f_i(x)$$

- convex hull of the union of subdifferentials of 'active' functions at  $x$
- if  $f_i$ 's are differentiable,  $\partial f(x) = \mathbf{conv}\{\nabla f_i(x) \mid i \in I(x)\}$

## Example: piecewise-linear function

$$f(x) = \max_{i=1,\dots,m} a_i^\top x + b_i$$



the subdifferential at  $x$  is a polyhedron

$$\partial f(x) = \mathbf{conv}\{a_i \mid i \in I(x)\}$$

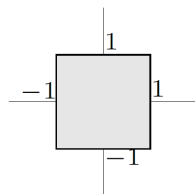
with  $I(x) = \{i \mid a_i^\top x + b_i = f(x)\}$

## Example: $\ell_1$ -norm

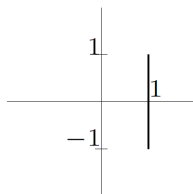
$$f(x) = \|x\|_1 = \max_{s \in \{-1, 1\}^n} s^\top x$$

the subdifferential is a product of intervals

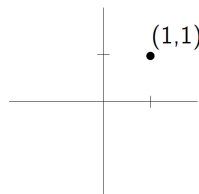
$$\partial f(x) = J_1 \times \cdots \times J_n, \quad J_k = \begin{cases} [-1, 1], & x_k = 0 \\ \{1\}, & x_k > 0 \\ \{-1\}, & x_k < 0 \end{cases}$$



$$\partial f(0, 0) = [-1, 1] \times [-1, 1]$$



$$\partial f(1, 0) = \{1\} \times [-1, 1]$$



$$\partial f(1, 1) = \{(1, 1)\}$$

# Pointwise supremum

$$f(x) = \sup_{\alpha \in \mathcal{A}} f_{\alpha}(x), \quad f_{\alpha}(x) \text{ convex in } x \text{ for every } \alpha$$

**weak result:** to find a subgradient at  $x$ ,

- find *any*  $\beta$  for which  $f(\hat{x}) = f_{\beta}(\hat{x})$  (assuming maximum is attained)
- choose *any*  $g \in \partial f_{\beta}(\hat{x})$

**strong result:** define  $I(x) = \{\alpha \in \mathcal{A} \mid f_{\alpha}(x) = f(x)\}$

$$\text{conv} \bigcup_{\alpha \in I(x)} \partial f_{\alpha}(x) \subseteq \partial f(x)$$

equality requires extra conditions (e.g.,  $\mathcal{A}$  compact,  $f_{\alpha}$  continuous in  $\alpha$ )

## Exercise: maximum eigenvalue

**problem:** explain how to find a subgradient of

$$f(x) = \lambda_{\max}(A(x)) = \sup_{\|y\|_2=1} y^\top A(x)y$$

where  $A(x) = A_0 + x_1A_1 + \cdots + x_nA_n$  with symmetric coefficients  $A_i$

**solution:** to find a subgradient at  $\hat{x}$ ,

- choose *any* unit eigenvector  $y$  with eigenvalue  $\lambda_{\max}(A(\hat{x}))$
- the gradient of  $y^\top A(x)y$  at  $\hat{x}$  is a subgradient of  $f$ :

$$(y^\top A_1 y, \cdots, y^\top A_n y) \in \partial f(\hat{x})$$

# Minimization

$$f(x) = \inf_y h(x, y), \quad h \text{ jointly convex in } (x, y)$$

**weak result:** to find a subgradient at  $\hat{x}$

- find  $\hat{y}$  that minimizes  $h(\hat{x}, y)$  (assuming minimum is attained)
- find subgradient  $(g, 0) \in \partial h(\hat{x}, \hat{y})$

*proof:* for all  $x, y$

$$\begin{aligned} h(x, y) &\geq h(\hat{x}, \hat{y}) + g^\top (x - \hat{x}) + 0^\top (y - \hat{y}) \\ &= f(\hat{x}) + g^\top (x - \hat{x}) \end{aligned}$$

therefore

$$f(x) = \inf_y h(x, y) \geq f(\hat{x}) + g^\top (x - \hat{x})$$

## Exercise: Euclidean distance to convex set

**problem:** explain how to find a subgradient of

$$f(x) = \inf_{y \in C} \|x - y\|_2$$

where  $C$  is a closed convex set

**solution:** to find a subgradient at  $\hat{x}$ ,

- if  $f(\hat{x}) = 0$  (that is,  $\hat{x} \in C$ ), take  $g = 0$
- if  $f(\hat{x}) > 0$ , find projection  $\hat{y} = P(\hat{x})$  on  $C$ ; take

$$g = \frac{1}{\|\hat{y} - \hat{x}\|_2} (\hat{x} - \hat{y}) = \frac{1}{\|\hat{x} - P(\hat{x})\|_2} (\hat{x} - P(\hat{x}))$$

# Composition

$f(x) = h(f_1(x), \dots, f_k(x))$ ,  $h$  convex nondecreasing,  $f_i$  convex

**weak result:** to find a subgradient at  $\hat{x}$ ,

- find  $z \in \partial h(f_1(\hat{x}), \dots, f_k(\hat{x}))$  and  $g_i \in \partial f_i(\hat{x})$
- then  $g = z_1 g_1 + \dots + z_k g_k \in \partial f(\hat{x})$

reduces to standard formula for differentiable  $h, f_i$

*proof:*

$$\begin{aligned} f(x) &\geq h\left(f_1(\hat{x}) + g_1^\top(x - \hat{x}), \dots, f_k(\hat{x}) + g_k^\top(x - \hat{x})\right) \\ &\geq h(f_1(\hat{x}), \dots, f_k(\hat{x})) + z^\top \left(g_1^\top(x - \hat{x}), \dots, g_k^\top(x - \hat{x})\right) \\ &= f(\hat{x}) + g^\top(x - \hat{x}) \end{aligned}$$



# Optimal value function

define  $h(u, v)$  as the optimal value of convex problem

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq u_i, \quad i = 1, \dots, m \\ & Ax = b + v \end{aligned}$$

(functions  $f_i$  are convex; optimization variable is  $x$ )

**weak result:** suppose  $h(\hat{u}, \hat{v})$  is finite, strong duality holds with the dual

$$\begin{aligned} \max \quad & \inf_x \left( f_0(x) + \sum_i \lambda_i (f_i(x) - \hat{u}_i) + \nu^\top (Ax - b - \hat{v}) \right) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

if  $\hat{\lambda}, \hat{\nu}$  are optimal dual variables (for r.h.s.  $\hat{u}, \hat{v}$ ) then  $(\hat{\lambda}, \hat{\nu}) \in \partial h(\hat{u}, \hat{v})$

*proof* : by weak duality for problem with r.h.s.  $u, v$

$$\begin{aligned}h(u, v) &\geq \inf_x \left( f_0(x) + \sum_i \hat{\lambda}_i (f_i(x - u_i)) + \hat{v}^\top (Ax - b - v) \right) \\&= \inf_x \left( f_0(x) + \sum_i \hat{\lambda}_i (f_i(x - \hat{u}_i)) + \hat{v}^\top (Ax - b - \hat{v}) \right) \\&\quad - \hat{\lambda}^\top (u - \hat{u}) - \hat{v}^\top (v - \hat{v}) \\&= h(\hat{u}, \hat{v}) - \hat{\lambda}^\top (u - \hat{u}) - \hat{v}^\top (v - \hat{v})\end{aligned}$$

# Expectation

$$f(x) = \mathbf{E}h(x, u) \quad u \text{ random, } h \text{ convex in } x \text{ for every } u$$

**weak result:** to find a subgradient at  $\hat{x}$

- choose a function  $u \mapsto g(u)$  with  $g(u) \in \partial_x h(\hat{x}, u)$
- then,  $g = \mathbf{E}_u g(u) \in \partial f(\hat{x})$

*proof* : by convexity of  $h$  and definition of  $g(u)$ ,

$$\begin{aligned} f(x) &= \mathbf{E}h(x, u) \\ &\geq \mathbf{E} \left( h(\hat{x}, u) + g(u)^\top (x - \hat{x}) \right) \\ &= f(\hat{x}) + g^\top (x - \hat{x}) \end{aligned}$$

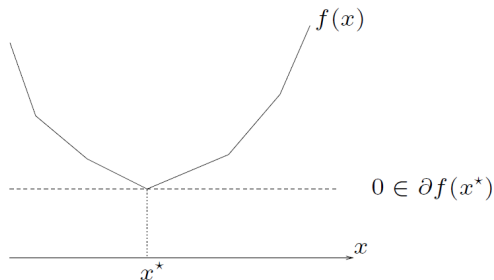
# Outline

- definition
- subgradient calculus
- **duality and optimality conditions**
- directional derivative

# Optimality conditions - unconstrained

$x^*$  minimizes  $f(x)$  if and only

$$0 \in \partial f(x^*)$$



*proof* : by definition

$$f(y) \geq f(x^*) + 0^\top (y - x^*) \text{ for all } y \iff 0 \in \partial f(x^*)$$

## Example: piecewise linear minimization

$$f(x) = \max_{i=1, \dots, m} (a_i^\top x + b_i)$$

### optimality condition

$$0 \in \mathbf{conv}\{a_i \mid i \in I(x^*)\} \quad (\text{where } I(x) = \{i \mid a_i^\top x + b_i = f(x)\})$$

in other words,  $x^*$  is optimal if and only if there is a  $\lambda$  with

$$\lambda \geq 0, \quad \mathbf{1}^\top \lambda = 1, \quad \sum_{i=1}^m \lambda_i a_i = 0, \quad \lambda_i = 0 \text{ for } i \notin I(x^*)$$

these are the optimality conditions for the equivalent linear program

$$\begin{array}{ll} \min & t \\ \text{s.t.} & Ax + b \leq t\mathbf{1} \end{array} \quad \begin{array}{ll} \max & b^\top \lambda \\ \text{s.t.} & A^\top \lambda = 0 \end{array}$$

$$\lambda \geq 0, \quad \mathbf{1}^\top \lambda = 1$$

# Optimality conditions - constrained

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad i = 1, \dots, m \end{array}$$

## from Lagrange duality

if strong duality holds, then  $x^*$ ,  $\lambda^*$  are primal, dual optimal if and only if

1.  $x^*$  is primal feasible
2.  $\lambda^* \geq 0$
3.  $\lambda_i^* f_i(x^*) = 0$  for  $i = 1, \dots, m$
4.  $x^*$  is a minimizer of

$$L(x, \lambda^*) = f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x)$$

**Karush-Kuhn-Tucker conditions** (if  $\text{dom } f_i = \mathbf{R}^n$ )  
conditions 1, 2, 3 and

$$0 \in \partial L_x(x^*, \lambda^*) = \partial f_0(x^*) + \sum_{i=1}^m \lambda_i^* \partial f_i(x^*).$$

The above condition generalizes

$$0 = \nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*)$$

for differentiable  $f_i$



- definition
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- **directional derivative**

# Directional derivative

**Definition** (general  $f$ ): directional derivative of  $f$  at  $x$  in the direction  $y$  is

$$\begin{aligned}f'(x; y) &= \lim_{\alpha \searrow 0} \frac{f(x + \alpha y) - f(x)}{\alpha} \\ &= \lim_{t \rightarrow \infty} \left( t(f(x + \frac{1}{t}y) - f(x)) \right)\end{aligned}$$

(if the limit exists)

- $f'(x; y)$  is the right derivative of  $g(\alpha) = f(x + \alpha y)$  at  $\alpha = 0$
- $f'(x; y)$  is homogeneous in  $y$ :

$$f'(x; \lambda y) = \lambda f'(x; y) \text{ for } \lambda \geq 0$$

# Directional derivative of a convex function

**equivalent definition** (convex  $f$ ): replace  $\lim$  with  $\inf$

$$\begin{aligned} f'(x; y) &= \inf_{\alpha > 0} \frac{f(x + \alpha y) - f(x)}{\alpha} \\ &= \inf_{t > 0} \left( t(f(x + \frac{1}{t}y) - f(x)) \right) \end{aligned}$$

*proof*

- the function  $h(y) = f(x + y) - f(x)$  is convex in  $y$ , with  $h(0) = 0$
- its perspective  $th(y/t)$  is nonincreasing in  $t$  (EE236B ex. A2.5); hence

$$f'(x; y) = \lim_{t \rightarrow \infty} th(y/t) = \inf_{t > 0} th(y/t)$$

# Properties

consequences of the expressions (for convex  $f$ )

$$\begin{aligned} f'(x; y) &= \inf_{\alpha > 0} \frac{f(x + \alpha y) - f(x)}{\alpha} \\ &= \inf_{t > 0} \left( t(f(x + \frac{1}{t}y) - f(x)) \right) \end{aligned}$$

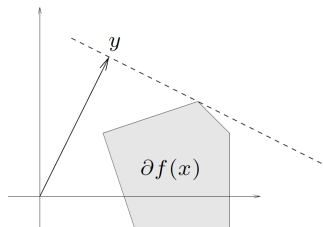
- $f'(x; y)$  is convex in  $y$  (partial minimization of a convex function in  $y, t$ )
- $f'(x; y)$  defines a lower bound on  $f$  in the direction  $y$ :

$$f(x + \alpha y) \geq f(x) + \alpha f'(x; y) \quad \forall \alpha \geq 0$$

# Directional derivative and subgradients

for convex  $f$  and  $x \in \mathbf{intdom} f$

$$f'(x; y) = \sup_{g \in \partial f(x)} g^\top y$$



$f'(x; y)$  is support function of  $\partial f(x)$

- generalizes  $f'(x; y) = \nabla f(x)^\top y$  for differentiable functions
- implies that  $f'(x; y)$  exists for all  $x \in \mathbf{intdom} f$ , all  $y$  (see page 6)

*proof* : if  $g \in \partial f(x)$  then from p 36

$$f'(x; y) \geq \inf_{\alpha > 0} \frac{f(x) + \alpha g^\top y - f(x)}{\alpha} = g^\top y$$

it remains to show that  $f'(x; y) = \hat{g}^\top y$  for at least one  $\hat{g} \in \partial f(x)$

- $f'(x; y)$  is convex in  $y$  with domain  $\mathbf{R}^n$ , hence subdifferentiable at all  $y$
- let  $\hat{g}$  be a subgradient of  $f'(x; y)$  at  $y$ : for all  $v, \lambda \geq 0$ ,

$$\lambda f'(x; v) = f'(x; \lambda v) \geq f'(x; y) + \hat{g}^\top (\lambda v - y)$$

- taking  $\lambda \rightarrow \infty$  shows  $f'(x; v) \geq \hat{g}^\top v$ ; from the lower bound on p 36

$$f(x + v) \geq f(x) + f'(x; v) \geq f(x) + \hat{g}^\top v \quad \forall v$$

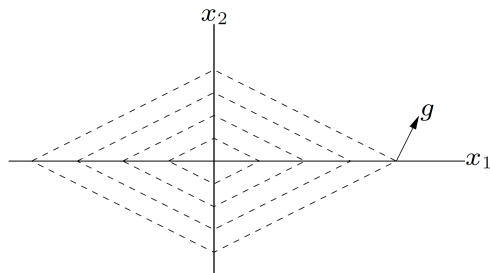
- hence  $\hat{g} \in \partial f(x)$ ; taking  $\lambda = 0$  we see that  $f'(x; y) \leq \hat{g}^\top y$

# Descent directions and subgradients

$y$  is a **descent direction** of  $f$  at  $x$  if  $f'(x; y) < 0$

- negative gradient of differentiable  $f$  is descent direction (if  $\nabla f(x) \neq 0$ )
- negative subgradient is **not** always a descent direction

**example:**  $f(x_1, x_2) = |x_1| + 2|x_2|$



$g = (1, 2) \in \partial f(1, 0)$ , but  $y = (-1, -2)$  is not a descent direction at  $(1, 0)$

# Steepest descent direction

**definition:** (normalized) steepest descent direction at  $x \in \text{int dom } f$  is

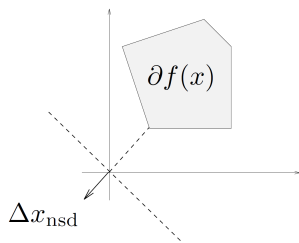
$$\Delta x_{\text{nsd}} = \underset{\|y\|_2 \leq 1}{\operatorname{argmin}} f'(x; y)$$

$\Delta x_{\text{nsd}}$  is the primal solution  $y$  of the pair of dual problems (BV S8.1.3)

$$\begin{aligned} \min (\text{over } y) \quad & f'(x; y) \\ \text{s.t.} \quad & \|y\|_2 \leq 1 \end{aligned}$$

$$\begin{aligned} \max (\text{over } g) \quad & -\|g\|_2 \\ \text{s.t.} \quad & g \in \partial f(x) \end{aligned}$$

- optimal  $g^*$  is subgradient with least norm
- $f'(x; \Delta x_{\text{nsd}}) = -\|g^*\|_2$
- if  $0 \notin \partial f(x)$ ,  $\Delta x_{\text{nsd}} = -g^*/\|g^*\|_2$








# Subgradients and distance to sublevel sets

if  $f$  is convex,  $f(y) < f(x)$ ,  $g \in \partial f(x)$ , then for small  $t > 0$ ,

$$\begin{aligned}\|x - tg - y\|_2^2 &= \|x - y\|_2^2 - 2tg^\top(x - y) + t^2\|g\|_2^2 \\ &\leq \|x - y\|_2^2 - 2t(f(x) - f(y)) + t^2\|g\|_2^2 \\ &< \|x - y\|_2^2\end{aligned}$$

- $-g$  is descent direction for  $\|x - y\|_2$ , for **any**  $y$  with  $f(y) < f(x)$
- in particular,  $-g$  is descent direction for distance to any minimizer of  $f$

# References

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-  Yu. Nesterov, *Introductory Lectures on Convex Optimization. A Basic Course* (2004), section 3.1.
-  B. T. Polyak, *Introduction to Optimization* (1987), section 5.1.