Lecture: Convex Sets

http://bicmr.pku.edu.cn/~wenzw/opt-2017-fall.html

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Introduction

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities
Affine set

line through $x_1$, $x_2$: all points

$$x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbb{R})$$

affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)
Convex set

**line segment** between $x_1$ and $x_2$: all points

$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \implies \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)
**Convex combination and convex hull**

**convex combination** of \(x_1, \ldots, x_k\): any point \(x\) of the form

\[
x = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_k x_k
\]

with \(\theta_1 + \ldots + \theta_k = 1\), \(\theta_i \geq 0\)

**convex hull** \(\text{conv} S\): set of all convex combinations of points in \(S\)
Convex cone

**conic (nonnegative) combination** of $x_1$ and $x_2$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

**convex cone**: set that contains all conic combinations of points in the set

![Diagram of a convex cone](image)
Hyperplanes and halfspaces

**Hyperplane**: set of the form \( \{ x \mid a^T x = b \} (a \neq 0) \)

**Halfspace**: set of the form \( \{ x \mid a^T x \leq b \} (a \neq 0) \)

- \( a \) is the normal vector
- Hyperplanes are affine and convex; halfspaces are convex
Euclidean balls and ellipsoids

(Euclidean) ball with center $x_c$ and radius $r$:

$$B(x_c, r) = \{x | \|x - x_c\|_2 \leq r\} = \{x_c + ru | \|u\|_2 \leq 1\}$$

ellipsoid: set of the form

$$\{x | (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with $P \in \mathbb{S}_+^n$ (i.e., $P$ symmetric positive definite)

other representation: $\{x_c + Au | \|u\|_2 \leq 1\}$ with $A$ square and nonsingular
Norm balls and norm cones

**norm**: a function \( \| \cdot \| \) that satisfies

- \( \| x \| \geq 0; \| x \| = 0 \) if and only if \( x = 0 \)
- \( \| tx \| = |t| \| x \| \) for \( t \in \mathbb{R} \)
- \( \| x + y \| \leq \| x \| + \| y \| \)

notation: \( \| \cdot \| \) is general (unspecified) norm; \( \| \cdot \|_{\text{symb}} \) is particular norm

**norm ball** with center \( x_c \) and radius \( r \): \( \{ x \mid \| x - x_c \| \leq r \} \)

**norm cone**: \( \{ (x, t) \mid \| x \| \leq t \} \)

Euclidean norm cone is called second-order cone

norm balls and cones are convex
Polyhedra

solution set of finitely many linear inequalities and equalities

\[ Ax \leq b, \quad Cx = d \]

(\( A \in \mathbb{R}^{m \times n}, \ C \in \mathbb{R}^{p \times n}, \leq \) is componentwise inequality)

polyhedron is intersection of finite number of halfspaces and hyperplanes
Positive semidefinite cone

notation:

- $S^n$ is set of symmetric $n \times n$ matrices
- $S^n_+ = \{ X \in S^n | X \succeq 0 \}$: positive semidefinite $n \times n$ matrices
  
  \[ X \in S^n_+ \iff z^T X z \geq 0 \text{ for all } z \]

- $S^n_+$ is a convex cone

- $S^n_{++} = \{ X \in S^n | X \succ 0 \}$: positive definite $n \times n$ matrices

example: 

\[ \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S^2_+ \]
Operations that preserve convexity

practical methods for establishing convexity of a set $C$

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

2. show that $C$ is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions
Intersection

The intersection of (any number of) convex sets is convex.

Example:

\[ S = \{ x \in \mathbb{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3 \} \]

Where \( p(t) = x_1 \cos t + x_2 \cos 2t + \ldots + x_m \cos mt \)

For \( m = 2 \):
Affine function

suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \) is affine \((f(x) = Ax + b \text{ with } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m)\)

- the image of a convex set under \( f \) is convex

\[
S \subseteq \mathbb{R}^n \text{ convex} \implies f(S) = \{f(x) | x \in S\} \text{ convex}
\]

- the inverse image \( f^{-1}(C) \) of a convex set under \( f \) is convex

\[
C \subseteq \mathbb{R}^m \text{ convex} \implies f^{-1}(C) = \{x \in \mathbb{R}^n | f(x) \in C\} \text{ convex}
\]

the image of a convex set under \( f \) is convex

- the inverse image \( f^{-1}(C) \) of a convex set under \( f \) is convex

examples

- scaling, translation, projection
- solution set of linear matrix inequality \( \{x | x_1A_1 + \ldots + x_mA_m \preceq B\} \) (with \( A_i, B \in \mathbb{S}^p \))
- hyperbolic cone \( \{x | x^TPx \leq (c^T x)^2, c^T x \geq 0\} \) (with \( P \in \mathbb{S}^n_+ \))
perspective function \( P : \mathbb{R}^{n+1} \to \mathbb{R}^n: \)

\[
P(x, t) = \frac{x}{t}, \quad \text{dom } P = \{(x, t) | t > 0\}
\]

images and inverse images of convex sets under perspective are convex

linear-fractional function \( f : \mathbb{R}^n \to \mathbb{R}^m: \)

\[
f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } f = \{x | c^T x + d > 0\}
\]

images and inverse images of convex sets under linear-fractional functions are convex
example of a linear-fractional function

\[ f(x) = \frac{1}{x_1 + x_2 + 1} \]
Separating hyperplane theorem

If $C$ and $D$ are disjoint convex sets, then there exists $a \neq 0, b$ such that

$$a^T x \leq b \text{ for } x \in \overline{C}, \quad a^T x \geq b \text{ for } x \in \overline{D}$$

where $\overline{C}$ and $\overline{D}$ are the closure of $C$ and $D$.

The hyperplane $\{x | a^T x = b\}$ separates $C$ and $D$

Strict separation requires additional assumptions (e.g., $C$ is closed, $D$ is a singleton)
Supporting hyperplane theorem

Supporting hyperplane to set $C$ at boundary point $x_0$:

$$\{x | a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

**supporting hyperplane theorem:** if $C$ is a nonempty convex set, then there exists a supporting hyperplane at every boundary point of $C$.
Generalized inequalities

A convex cone $K \subseteq \mathbb{R}^n$ is a proper cone if

- $K$ is closed (contains its boundary)
- $K$ is solid (has nonempty interior)
- $K$ is pointed (contains no line)

Examples

- Nonnegative orthant $K = \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n \}$
- Positive semidefinite cone $K = \mathbb{S}^n_+$
- Nonnegative polynomials on $[0, 1]$: $K = \{ x \in \mathbb{R}^n | x_1 + x_2 t + x_3 t^2 + \ldots + x_n t^{n-1} \geq 0$ for $t \in [0, 1] \}$
**generalized inequality** defined by a proper cone $K$:

\[ x \preceq_K y \iff y - x \in K, \quad x \prec_K y \iff y - x \in \text{int } K \]

**examples**

- componentwise inequality ($K = \mathbb{R}^n_+$)
  
  \[ x \preceq_{\mathbb{R}^n_+} y \iff x_i \leq y_i, \quad i = 1, \ldots, n \]

- matrix inequality ($K = \mathbb{S}^n_+$)
  
  \[ X \preceq_{\mathbb{S}^n_+} Y \iff Y - X \text{ positive semidefinite} \]

these two types are so common that we drop the subscript in $\preceq_K$

**properties:** many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$, e.g.,

\[ x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v \]
Dual cones and generalized inequalities

**Dual cone** of a cone $K$:

$$K^* = \{ y | y^T x \geq 0 \text{ for all } x \in K \}$$

**Examples**

- $K = \mathbb{R}^n_+ : K^* = \mathbb{R}^n_+$
- $K = \mathbb{S}^n_+ : K^* = \mathbb{S}^n_+$
- $K = \{ (x, t) | \|x\|_2 \leq t \} : K^* = \{ (x, t) | \|x\|_2 \leq t \}$
- $K = \{ (x, t) | \|x\|_1 \leq t \} : K^* = \{ (x, t) | \|x\|_\infty \leq t \}$

First three examples are **self-dual** cones

Dual cones of proper cones are proper, hence define generalized inequalities:

$$y \geq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \geq_K 0$$
Minimum and minimal elements

\( \preceq_K \) is not in general a linear ordering: we can have \( x \not\preceq_K y \) and \( y \not\preceq_K x \)

\( x \in S \) is **the minimum element** of \( S \) with respect to \( \preceq_K \) if

\[
y \in S \quad \implies \quad x \preceq_K y
\]

\( x \in S \) is **a minimal element** of \( S \) with respect to \( \preceq_K \) if

\[
y \in S, \quad y \preceq_K x \quad \implies \quad y = x
\]

**example** \((K = \mathbb{R}^2_+)\)

\( x_1 \) is the minimum element of \( S_1 \)

\( x_2 \) is a minimal element of \( S_2 \)
Minimum and minimal elements via dual inequalities

**minimum element** w.r.t. $\preceq_K$

$x$ is minimum element of $S$ iff for all $\lambda \succ_K 0$, $x$ is the unique minimizer of $\lambda^T z$ over $S$

**minimal element** w.r.t. $\preceq_K$

- if $x$ minimizes $\lambda^T z$ over $S$ for some $\lambda \succ_K 0$, then $x$ is minimal
- if $x$ is a minimal element of a convex set $S$, then there exists a nonzero $\lambda \succeq_K 0$ such that $x$ minimizes $\lambda^T z$ over $S$
optimal production frontier

- different production methods use different amounts of resources
  \[ x \in \mathbb{R}^n \]

- production set \( P \) : resource vectors \( x \) for all possible production methods

- efficient (Pareto optimal) methods correspond to resource vectors \( x \) that are minimal w.r.t. \( \mathbb{R}^n_+ \)

example \((n = 2)\)
\( x_1, x_2, x_3 \) are efficient; \( x_4, x_5 \) are not