Lecture: Algorithms for LP, SOCP and SDP

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some parts are based on Prof. Farid Alizadeh lecture notes

Outline

1. Properties of LP
2. Primal Simplex method
3. Dual Simplex method
4. Interior Point method

Standard form LP

$$
\textbf{(P)} \quad \begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}
\qquad\qquad
\textbf{(D)} \quad \begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \geq 0 \end{aligned}
$$

- KKT condition

$$
\begin{aligned}
Ax &= b, \quad x \geq 0 \\
A^\top y + s &= c, \quad s \geq 0 \\
x_i s_i &= 0 \quad \text{for } i = 1, \ldots, n
\end{aligned}
$$

- **Strong duality**: If a LP has an optimal solution, so does its dual, and their objective fun. are equal.

dual \ primal	finite	unbounded	infeasible
finite	$\surd$	$\times$	$\times$
unbounded	$\times$	$\times$	$\surd$
infeasible	$\times$	$\surd$	$\surd$

Geometry of the feasible set

- Assume that $A \in \mathbb{R}^{m\times n}$ has **full row rank.** Let A_i be the ith column of A:

$$A = \begin{pmatrix} A_1 & A_2 & \dots & A_n \end{pmatrix}$$

- A vector x is a **basic feasible solution (BFS)** if x is feasible and there exists a subset $\mathcal{B} \subset \{1, 2, \dots, n\}$ such that
  - $\mathcal{B}$ contains exactly m indices
  - $i \notin \mathcal{B} \implies x_i = 0$
  - The $m \times m$ submatrix $B = [A_i]_{i\in\mathcal{B}}$ is nonsingular

  $\mathcal{B}$ is called a basis and B is called the basis matrix

Properties:

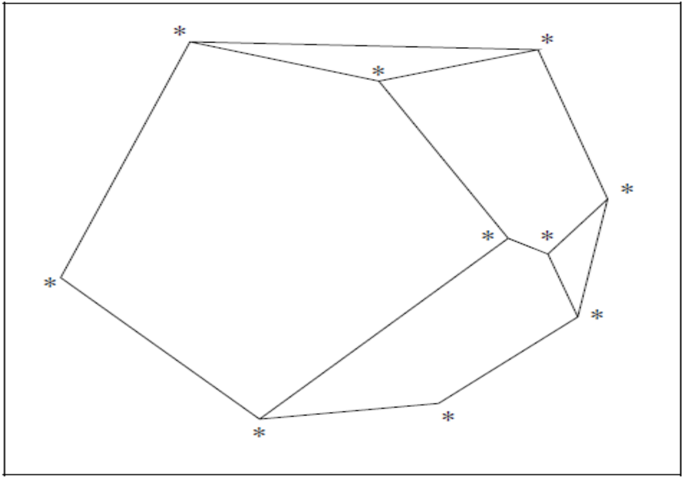
- If (P) has a nonempty feasible region, then there is at least one basic feasible point;
- If (P) has solutions, then at least one such solution is a basic optimal point.
- If (P) is feasible and bounded, then it has an optimal solution.

If (P) has a nonempty feasible region, then there is at least one BFS;

- Choose a feasible x with the minimal number (p) of nonzero x_i: $\sum_{i=1}^{p} A_i x_i = b$
- Suppose that $A_1, \ldots, A_p$ are linearly dependent $A_p = \sum_{i=1}^{p-1} z_i A_i$. Let $x(\epsilon) = x + \epsilon(z_1, \ldots, z_{p-1}, -1, 0, \ldots, 0)^\top = x + \epsilon z$. Then $Ax(\epsilon) = b$, $x_i(\epsilon) > 0$, $i = 1, \ldots, p$, for ϵ sufficiently small. There exists $\bar{\epsilon}$ such that $x_i(\bar{\epsilon}) = 0$ for some $i = 1, \ldots, p$. Contradiction to the choice of x.
- If $p = m$, done. Otherwise, choose $m - p$ columns from among $A_{p+1}, \ldots, A_n$ to build up a set set of m linearly independent vectors.

Polyhedra, extreme points, vertex, BFS

- A **Polyhedra** is a set that can be described in the form $\{x \in \mathbb{R}^n \mid Ax \geq b\}$
- Let P be a polyhedra. A vector $x \in P$ is an **extreme point** if we cannot find two vectors $y, z \in P$ (both different from x) such that $x = \lambda y + (1 - \lambda)z$ for $\lambda \in [0, 1]$
- Let P be a polyhedra. A vector $x \in P$ is a **vertex** if there exists some c such that $c^\top x < c^\top y$ for all $y \in P$ and $y \neq x$
- Let P be a nonempty polyhedra. Let $x \in P$. The following statements are equivalent: (i) x is vertex; (ii) x is an extreme point; (iii) x is a BFS
- A basis $\mathcal{B}$ is said to be **degenerate** if $x_i = 0$ for some $i \in \mathcal{B}$, where x is the BFS corresponding to $\mathcal{B}$. A linear program (P) is said to be degenerate if it has at least one degenerate basis.



Vertices of a three-dimensional polyhedron (indicated by $*$)

Outline

The Simplex Method For LP

Basic Principle

Move from a BFS to its adjacent BFS unitil convergence (either optimal or unbounded)

- Let x be a BFS and $\mathcal{B}$ be the corresponding basis
- Let $\mathcal{N} = \{1, 2, \ldots, n\} \backslash \mathcal{B}$, $N = [A_i]_{i \in \mathcal{N}}$, $x_B = [x_i]_{i \in \mathcal{B}}$ and $x_N = [x_i]_{i \in \mathcal{N}}$
- Since x is a BFS, then $x_N = 0$ and $Ax = Bx_B + Nx_N = b$:

$$x_B = B^{-1} b$$

- Find exactly one $q \in \mathcal{N}$ and exactly one $p \in \mathcal{B}$ such that

$$\mathcal{B}^+ = \{q\} \cup (\mathcal{B} \backslash \{p\})$$

Finding $q \in \mathcal{N}$ to enter the basis

Let x^+ be the new BFS:

$$x^+ = \begin{pmatrix} x^+_{\mathcal{B}} \\ x^+_{\mathcal{N}} \end{pmatrix}, \quad Ax^+ = b \Longrightarrow x^+_{\mathcal{B}} = B^{-1}b - B^{-1}Nx^+_{\mathcal{N}}$$

The cost at x^+ is

$$\begin{aligned} c^\top x^+ &= c_B^\top x^+_{\mathcal{B}} + c_N^\top x^+_{\mathcal{N}} \\ &= c_B^\top B^{-1}b - c_B^\top B^{-1}Nx^+_{\mathcal{N}} + c_N^\top x^+_{\mathcal{N}} \\ &= c^\top x + (c_N^\top - c_B^\top B^{-1}N)x^+_{\mathcal{N}} \\ &= c^\top x + \sum_{j \in \mathcal{N}} \underbrace{(c_j - c_B^\top B^{-1}A_j)}_{s_j} x^+_j \end{aligned}$$

- s_j is also called **reduced cost.** It is actually the dual slackness
- If $s_j \geq 0$, $\forall j \in \mathcal{N}$, then x is optimal as $c^\top x^+ \geq c^\top x$
- Otherwise, find q such that $s_q < 0$. Then $c^\top x^+ = c^\top x + s_q x^+_q \leq c^\top x$

Finding $p \in \mathcal{B}$ to exit the basis

What is x^+: select $q \in \mathcal{N}$ and $p \in \mathcal{B}$ such that

$$x_{\mathcal{B}}^+ = B^{-1}b - B^{-1}A_q x_q^+, \quad x_q^+ \geq 0, x_p^+ = 0, x_j^+ = 0, j \in \mathcal{N} \setminus \{q\}$$

Let $u = B^{-1}A_q$. Then $x_{\mathcal{B}}^+ = x_{\mathcal{B}} - u x_q^+$

- If $u \leq 0$, then $c^\top x^+ = c^\top x + s_q x_q^+ \to -\infty$ as $x_q^+ \to +\infty$ and x^+ is feasible. **(P) is unbounded**

- If $\exists u_k > 0$, then find x_q^+ and p such that

  $$x_{\mathcal{B}}^+ = x_{\mathcal{B}} - u x_q^+ \geq 0, \quad x_p^+ = 0$$

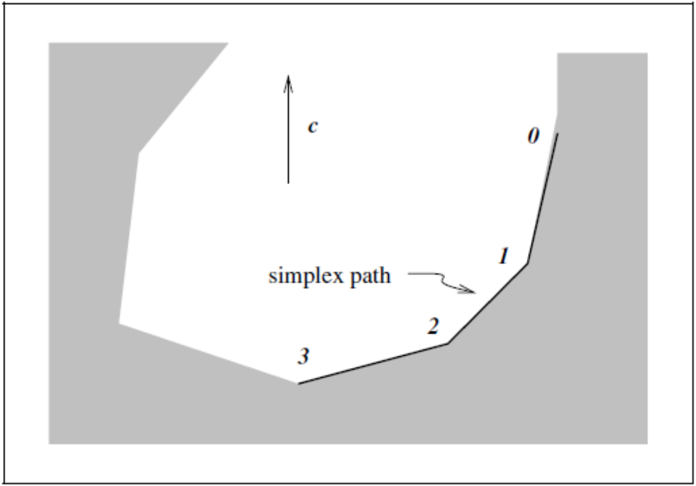
  Let p be the index corresponding to

  $$x_q^+ = \min_{i=1,\ldots,m | u_i > 0} \frac{x_{\mathcal{B}(i)}}{u_i}$$

An iteration of the simplex method

Typically, we start from a BFS x and its associate basis $\mathcal{B}$ such that $x_B = B^{-1}b$ and $x_N = 0$.

- Solve $y^\top = c_B^\top B^{-1}$ and then the reduced costs $s_N = c_N - N^\top y$
- If $s_N \geq 0$, x is optimal and stop; Else, choose $q \in \mathcal{N}$ with $s_q < 0$.
- Compute $u = B^{-1}A_q$. If $u \leq 0$, then (P) is unbounded and stop.
- If $\exists u_k > 0$, then find $x_q^+ = \min_{i=1,\dots,m | u_i > 0} \frac{x_{\mathcal{B}(i)}}{u_i}$ and use p to denote the minimizing i. Set $x_{\mathcal{B}}^+ = x_{\mathcal{B}} - u x_q^+$.
- Change $\mathcal{B}$ by adding q and removing the basic variable corresponding to column p of B.



Simplex iterates for a two-dimensional problem

Finite Termination of the simplex method

Theorem

Suppose that the LP (P) is nondegenerate and bounded, the simplex method terminates at a basic optimal point.

- nondegenerate: $x_{\mathcal{B}} > 0$ and $c^\top x$ is bounded
- A strict reduction of $c^\top x$ at each iteration
- There are only a finite number of BFS since the number of possible bases $\mathcal{B}$ is finite (there are only a finite number of ways to choose a subset of m indices from $\{1, 2, \ldots, n\}$), and since each basis defines a single basic feasible point

Finite termination does not mean a polynomial-time algorithm

Linear algebra in the simplex method

- Given B^{-1}, we need to compute $\bar{B}^{-1}$, where

$$B = [A_1, \ldots, A_m], \quad \bar{B} := B^+ = [A_1, \ldots, A_{p-1}, A_q, A_{p+1}, \ldots, A_m]$$

- the cost of inversion $\bar{B}^{-1}$ from scratch is $O(m^3)$
- Since $BB^{-1} = I$, we have

$$\begin{aligned} B^{-1}\bar{B} &= [e_1, \ldots e_{p-1}, u, e_{p+1}, \ldots, e_m] \\ &= \begin{pmatrix} 1 & & u_1 & & \\ & \ddots & \vdots & & \\ & & u_p & & \\ & & \vdots & \ddots & \\ & & u_m & & 1 \end{pmatrix}, \end{aligned}$$

where e_i is the ith column of I and $u = B^{-1}A_q$

Linear algebra in the simplex method

- Apply a sequence of "elementary row operation"
  - For each $j \neq p$, we add the p-th row times $-\frac{u_j}{u_p}$ to the jth row. This replaces u_j by zero.
  - We divide the pth row by u_p. This replaces u_p by one.

$$Q_{ip} = I + D_{ip}, \quad (D_{ip})_{jl} = \begin{cases} -\frac{u_j}{u_p}, & (j,l) = (i,p) \\ 0, & \text{otherwise} \end{cases}, \text{ for } i \neq p$$

- Find Q such that $QB^{-1}\bar{B} = I$. Computing $\bar{B}^{-1}$ needs only $O(m^2)$
- What if B^{-1} is computed by the LU factorization, i.e., $B = LU$?
  L is is unit lower triangular, U is upper triangular.
  Read section 13.4 in "Numerical Optimization", Jorge Nocedal and Stephen Wright,

An iteration of the revised simplex method

Typically, we start from a BFS x and its associate basis $\mathcal{B}$ such that $x_B = B^{-1}b$ and $x_N = 0$.

- Solve $y^\top = c_B^\top B^{-1}$ and then the reduced costs $s_N = c_N - N^\top y$
- If $s_N \geq 0$, x is optimal and stop; Else, choose $q \in \mathcal{N}$ with $s_q < 0$.
- Compute $u = B^{-1}A_q$. If $u \leq 0$, then (P) is unbounded and stop.
- If $\exists u_k > 0$, then find $x_q^+ = \min\limits_{i=1,\dots,m|u_i>0} \frac{x_{\mathcal{B}(i)}}{u_i}$ and use p to denote the minimizing i. Set $x_{\mathcal{B}}^+ = x_{\mathcal{B}} - u x_q^+$.
- Form the $m \times (m+1)$ matrix $[B^{-1} \mid u]$. Add to each one of its rows a multiple of the pth row to make the last column equal to the unit vector e_p. The first m columns of the result is the matrix $\bar{B}^{-1}$.

Selection of the entering index (pivoting rule)

Reduced costs $s_N = c_N - N^\top y$, $c^\top x^+ = c^\top x + s_q x_q^+$

- Dantzig: chooses $q \in \mathcal{N}$ such that s_q is the most negative component
- Bland's rule: choose the smallest $j \in \mathcal{N}$ such that $s_j < 0$; out of all variables x_i that are tied in the test for choosing an exiting variable, select the one with with the smallest value i.
- Steepest-edge: choose $q \in \mathcal{N}$ such that $\frac{c^\top \eta_q}{\|\eta_q\|}$ is minimized, where

$$x^+ = \begin{pmatrix} x_B^+ \\ x_N^+ \end{pmatrix} = \begin{pmatrix} x_B \\ x_N \end{pmatrix} + \begin{pmatrix} -B^{-1}A_q \\ e_q \end{pmatrix} x_q = x + \eta_q x_q^+$$

  efficient computation of this rule is available

Degenerate steps and cycling

Let q be the entering variable:

$$x_{\mathcal{B}}^{+} = B^{-1}b - B^{-1}A_q x_q^{+} = x_B - x_q^{+}u, \text{ where } u = B^{-1}A_q$$

- Degenerate step: there exists $i \in \mathcal{B}$ such that $x_i = 0$ and $u_i > 0$. Then $x_i^{+} < 0$ if $x_q^{+} > 0$. Hence, $x_q^{+} = 0$ and do the pivoting
- Degenerate step may still be useful because they change the basis $\mathcal{B}$, and the updated $\mathcal{B}$ may be closer to the optimal basis.
- cycling: after a number of successive degenerate steps, we may return to an earlier basis $\mathcal{B}$
- Cycling has been observed frequently in the large LPs that arise as relaxations of integer programming problems
- Avoid cycling: Bland's rule and Lexicographically pivoting rule

Finding an initial BFS

The two-phase simplex method

$$\textbf{(P)} \quad \begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad \textbf{(P0)} \quad \begin{aligned} \tilde{f} = \min \quad & z_1 + z_2 + \ldots + z_m \\ \text{s.t.} \quad & Ax + z = b \\ & x \geq 0, \quad z \geq 0 \end{aligned}$$

- A BFS to (P0): $x = 0$ and $z = b$
- If x is feasible to (P), then $(x, 0)$ is feasible to (P0)
- If the optimal cost $\tilde{f}$ of (P0) is nonzero, then (P) is infeasible
- If $\tilde{f} = 0$, then its optimal solution must satisfies: $z = 0$ and x is feasible to (P)
- An optimal basis $\mathcal{B}$ to (P0) may contain some components of z

Finding an initial BFS

(x, z) is optimal to (P0) with some components of z in the basis

- Assume $A_1, \ldots, A_k$ are in the basis matrix with $k < m$. Then

$$B = [A_1, \ldots, A_k \mid \text{ some columns of } I]$$

$$B^{-1}A = [e_1, \ldots, e_k, B^{-1}A_{k+1}, \ldots, B^{-1}A_n]$$

- Suppose that ℓth basic variable is an artificial variable

- If the ℓth row of $B^{-1}A$ is zero, then $g^\top A = 0^\top$, where $g^\top$ is the ℓth row of B^{-1}. If $g^\top b \neq 0$, (P) is infeasible. Otherwise, A has linearly dependent rows. Remove the ℓth row.

- There exists j such that the ℓth entry of $B^{-1}A_j$ is nonzero. Then A_j is linearly independent to $A_1, \ldots, A_k$. Perform elementary row operation to replace $B^{-1}A_j$ to be the ℓth unit vector. Driving one of z out of the basis

The primal simplex method for LP

$$\textbf{(P)}\quad \begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = b \\ & x \ge 0 \end{array} \qquad \textbf{(D)}\quad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y + s = c \\ & s \ge 0 \end{array}$$

- KKT condition

$$\begin{array}{rcll} Ax & = & b, & x \ge 0 \\ A^\top y + s & = & c, & s \ge 0 \\ x_i s_i & = & 0 & \text{for } i = 1, \ldots, n \end{array}$$

- The primal simplex method generates

$$\begin{array}{rcl} x_B & = & B^{-1} b \ge 0, \quad x_N = 0, \\ y & = & B^{-T} c_B, \\ s_B & = & c_B - B^\top y = 0, s_N = c_N - N^\top y ? 0 \end{array}$$

Outline

The dual simplex method for LP

- The dual simplex method generates

$$\begin{aligned} x_B &= B^{-1}b ? 0, \quad x_N = 0, \\ y &= B^{-T}c_B, \\ s_B &= c_B - B^\top y = 0, s_N = c_N - N^\top y \geq 0 \end{aligned}$$

- If $x_B \geq 0$, then (x, y, s) is optimal
- Otherwise, select $q \in \mathcal{B}$ such that $x_q < 0$ to exit the basis, select $r \in \mathcal{N}$ to enter the basis, i.e., $s_r^+ = 0$
- The update is of the form

$$\begin{aligned} s_B^+ &= s_B + \alpha e_q \quad \text{obvious} \\ y^+ &= y + \alpha v \quad \text{requirement} \end{aligned}$$

The dual simplex method for LP

- What is v? Since $A^\top y^+ + s^+ = c$, it holds

$$s_B^+ = c_B - B^\top y^+$$
$$\Longrightarrow \quad s_B + \alpha e_q = c_B - B^\top (y + \alpha v) \Longrightarrow e_q = -B^\top v$$

- The update of the dual objective function

$$\begin{aligned} b^\top y^+ &= b^\top y + \alpha b^\top v \\ &= b^\top y - \alpha b^\top B^{-T} e_q \\ &= b^\top y - \alpha x_B^\top e_q \\ &= b^\top y - \alpha x_q \end{aligned}$$

- Since $x_q < 0$ and we maximize $b^\top y^+$, we choose α as large as possible, but require $s_N^+ \geq 0$

The dual simplex method for LP

- Let $w = N^\top v = -N^\top B^{-T} e_q$. Since $Ay + s = c$ and $A^\top y^+ + s^+ = c$, it holds

$$s_N^+ = c_N - N^\top y^+ = s_N - \alpha N^\top v = s_N - \alpha w \geq 0$$

- The largest α is

$$\alpha = \min_{j \in \mathcal{N}, w_j > 0} \frac{s_j}{w_j}.$$

Let r be the index at which the minimum is achieved.

$$s_r^+ = 0, \quad w_r = A_r^\top v > 0$$

- (D) is unbounded if $w \leq 0$

The dual simplex method for LP: update of x^+

We have: $Bx_B = b$, $x_q^+ = 0$, $x_r^+ = \gamma$ and $Ax^+ = b$, i.e.,

$$Bx_{\mathcal{B}}^+ + \gamma A_r = b \Longrightarrow x_{\mathcal{B}}^+ = B^{-1}b - \gamma B^{-1}A_r,$$

where $Bd = A_r$. Then $Ax^+ = b$ gives

$$B(x_{\mathcal{B}} - \gamma d) + \gamma A_r = b \text{ for any } \gamma.$$

Since it is required $x_q^+ = 0$, we set

$$\gamma = \frac{x_q}{d_q}, \text{ where } d_q = d^\top e_q = A_r^\top B^{-T} e_q = -A_r^\top v = -w_r < 0.$$

Therefore

$$x_i^+ = \begin{cases} x_i - \gamma d_i, & \text{for } i \in \mathcal{B} \text{ with } i \neq q, \\ 0, & i = q, \\ 0, & i \in \mathcal{N} \text{ with } i \neq r, \\ \gamma, & i = r \end{cases}$$

An iteration of the dual simplex method

Typically, we start from a dual feasible (y, s) and its associate basis $\mathcal{B}$ such that $x_B = B^{-1}b$ and $x_N = 0$.

- If $x_B \geq 0$, then x is optimal and stop. Else, choose q such that $x_q < 0$.
- Compute $v = -B^{-T}e_q$ and $w = N^\top v$. If $w \leq 0$, then (D) is unbounded and stop.
- If $\exists w_k > 0$, then find $\alpha = \min_{j \in \mathcal{N}, w_j > 0} \frac{s_j}{w_j}$ and use r to denote the minimizing j. Set $s_B^+ = s_B + \alpha e_q$, $s_N^+ = s_N - \alpha w$ and $y^+ = y + \alpha v$.
- Change $\mathcal{B}$ by adding r and removing the basic variable corresponding to column q of B.

Outline

Primal-Dual Methods for LP

$$
\textbf{(P)} \quad \begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad\qquad \textbf{(D)} \quad \begin{aligned} \max \quad & b^\top y \\ \text{s.t.} \quad & A^\top y + s = c \\ & s \geq 0 \end{aligned}
$$

- KKT condition

$$
\begin{aligned} Ax &= b, \quad x \geq 0 \\ A^\top y + s &= c, \quad s \geq 0 \\ x_i s_i &= 0 \quad \text{for } i = 1, \ldots, n \end{aligned}
$$

- Perturbed system

$$
\begin{aligned} Ax &= b, \quad x \geq 0 \\ A^\top y + s &= c, \quad s \geq 0 \\ x_i s_i &= \sigma \mu \quad \text{for } i = 1, \ldots, n \end{aligned}
$$

Newton's method

- Let (x, y, s) be the current estimate with $(x, s) > 0$
- Let $(\Delta x, \Delta y, \Delta s)$ be the search direction
- Let $\mu = \frac{1}{n} x^\top s$ and $\sigma \in (0, 1)$. Hope to find

$$
\begin{aligned}
A(x + \Delta x) &= b \\
A^\top (y + \Delta y) + s + \Delta s &= c \\
(x_i + \Delta x_i)(s_i + \Delta s_i) &= \sigma \mu
\end{aligned}
$$

- dropping the nonlinaer term $\Delta x_i \Delta s_i$ gives

$$
\begin{aligned}
A \Delta x &= r_p := b - Ax \\
A^\top \Delta y + \Delta s &= r_d := c - A^\top y - s \\
x_i \Delta s_i + \Delta x_i s_i &= (r_c)_i := \sigma \mu - x_i s_i
\end{aligned}
$$

Newton's method

- Let $\mathsf{L}_x = \mathrm{Diag}(x)$ and $\mathsf{L}_s = \mathrm{Diag}(s)$. The matrix form is:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ \mathsf{L}_s & 0 & \mathsf{L}_x \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = \begin{pmatrix} r_p \\ r_d \\ r_c \end{pmatrix}$$

- Solving this system we get

$$\begin{aligned} \Delta y &= (A\mathsf{L}_s^{-1}\mathsf{L}_x A^\top)^{-1}(r_p + A\mathsf{L}_s^{-1}(\mathsf{L}_x r_d - r_c)) \\ \Delta s &= r_d - A^\top \Delta y \\ \Delta x &= -\mathsf{L}_s^{-1}(\mathsf{L}_x \Delta s - r_c) \end{aligned}$$

- The matrix $A\mathsf{L}_s^{-1}\mathsf{L}_x A^\top$ is symmetric and positive definite if A is full rank

The Primal-Dual Path-following Method

Given (x^0, y^0, s^0) with $(x^0, s^0) \geq 0$. A typical iteration is

- Choose $\mu = (x^k)^\top s^k/n$, $\sigma \in (0,1)$ and solve

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ \mathsf{L}_{s^k} & 0 & \mathsf{L}_{x^k} \end{pmatrix} \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} r_p^k \\ r_d^k \\ r_c^k \end{pmatrix}$$

- Set

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha_k(\Delta x^k, \Delta y^k, \Delta s^k),$$

  choosing α_k such that $(x^{k+1}, s^{k+1}) > 0$

The choices of centering parameter σ and step length α_k are crucial to the performance of the method.

The Central Path

- The primal-dual feasible and strictly feasible sets:

$$\begin{aligned} \mathcal{F} &= \{(x,y,s) \mid Ax = b, A^\top y + s = c, (x,s) \geq 0\} \\ \mathcal{F}^o &= \{(x,y,s) \mid Ax = b, A^\top y + s = c, (x,s) > 0\} \end{aligned}$$

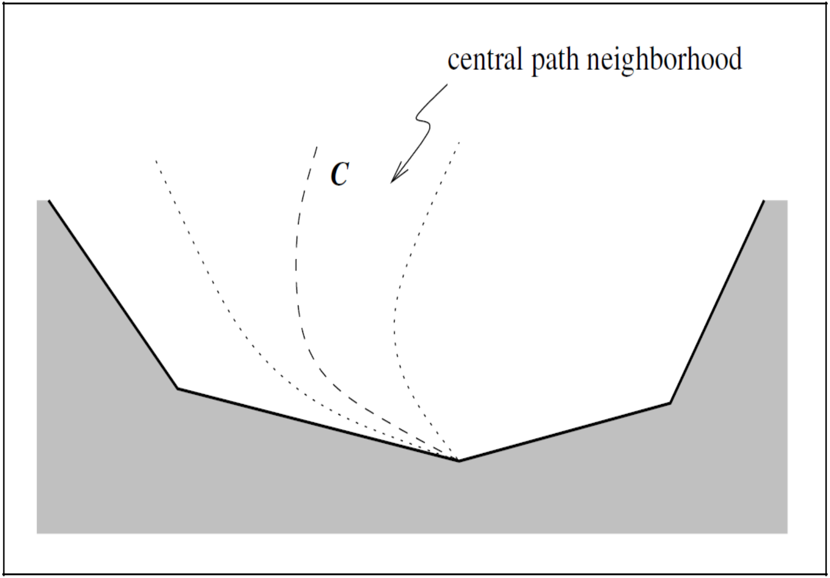
- The central path is $\mathcal{C} = \{(x_\tau, y_\tau, s_\tau) \mid \tau > 0\}$, where

$$\begin{aligned} Ax_\tau &= b, \quad x_\tau > 0 \\ A^\top y_\tau + s_\tau &= c, \quad s_\tau > 0 \\ (x_\tau)_i (s_\tau)_i &= \tau \quad \text{for } i = 1, \ldots, n \end{aligned}$$

- Central path neighborhoods, for $\theta, \gamma \in [0, 1)$:

$$\begin{aligned} \mathcal{N}_2(\theta) &= \{(x,y,s) \in \mathcal{F}^o \mid \|\mathsf{L}_x \mathsf{L}_s e - \mu e\|_2 \leq \theta\mu\} \\ \mathcal{N}_{-\infty}(\gamma) &= \{(x,y,s) \in \mathcal{F}^o \mid x_i s_i \geq \gamma\mu\} \end{aligned}$$

Tyically, $\theta = 0.5$ and $\gamma = 10^{-3}$



Central path, projected into space of primal variables x, showing a typical neighborhood $\mathcal{N}$

The Long-Step Path-following Method

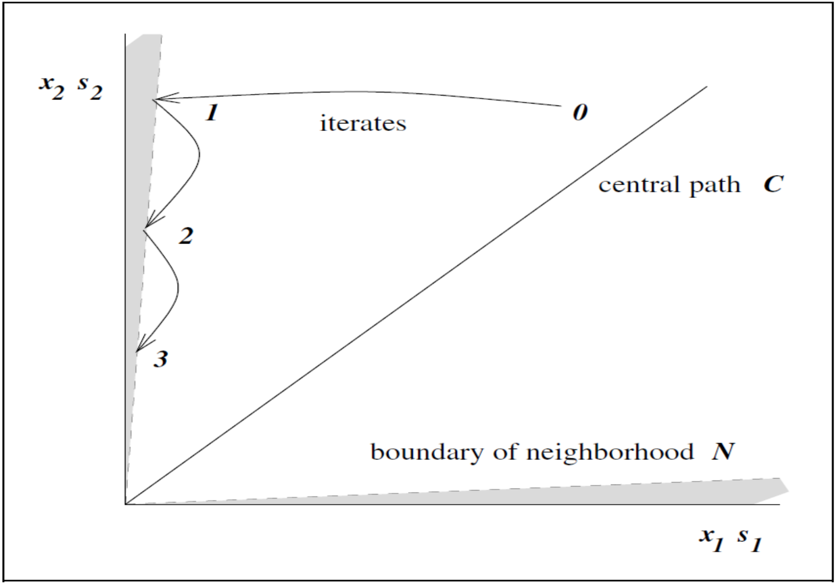
Given $(x^0, y^0, s^0) \in \mathcal{N}_{-\infty}(\gamma)$. A typical iteration is

- Choose $\mu = (x^k)^\top s^k/n$, $\sigma \in (0,1)$ and solve

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ \mathsf{L}_{s^k} & 0 & \mathsf{L}_{x^k} \end{pmatrix} \begin{pmatrix} \Delta x^k \\ \Delta y^k \\ \Delta s^k \end{pmatrix} = \begin{pmatrix} r_p^k \\ r_d^k \\ r_c^k \end{pmatrix}$$

- Set α_k be the largest value of $\alpha \in [0,1]$ such that $(x^{k+1}, y^{k+1}, s^{k+1}) \in \mathcal{N}_{-\infty}(\gamma)$ where

$$(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + \alpha_k(\Delta x^k, \Delta y^k, \Delta s^k),$$


$x_2\ s_2$
1
iterates
0
2
3
central path C
boundary of neighborhood N
$x_1\ s_1$

Analysis of Primal-Dual Path-Following

1. If $(x, y, s) \in \mathcal{N}_{-\infty}(\gamma)$, then $\|\Delta x \circ \Delta s\| \leq 2^{-3/2}(1 + 1/\gamma)n\mu$
2. The long-step path-following method yields
$$\mu_{k+1} \leq \left(1 - \frac{\delta}{n}\right)\mu_k,$$
where $\delta = 2^{3/2}\gamma\frac{1-\gamma}{1+\gamma}\sigma(1-\sigma)$
3. Given $\epsilon, \gamma \in (0, 1)$, suppose that the starting point $(x^0, y^0, s^0) \in \mathcal{N}_{-\infty}(\gamma)$. Then there exists $K = O(nlog(1/\epsilon))$ such that
$$\mu_k \leq \epsilon\mu_0, \quad \text{for all } k \geq K$$

Proof of 3:

$$\begin{aligned} \log(\mu_{k+1}) &\leq \log\left(1 - \frac{\delta}{n}\right) + \log(\mu_k) \\ \log(1+\beta) &\leq \beta, \quad \forall\beta > -1 \end{aligned}$$

Barrier Methods

A general strategy for solving convex optimization problem:

$$
\begin{array}{lll}
(P) & \min & c^\top x \\
& \text{s.t.} & x \in C,
\end{array}
$$

where C is convex. Find a barrier function $b(x)$: Int $C \to \mathbb{R}$

- $b(x)$ is convex on Int C
- for any sequence of points $\{x_i\}$ approaching boundary bd(C), $b(x_i) \to \infty$
- We can replace the problem

$$
(II) \quad \min \quad c^\top x + \mu b(x)
$$

- If x_μ is the optimum of (II) and x^* of (I) then
  - $x_\mu \in$ Int$C$
  - As $\mu \to 0$, $x_\mu \to x^*$

- For the positive orthant $\{x \mid x \geq 0\}$, a barrier is

$$b(x) = -\sum_i \ln(x_i)$$

- For the semidefinite cone $\{X \mid X \succeq 0\}$, a barrier is

$$b(x) = -\ln\det(X)$$

- We will discuss the second order cone shortly

Barriers for LP and SDP

Thus LP can be replaced by

Primal $(\mathrm{P})_\mu$

$$\begin{aligned} \min \quad & c^\top x - \mu \sum_i \ln x_i \\ \text{s.t.} \quad & Ax = b \\ & x > 0 \end{aligned}$$

Dual $(\mathrm{D})_\mu$

$$\begin{aligned} \max \quad & b^\top y + \mu \sum_i \ln s_i \\ \text{s.t.} \quad & A^\top y + s = c \\ & s > 0 \end{aligned}$$

Thus SDP can be replaced by

Primal $(\mathrm{P})_\mu$

$$\begin{aligned} \min \quad & \langle C, X \rangle - \mu \ln \det(X) \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i \\ & X \succ 0 \end{aligned}$$

Dual $(\mathrm{D})_\mu$

$$\begin{aligned} \max \quad & b^\top y + \mu \ln \det(S) \\ \text{s.t.} \quad & \sum_i y_i A_i + S = C \\ & S \succ 0 \end{aligned}$$

Applying standard optimality condition we get

- LP: $\mathcal{L}(x,y) = c^\top x - \mu \sum_i \ln x_i - y^\top (b - Ax)$
- SDP: $\mathcal{L}(x,y) = \langle C, X \rangle - \mu \ln \det(X) - \sum_i y_i (b_i - \langle A_i, X \rangle)$

The Karush-Kuhn-Tucker condition requires that at the optimum

$$\nabla_X \mathcal{L} = 0$$

which translates into

(LP)

$$\nabla_y \mathcal{L} = b - Ax = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = c_i - \frac{\mu}{x_i} - (y^\top A)_i = 0$$

(SDP)

$$\nabla_y \mathcal{L} = (b_i - \langle A_i, X \rangle) = 0$$

$$\nabla_X = C - \mu X^{-1} - \sum_i y_i A_i = 0$$

- In LP: define $s_i = \frac{\mu}{x_i}$, then s is dual feasible
- In SDP: define $S = \mu S^{-1}$, then S is dual feasible

The optimality conditions result in the square system

(LP)	(SDP)
$Ax = b$	$\langle A_i, X\rangle = b_i$
$A^\top y + s = c$	$\sum_i y_i A_i + S = C$
$x_i = \frac{\mu}{s_i}$	$X = \mu S^{-1}$

- In LP: if we write $x_i s_i = \mu$, we get relaxed complementarity
- In SDP: if we write $XS = \mu I$, we get relaxed complementarity

Newton's method for SDP

- Let X, y, S be initial estimates Then
- If we use $XS = \mu I$, ΔX is not symmetric
- Since $X, S \succ 0$ then $XS = \mu I$ iff $X \circ S = \frac{XS+SX}{2} = \mu I$
- Now applying Newton, we get

$$
\begin{aligned}
\langle A_i, X + \Delta X \rangle &= b_i \\
\sum_i (y_i + \Delta y_i) A_i + S + \Delta S &= C \\
(X + \Delta X) \circ (S + \Delta S) &= \mu I
\end{aligned}
$$

Newton's method

Expanding and throwing out nonlinear terms

$$\begin{aligned} \langle A_i, \Delta X\rangle &= (r_p)_i \\ \sum_i \Delta y_i A_i + \Delta S &= R_d \\ S \circ \Delta X + \Delta S \circ X &= R_c \end{aligned}$$

where

$$\begin{aligned} (r_p)_i &= b_i - \langle A_i, X\rangle \\ R_d &= C - \sum_i y_i A_i - S \\ R_c &= \mu I - X \circ S \end{aligned}$$

- In matrix form

$$\begin{pmatrix} \mathcal{A} & 0 & 0 \\ 0 & \mathcal{A}^\top & I \\ \mathcal{L}_S & 0 & \mathcal{L}_X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = \begin{pmatrix} r_p \\ r_d \\ r_c \end{pmatrix}$$

- $vec(A)$ is a vector made by stacking columns of a matrix A
- $\mathcal{A}$ is a matrix whose rows are $vec(A_i)$
- $x = vec(X)$, $s = vec(S)$ ...
- $\mathcal{L}_X$ (and $\mathcal{L}_S$) are matrix representations of L_X (and L_S) operators
- $\mathcal{L}_X = X \otimes I + I \otimes X$ and $\mathcal{L}_S = S \otimes I + I \otimes S$
- Kronecker product: $A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}$

  http://en.wikipedia.org/wiki/Kronecker_product

- Solving this system we get

$$\begin{aligned}
\Delta y &= (\mathcal{A}\mathcal{L}_S^{-1}\mathcal{L}_X\mathcal{A}^\top)^{-1}(r_p + \mathcal{A}\mathcal{L}_S^{-1}(\mathcal{L}_X r_d - r_c)) \\
\Delta s &= r_d - \mathcal{A}^\top \Delta y \\
\Delta x &= -\mathcal{L}_S^{-1}(\mathcal{L}_X \Delta s - r_c)
\end{aligned}$$

- The matrix $\mathcal{A}\mathcal{L}_s^{-1}\mathcal{L}_x\mathcal{A}^\top$ is not symmetric because $\mathcal{L}_S$ and $\mathcal{L}_X$ do not commute!
- In LP, it is quite easy to compute $A\mathsf{L}_s^{-1}\mathsf{L}_x A^\top$
- Most computational work in LP involves solving the system

$$(A\mathsf{L}_s^{-1}\mathsf{L}_x A^\top)v = u$$

- in SDP even computing $\mathcal{A}\mathcal{L}_s^{-1}\mathcal{L}_x\mathcal{A}^\top$ is fairly expensive (in this form requires solving Lyapunov equations)

- How about SOCP?
- What is an appropriate barrier for the convex cone

$$\mathcal{Q} = \{x \mid x_0 \geq \|\bar{x}\|\}?$$

- By analogy we expect relaxed complementary conditions turn out to be $x \circ s = \mu e$

Algebra Associated with SOCP

In SDP

- The barrier $\ln\det(X) = \sum_i \ln\lambda_i(X)$
- For each symmetric $n \times n$ matrix X, there is a characteristic polynomial, such that
  - $p(t) = p_0 + p_1 t + \ldots + p_{n-1}t^{n-1} + t^n$
  - roots of $p(t)$ are eigenvalues of X
  - $\mathrm{Tr}(X) = p_{n-1}$, $\det(X) = p_0$
  - roots of $p(t)$ are real numbers
  - $p(X) = 0$ by Cayley-Hamilton Theorem
  - There is orthogonal matrix Q: $X = Q\Lambda Q^\top = \lambda_1 q_1 q_1^\top + \ldots + \lambda_n q_n q_n^\top$

SOCP

- Remember

$$x \circ s = \begin{pmatrix} x^\top s \\ x_0 \bar{s} + s_0 \bar{x} \end{pmatrix} \quad \mathsf{L}_x = \mathrm{Arw}(x) = \begin{pmatrix} x_0 & \bar{x}^\top \\ \bar{x} & x_0 I \end{pmatrix} \quad e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- It is easy to verify

$$x \circ x - 2x_0 x + (x_0^2 - \|\bar{x}\|^2)e = 0$$

- Define the characteristic polynomial

$$p(t) = t^2 - 2x_0 t + (x_0^2 - \|\bar{x}\|^2) = (t - (x_0 + \|\bar{x}\|)(t - (x_0 - \|\bar{x}\|)$$

- Define eigenvalues of x roots of $p(t)$: $\lambda_{1,2} = x_0 \pm \|\bar{x}\|$
- Define $\mathrm{Tr}(x) = 2x_0$ and $\det(x) = x_0^2 - \|\bar{x}\|^2$

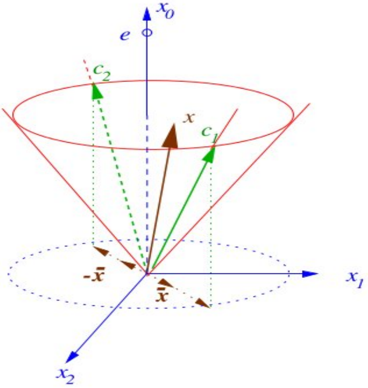
- For each x define

$$c_1 = \frac{1}{2}\begin{pmatrix} 1 \\ \frac{\bar{x}}{\|\bar{x}\|} \end{pmatrix} \quad c_2 = \frac{1}{2}\begin{pmatrix} 1 \\ -\frac{\bar{x}}{\|\bar{x}\|} \end{pmatrix}$$

- We can verify

$$x = \lambda_1 c_1 + \lambda_2 c_2$$

- This relation is the spectral decomposition of the vectors in SOCP Algebra



to get c_1 and c_2, (i) project x to $x_1, \ldots, x_n$ plane, (ii) normalize $\bar{x}$ and $-\bar{x}$ (iii) lift the normalized vectors up to touch the boundary of the cone

- Define for any real number t, $x^t = \lambda_1^t c_1 + \lambda_2^t c_2$ whenever λ_i^t is defined

$$x^{-1} = \frac{1}{\lambda_1} c_1 + \frac{1}{\lambda_2} c_2 = \frac{1}{\det(x)} \begin{pmatrix} x_0 \\ -\bar{x} \end{pmatrix}$$

- Now we can define an appropriate barrier for $\mathcal{Q}$

$$-\ln\det(x) = -\ln(x_0^2 - \|\bar{x}\|^2)$$

$$\nabla_x(-\ln\det x) = \frac{2}{\det(x)} \begin{pmatrix} x_0 \\ -\bar{x} \end{pmatrix} = 2x^{-1}$$

- we can replace SOCP problem with

$$\begin{array}{ll} \max & c^\top x - \mu \ln \det x \\ \text{s.t.} & Ax = b \\ & x \succ_{\mathcal{Q}} 0 \end{array}$$

- The Lagrangian

$$\mathcal{L}(x, y) = c^\top x - \mu \ln \det x - y^\top (b - Ax)$$

- Applying KKT

$$\begin{aligned} b - Ax &= 0 \\ c - \mu x^{-1} - A^\top y &= 0 \end{aligned}$$

- Setting $s = \mu x^{-1}$ we can see that s is dual feasible

Newton's method

Thus we have to solve the following system

$$\begin{aligned} Ax &= b \\ A^\top y + s &= c \\ x \circ s &= 2\mu e \end{aligned}$$

Using Newton's method, we get

$$\begin{aligned} A(x + \Delta x) &= b \\ A^\top (y + \Delta y) + s + \Delta s &= c \\ (x_i + \Delta x_i) \circ (s_i + \Delta s_i) &= 2\mu e \end{aligned}$$

Newton's method

Now expanding and dropping nonlinear terms

$$\begin{aligned} A\Delta x &= b - Ax \\ A^\top \Delta y + \Delta s &= c - A^\top y - s \\ x \circ \Delta s + \Delta x \circ s &= 2\mu e - x \circ s \quad \text{nonlinear term } \Delta x \circ \Delta s \text{ was dropped} \end{aligned}$$

- In matrix form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^\top & I \\ \mathsf{L}_s & 0 & \mathsf{L}_x \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = \begin{pmatrix} r_p \\ r_d \\ r_c \end{pmatrix} \quad \text{where} \quad \begin{aligned} r_p &= b - Ax \\ r_d &= c - A^\top y - s \\ r_c &= 2\mu e - x \circ s \end{aligned}$$