Parallel/Distributed Optimization

http://bicmr.pku.edu.cn/~wenzw/bigdata2018.html

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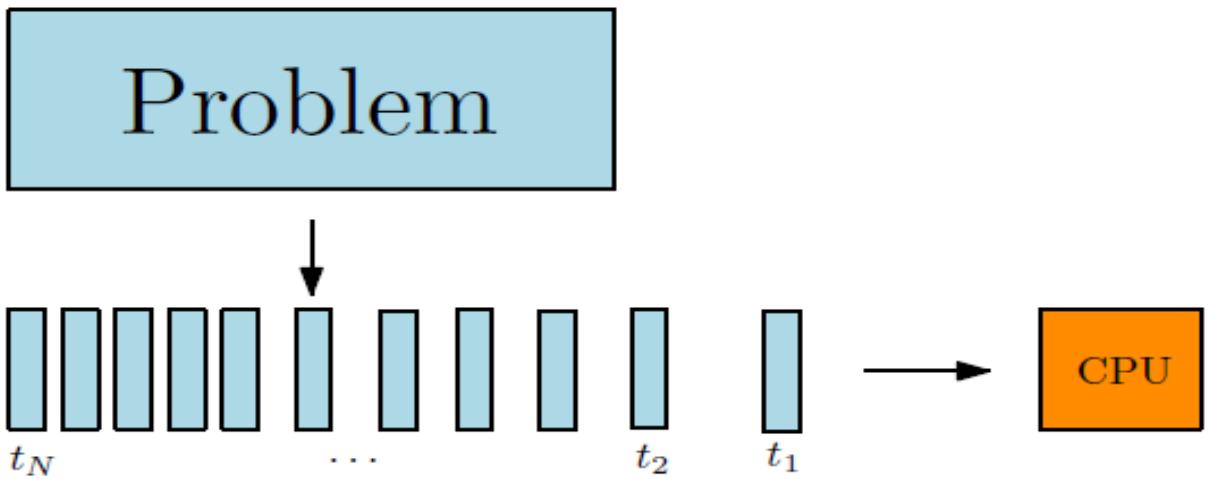
Thanks Haoyang Liu for preparing the part on HOGWILD! and CYCLADES

Outline

Serial computing

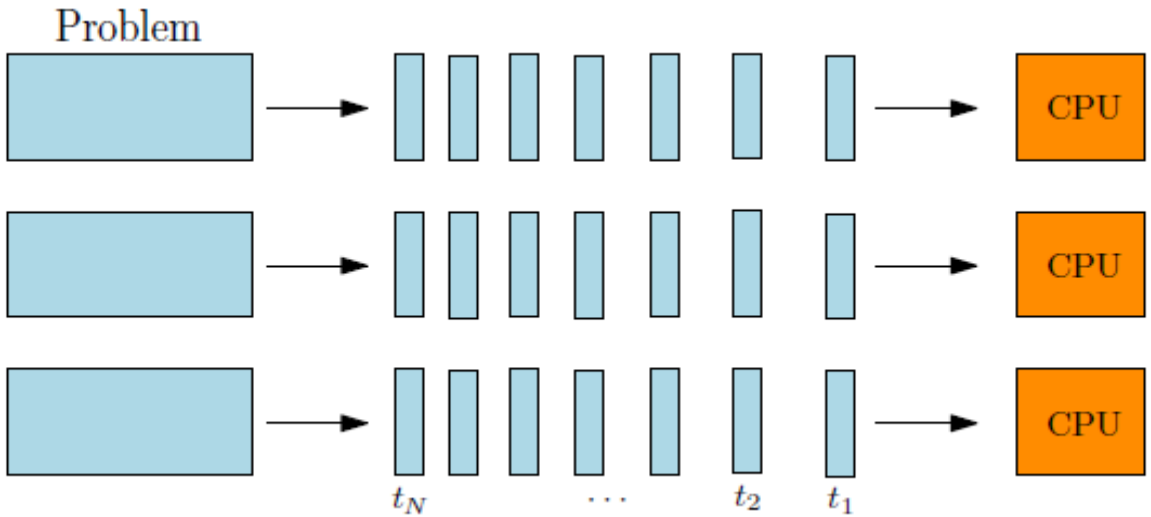
Traditional computing is **serial**

- a single CPU, one instruction is executed at a time
- a problem is broken into a series of instructions, executed one after another



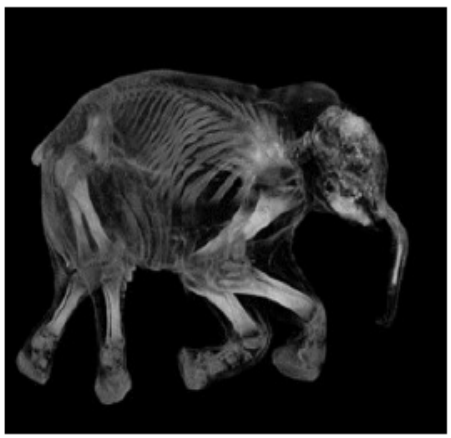
Parallel computing

- a problem is broken into concurrent parts, executed simultaneously, coordinated by a controller
- uses multiple cores/CPUs/networked computers, or their combinations
- expected to solve a problem in less time, or a larger problem



Commercial applications

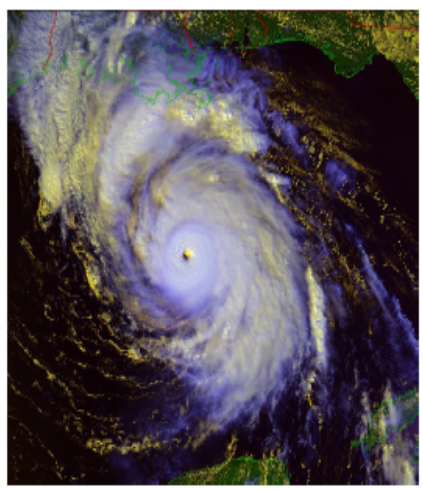
Examples: internet data mining, logistics/supply chain, finance, online advertisement, recommendation system, sales data analysis, virtual reality, computer graphics (cartoon movies), medicine design, medical imaging, network video, ...



Applications in science and engineering

- **simulation**: many real-world events happen concurrently, interrelated
  examples: galaxy, ocean, weather, traffic, assembly line, queues
- **use in science/engineering**: environment, physics, bioscience, chemistry, geoscience, mechanical engineering, mathematics (computerized proofs), defense, intelligence





Benefits of being parallel

- break single-CPU limits
- save time
- process big data
- access non-local resource
- handle concurrent data streams / enable collaboration

Parallel overhead

- **computing overhead**: start up time, synchronization wait time, data communication time, termination (data collection time)
- **I/O overhead**: slow read and write of non-local data
- **algorithm overhead**: extra variables, data duplication
- **coding overhead**: language, library, operating system, debug, maintenance

Different parallel computers

- basic element is like a single computer, a 70-year old architecture components: CPU (control/arithmetic units), memory, input/output
- **SIMD**: single instruction multiple data, GPU-like applications: image filtering, PDE on grid, ...
- **MISD**: multiple instruction single data, rarely seen conceivable applications: cryptography attack, global optimization
- **MIMD**: multiple instruction multiple data most of today's supercomputer, clusters, clouds, multi-core PCs

Tech jargons

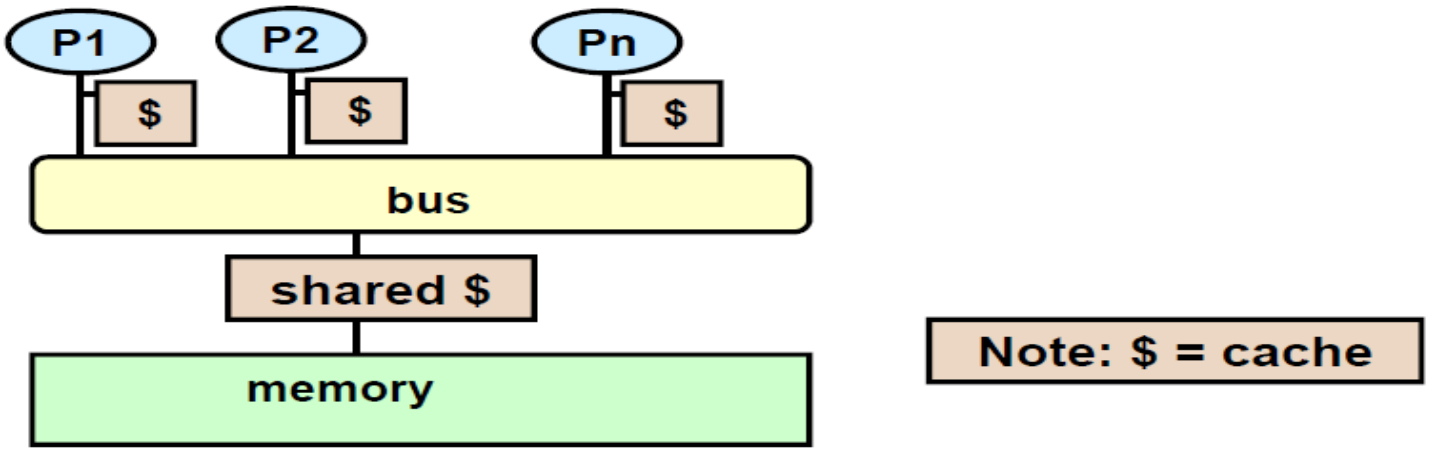
- HPC: high-performance computing
- node/CPU/processor/core
- **shared memory**: either all processors access a common physical memory, or parallel tasks have direct address of logical memory
- SMP: multiple processors share a single memory, shared-memory computing
- **distributed memory**: parallel tasks see local memory and use communication to access remote memory
- communication: exchange of data or synchronization controls; either through shared memory or network
- **synchronization**: coordination, one processor awaits others

Tech jargons

- granularity: coarse — more work between communication events; fine — less work between communication events
- Embarrassingly parallel: main tasks are independent, little need of coordination
- **speed scalability**: how speedup can increases with additional resources. **data scalability**: how problem size can increase with additional resources. **scalability affected by**: problem model, algorithm, memory bandwidth, network speed, parallel overhead. sometimes, adding more processors/memory may not save time!
- **memory architectures**: uniform or non-uniform shared, distributed, hybrid
- **parallel models**: shared memory, message passing, data parallel, SPMD (single program multiple data), MPMD, etc.

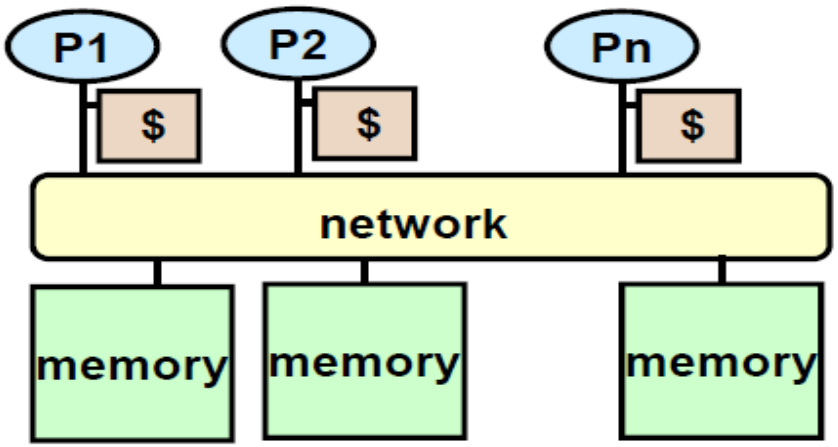
Machine Model: Shared Memory

- Processors all connected to a large shared memory.
  - Typically called Symmetric Multiprocessors (SMPs)
  - SGI, Sun, HP, Intel, IBM SMPs
  - Multicore chips, except that all caches are shared
- Advantage: uniform memory access (UMA)
- Cost: much cheaper to access data in cache than main memory
- Difficulty scaling to large numbers of processors
  $\leq$ 32 processors typical



Machine Model: Distributed Shared Memory

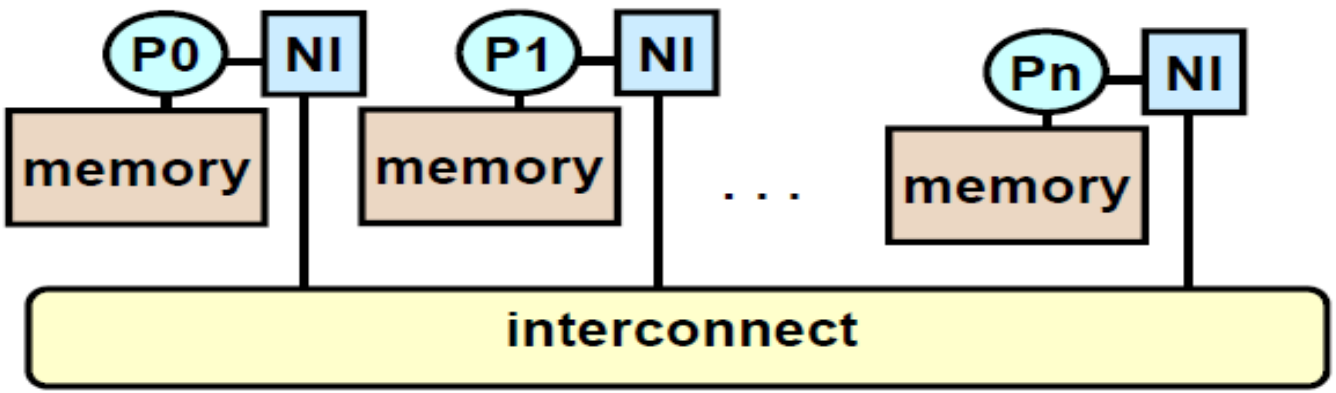
- Memory is logically shared, but physically distributed
  - Any processor can access any address in memory
  - Cache lines (or pages) are passed around machine
- SGI is canonical example (+ research machines)
  - Scales to 512 (SGI Altix (Columbia) at NASA/Ames)
  - Limitation is cache coherency protocols — how to keep cached copies of the same address consistent



Cache lines (pages) must be large to amortize overhead
➔
locality still critical to performance

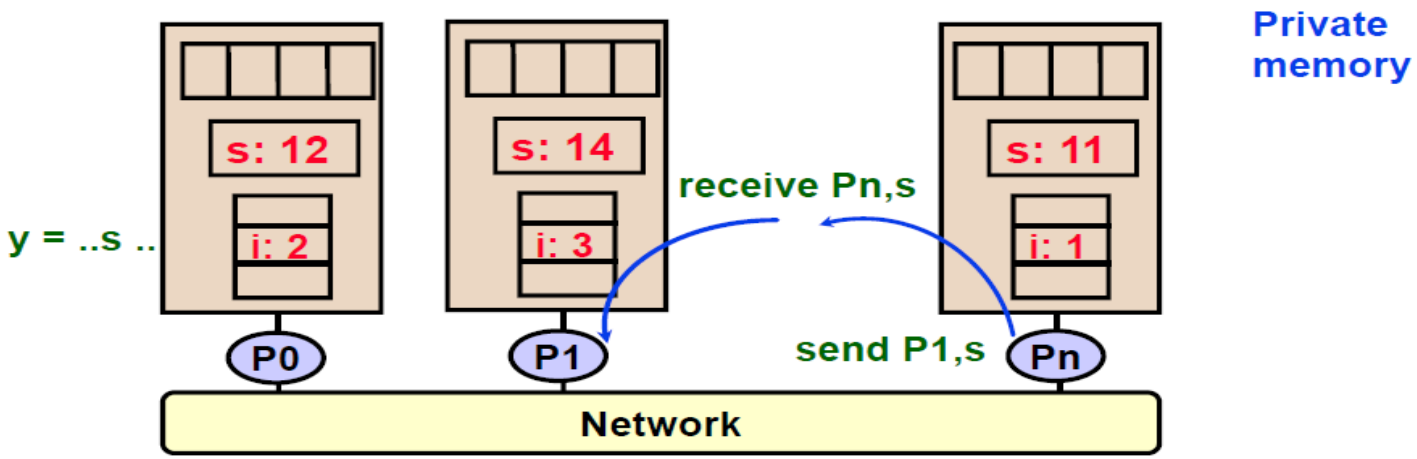
Machine Model: Distributed Memory

- Cray XE6 (Hopper), Cray XC30 (Edison)
- PC Clusters (Berkeley NOW, Beowulf)
- Edison, Hopper, most of the Top500, are distributed memory machines, but the nodes are SMPs.
- Each processor has its own memory and cache but cannot directly access another processor's memory.
- Each "node" has a Network Interface (NI) for all communication and synchronization.



Programming Model: Message Passing

- Program consists of a collection of named processes
  - Usually fixed at program startup time
  - Thread of control plus local address space — NO shared data.
  - Logically shared data is partitioned over local processes
- Processes communicate by explicit send/receive pairs
  - Coordination is implicit in every communication event.
  - MPI (Message Passing Interface) is the most commonly used SW



MPI References

- The Standard itself:
  - at `http://www.mpi-forum.org`
  - All MPI official releases, in both postscript and HTML
  - Latest version MPI 3.0, released Sept 2012
- Other information on Web
  - at `http://www.mcs.anl.gov/mpi`
  - pointers to lots of stuff, including other talks and tutorials, a FAQ, other MPI pages

Simple Example: MPI

```
#include "mpi.h"
#include <stdio.h>
int main( int argc, char *argv[] )
{
    int rank, size;
    MPI_Init( &argc, &argv );
    MPI_Comm_rank( MPI_COMM_WORLD, &rank );
    MPI_Comm_size( MPI_COMM_WORLD, &size );
    printf( "I am %d of %d\n", rank, size );
    MPI_Finalize();
    return 0;
}
```

```
mpirun -np 4 ./mpi_hello_world
I am 1 of 4
I am 0 of 4
I am 2 of 4
I am 3 of 4
```

Notes on Hello World

- All MPI programs begin with MPI_Init and end with MPI_Finalize
- MPI_COMM_WORLD is defined by mpi.h (in C) or mpif.h (in Fortran) and designates all processes in the MPI “job”
- Each statement executes independently in each process including the printf/print statements
- The MPI-1 Standard does not specify how to run an MPI program, but many implementations provide
  mpirun -np 4 a.out

MPI is Simple

Claim: most MPI applications can be written with only 6 functions (although which 6 may differ)

Using point-to-point

- MPI_INIT
- MPI_FINALIZE
- MPI_COMM_SIZE
- MPI_COMM_RANK
- MPI_SEND
- MPI_RECEIVE

Using collectives

- MPI_INIT
- MPI_FINALIZE
- MPI_COMM_SIZE
- MPI_COMM_RANK
- MPI_BCAST
- MPI_REDUCE

You may use more for convenience or performance

Parallel speedup

- definition:

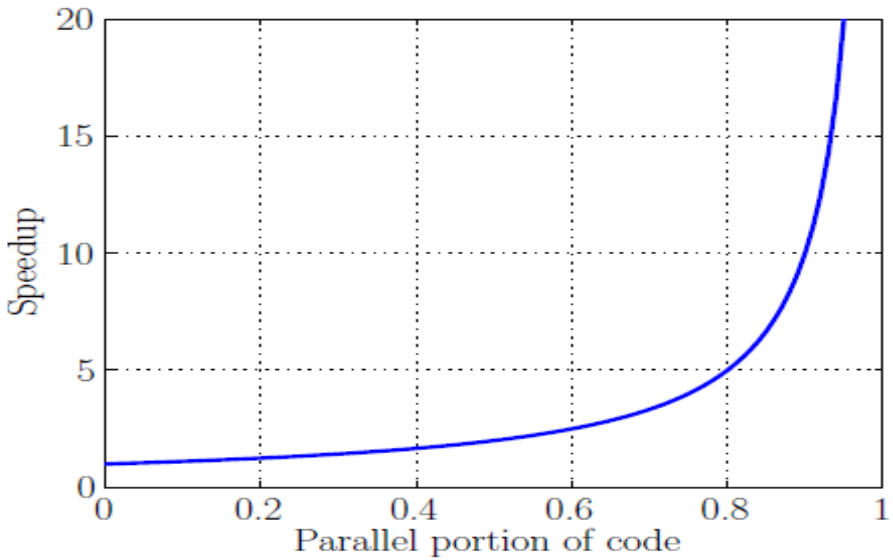
$$\text{speedup} = \frac{\text{serial time}}{\text{parallel time}}$$

  time is in the wall-clock (in contrast to CPU time) sense

- **Amdahl's Law**: assume infinite processors and no overhead

$$\text{ideal max speedup} = \frac{1}{1-\rho}$$

  where $\rho =$ percentage of parallelized code, can depend on input size

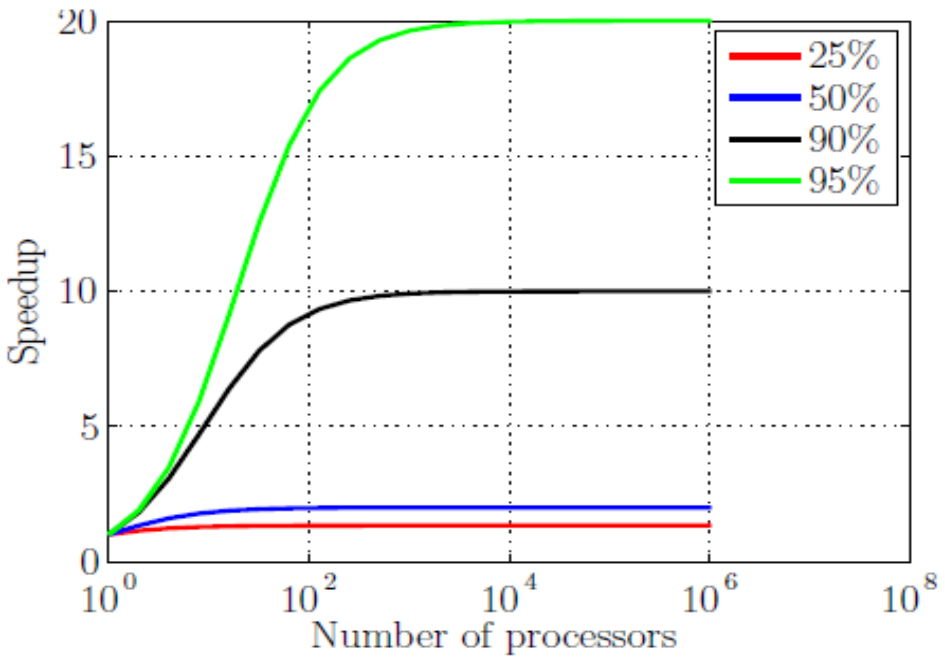


Parallel speedup

- assume N processors and no overhead:

$$\text{ideal speedup} = \frac{1}{(\rho/N) + (1-\rho)}$$

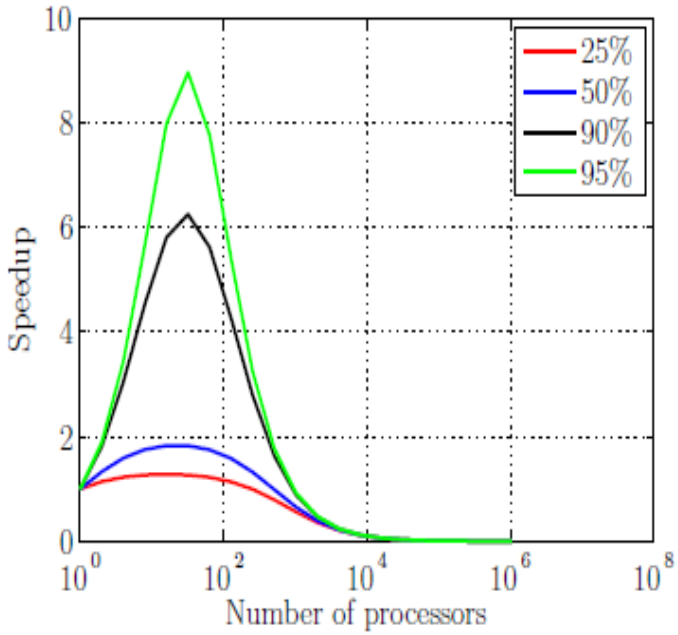
- ρ may also depend on N



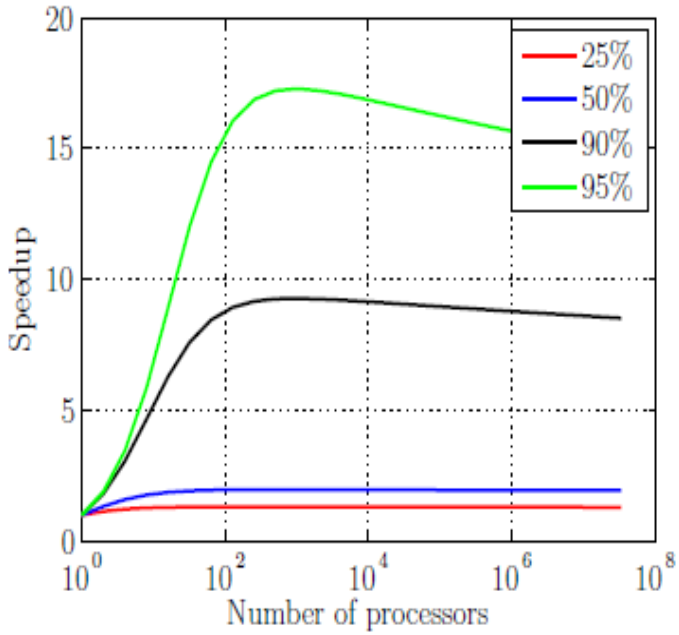
Parallel speedup

- $\epsilon :=$ parallel overhead
- in the real world

$$\text{actual speedup} = \frac{1}{(\rho/N) + (1-\rho) + \epsilon}$$

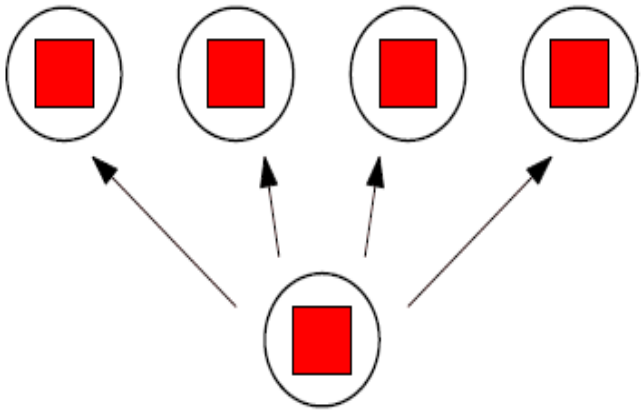


when $\varepsilon = N$

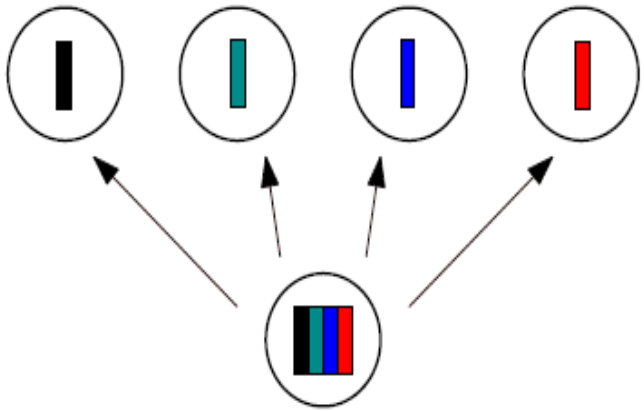


when $\varepsilon = \log(N)$

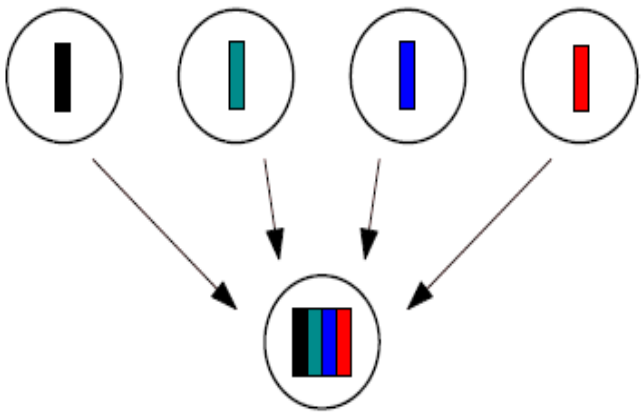
Basic types of communication



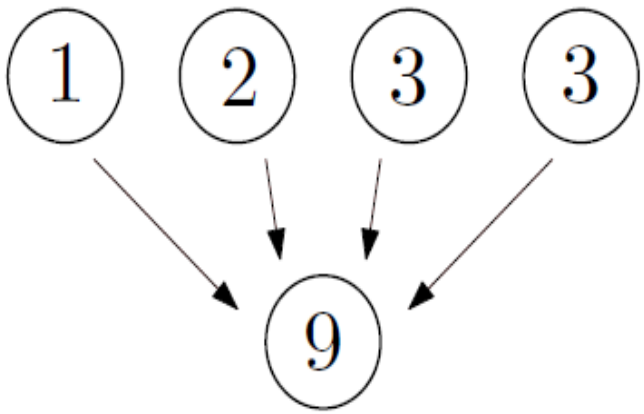
Broadcast



Scatter



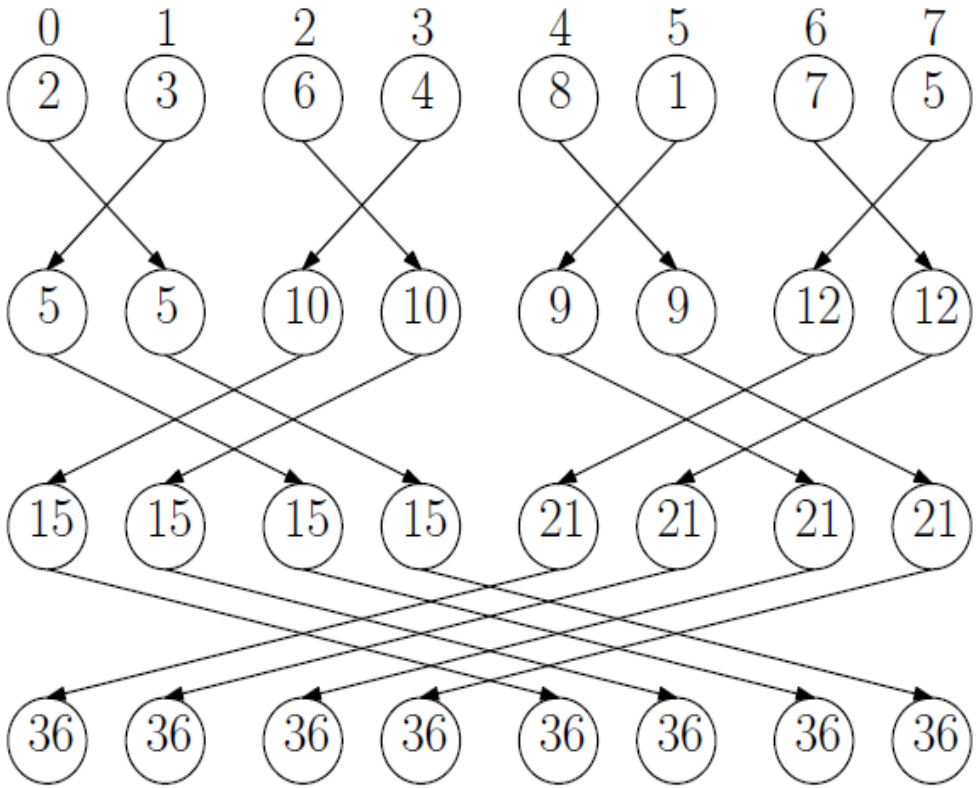
Gather



Reduce

Allreduce

Allreduce *sum* via butterfly communication

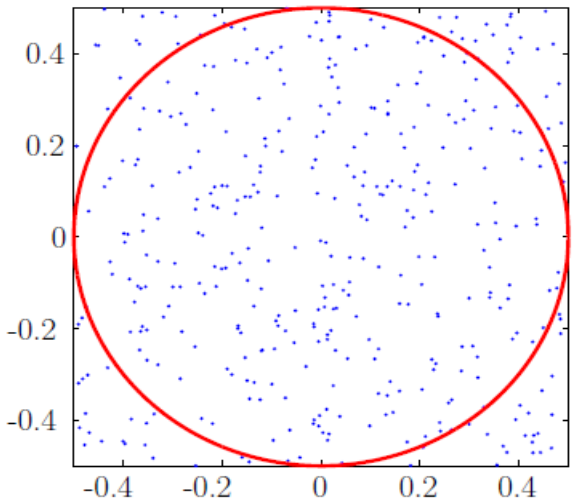


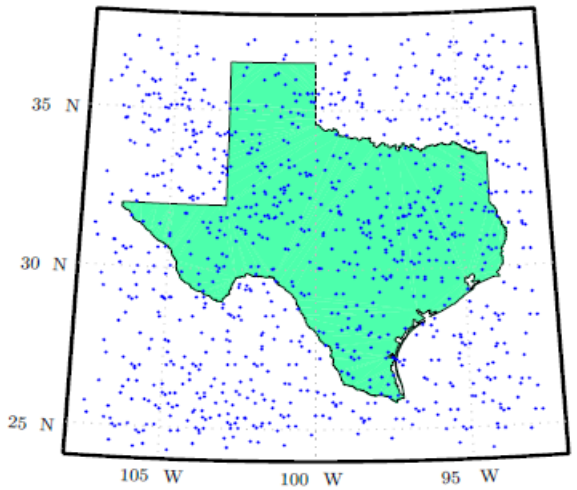
$\log(N)$ layers, N parallel communications per layer

Additional important topics

- **synchronization types**: barrier, lock, synchronous communication
- **load balancing**: static or dynamic distribution of tasks/data so that all processors are busy all time
- **granularity**: fine grain vs coarse grain
- **I/O**: a traditional parallelism killer, but modern database/parallel file system alleviates the problem (e.g., Hadoop Distributed File System (HDFS))

Example: π, area computation



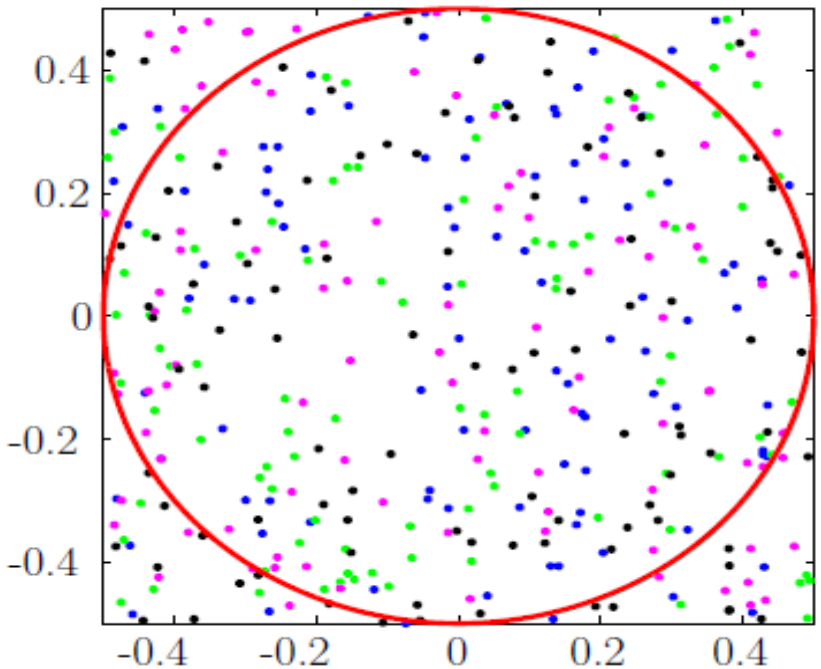


- box area is 1×1
- disc area is $\pi/4$
- ratio of area $\approx$ % inside points
- $\hat{\pi} = 4 \frac{\#\text{ points in the disc}}{\#\text{all the points}}$

Serial pseudo-code

1. total = 100000
2. in_disc = 0
3. **for** k = 1 to total do
4. x = a uniformly random point in $[-1/2, 1/2]^2$
5. **if** x is in the disc then
6. in_disc++
7. **end if**
8. **end for**
9. **return** 4*in_disc/total

Parallel π computation



- each sample is independent of others
- use multiple processors to run the for-loop
- use SPMD, process #1 collects results and return $\hat{\pi}$

SPMD pseudo-code

```
1  total = 100000
2  in_disc = 0
3  P= find total parallel processors
4  i= find my processor id
5  total = total/P
6  for k = 1 to total do
7      x = a uniformly random point in [0, 1]^2
8      if x is in the disc then
9          in_disc++
10     end if
11 end for
12 if i == 1 then
13     total_in_disc = reduce(in disc, SUM)
14     return 4*total_in_disc/total
15 end if
```

- requires only one collective communication;
- speedup = $\frac{1}{1/N+O(\log(N))} = \frac{N}{1+cN\log(N)}$, where c is a very small number;
- main loop doesn't require any synchro-nization.

Shared Memory Programming: OpenMP

- What is OpenMP?
  - Open specification for Multi-Processing, latest version 4.0, July 2013
  - "Standard" API for defining multi-threaded shared-memory programs
  - `openmp.org` — Talks, examples, forums, etc.
  - See `parlab.eecs.berkeley.edu/2012bootcampagenda` 2 OpenMP lectures (slides and video) by Tim Mattson
  - `computing.llnl.gov/tutorials/openMP/`
  - `portal.xsede.org/online-training`
  - `www.nersc.gov/assets/Uploads/XE62011OpenMP.pdf`
- High-level API
  - Preprocessor (compiler) directives (80%)
  - Library Calls (19%)
  - Environment Variables (1%)

Simple Example: OpenMP

```
int main() {
    for( i=0; i < 25; i++ ){
        printf(``Foo'');
    }
    return 0;
}
```

Parallel code by OpenMP:

```
int main() {
    omp_set_num_threads(16);
    // Do this part in parallel
    #pragma omp parallel for
    for( i=0; i < 25; i++ ){
        printf(``Foo'');
    }
    return 0;
}
```

Outline

Decomposable objective and constraints enables parallelism

- variables $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]^T$
- embarrassingly parallel (not an interesting case): $\min \sum_i f_i(\mathbf{x}_i)$
- consensus minimization, no constraints

$$\min f(\mathbf{x}) = \sum_{i=1}^{N} f_i(\mathbf{x}) + r(\mathbf{x})$$

LASSO example: $f(x) = \frac{\lambda}{2} \sum_i \|\mathbf{A}_{(i)}\mathbf{x} - \mathbf{b}_i\|_2^2 + \|\mathbf{x}\|_1$

- coupling variable $\mathbf{z}$

$$\min_{\mathbf{x},\mathbf{z}} f(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^{N} [f_i(\mathbf{x}_i, \mathbf{z}) + r_i(\mathbf{x}_i)] + r(\mathbf{z})$$

- coupling constraints

$$\min f(x) = \sum_{i=1}^{N} f_i(\mathbf{x}_i) + r_i(\mathbf{x}_i)$$

subject to

- equality constraints: $\sum_{i=1}^{N} \mathbf{A}_i \mathbf{x}_i = b$
- inequality constraints: $\sum_{i=1}^{N} \mathbf{A}_i \mathbf{x}_i \leq b$
- graph-coupling constraints $\mathbf{A}\mathbf{x} = b$, where $\mathbf{A}$ is sparse

- combinations of independent variables, coupling variables and constraints. example:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{w},\mathbf{z}} \quad & \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}) \\ \text{s.t.} \quad & \sum_{i} \mathbf{A}_i \mathbf{x}_i \leq \mathbf{b} \\ & \mathbf{B}_i \mathbf{x}_i \leq \mathbf{w}, \forall i \\ & \mathbf{w} \in \Omega \end{aligned}$$

- combinations of independent variables, coupling variables and constraints. example:

$$\begin{aligned} \min_{\mathbf{x},\mathbf{w},\mathbf{z}} \quad & \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}) \\ \text{s.t.} \quad & \sum_{i} \mathbf{A}_i \mathbf{x}_i \leq \mathbf{b} \\ & \mathbf{B}_i \mathbf{x}_i \leq \mathbf{w}, \forall i \\ & \mathbf{w} \in \Omega \end{aligned}$$

- variable coupling $\longleftrightarrow$ constraint coupling:

$$\min \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}) \longrightarrow \min \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}_i), \text{ s.t. } z_i = z, \forall i$$

Approaches toward parallelism

- Keep an existing serial algorithm, parallelize its expensive part(s)
- Introduce a new parallel algorithm by
  - formulating an equivalent model
  - replacing the model by an approximate model

  where the new model lands itself for parallel computing

Example: parallel ISTA

$$\min_{\mathbf{x}} \quad \lambda\|\mathbf{x}\|_1 + f(\mathbf{x})$$

- Prox-linear approximation:

$$\arg\min_{\mathbf{x}} \lambda\|\mathbf{x}\|_1 + \left\langle \nabla f(\mathbf{x}^k), \mathbf{x} - \mathbf{x}^k \right\rangle + \frac{1}{2\delta_k}\|\mathbf{x} - \mathbf{x}^k\|_2^2$$

- Iterative soft-threshold algorithm:

$$\mathbf{x}^{k+1} = \text{shrink}(\mathbf{x}^k - \delta_k \nabla f(\mathbf{x}^k), \lambda\delta_k)$$

  - shrink(y, t) has a simple close form: max(abs(y)-t,0).*sign(y)
  - the dominating computation is $\nabla f(\mathbf{x})$

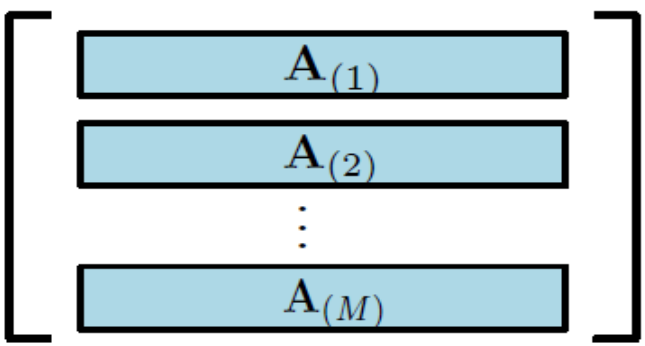
- example: $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$, logistic loss of $\mathbf{A}\mathbf{x} - \mathbf{b}$, hinge loss, ..., where $\mathbf{A}$ is often dense and can go very large-scale

Parallel gradient computing of $f(\mathbf{x})$: row partition

Assumption: $f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b})$, $\mathbf{A}$ is big and dense; $g(\mathbf{y}) = \sum_i g_i(\mathbf{y}_i)$ decomposable. Then

$$f(\mathbf{x}) = g(\mathbf{A}\mathbf{x} + \mathbf{b}) = \sum g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i)$$

- Let $\mathbf{A}_{(i)}$ and $\mathbf{b}_i$ be the ith row block of $\mathbf{A}$ and $\mathbf{b}$, respectively



- store $\mathbf{A}_{(i)}$ and $\mathbf{b}_i$ on node i
- goal: to compute $\nabla f(\mathbf{x}) = \sum_i \mathbf{A}_{(i)}^T \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i)$, or similarly, $\partial f(\mathbf{x})$

Parallel gradient computing of $f(\mathbf{x})$: row partition

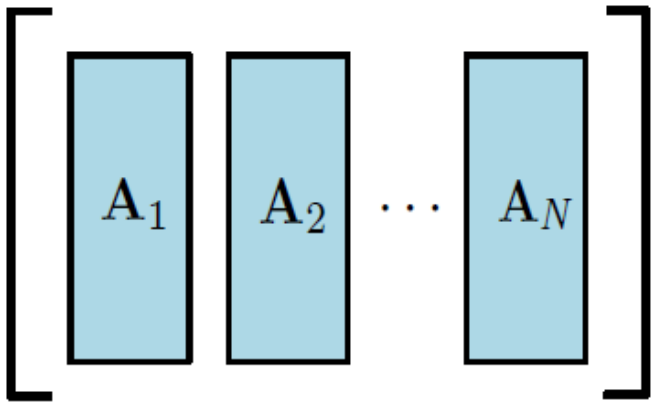
Steps:

1. Compute $\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i$, $\forall i$ in parallel
2. Compute $\mathbf{g}_i = \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i)$, $\forall i$ in parallel
3. compute $\mathbf{A}_{(i)}^T \mathbf{g}_i$, $\forall i$ in parallel
4. allreduce $\sum_{i=1}^{M} \mathbf{A}_{(i)}^T \mathbf{g}_i$

Notes:

- step 1, 2 and 3 are local computation and communication-free;
- step 4 requires an allreduce sum communication.

Parallel gradient computing of $f(\mathbf{x})$: column partition



column block partition

$$\mathbf{Ax} = \sum_{i=1}^{N} \mathbf{A}_i \mathbf{x}_i \Longrightarrow f(\mathbf{x}) = g(\mathbf{Ax} + \mathbf{b}) = g(\sum_{i} \mathbf{A}_i \mathbf{x}_i + \mathbf{b})$$

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \mathbf{A}_1^T \nabla g(\mathbf{Ax} + \mathbf{b}) \\ \mathbf{A}_2^T \nabla g(\mathbf{Ax} + \mathbf{b}) \\ \cdots \\ \mathbf{A}_N^T \nabla g(\mathbf{Ax} + \mathbf{b}) \end{pmatrix}$$

Parallel gradient computing of $f(\mathbf{x})$: column partition

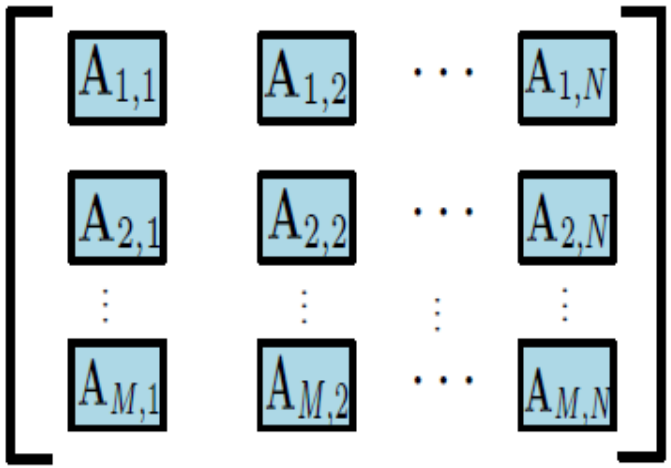
Steps:

1. Compute $\mathbf{A}_i\mathbf{x}_i + \frac{1}{N}\mathbf{b}$, $\forall i$ in parallel
2. allreduce $\mathbf{A}\mathbf{x} + \mathbf{b} = \sum_{i=1}^{N}(\mathbf{A}_i\mathbf{x}_i + \frac{1}{N}\mathbf{b})$
3. Compute $\nabla g(\mathbf{A}\mathbf{x} + \mathbf{b})$, $\forall i$ in each process
4. compute $\mathbf{A}_i^T \nabla g(\mathbf{A}\mathbf{x} + \mathbf{b})$, $\forall i$ in parallel

Notes:

- each processor keeps $\mathbf{A}_i$ and $\mathbf{x}_i$, as well as a copy of $\mathbf{b}$;
- step 2 requires an allreduce sum communication;
- step 3 involves duplicated computation in each process (hopefully, g is simple).

Parallel gradient computing of $f(\mathbf{x})$: Two-way partition



$$f(\mathbf{x}) = \sum_i g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i) = \sum_i g_i(\sum_j \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{b}_i)$$

$$\nabla f(\mathbf{x}) = \sum_i \mathbf{A}_{(i)}^T \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i) = \sum_i \mathbf{A}_{(i)}^T \nabla g_i(\sum_j \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{b}_i)$$

Parallel gradient computing of $f(\mathbf{x})$: Two-way partition

Step:

1. Compute $\mathbf{A}_{ij}\mathbf{x}_j + \frac{1}{N}\mathbf{b}_i$, $\forall i$ in parallel
2. allreduce $\sum_{j=1}^{N} \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{b}_i$ for every i
3. compute $\mathbf{g}_i = \nabla g_i(\mathbf{A}_{(i)}\mathbf{x} + \mathbf{b}_i)$, $\forall i$ in parallel
4. compute $\mathbf{A}_{j,i}^T \mathbf{g}_i$, $\forall i$ in parallel
5. allreduce $\sum_{i=1}^{N} \mathbf{A}_{j,i}^T \mathbf{g}_i$ for every j

Notes:

- processor (i,j) keeps $\mathbf{A}_{i,j}, \mathbf{x}_j$ and $\mathbf{b}_i$;
- step 2 and 5 require parallel allreduce sum communication;

Example: parallel ISTA for LASSO

LASSO

$$\min f(\mathbf{x}) = \lambda \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$$

algorithm:

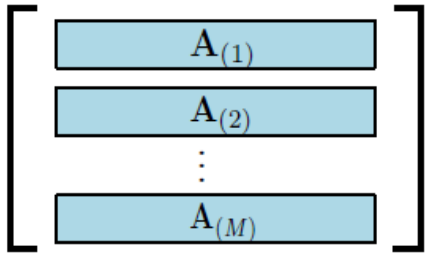
$$\mathbf{x}^{k+1} = \text{shrink}(\mathbf{x}^k - \delta_k \mathbf{A}^T \mathbf{A}\mathbf{x}^k + \delta_k \mathbf{A}^T \mathbf{b}, \lambda \delta_k)$$

Serial pseudo-code

1. initialize $\mathbf{x} = 0$ and δ
2. pre-compute $\mathbf{A}^T\mathbf{b}$
3. **while** not converged **do**
4. $\quad \mathbf{g} = \mathbf{A}^T\mathbf{A}\mathbf{x} - \mathbf{A}^T\mathbf{b}$
5. $\quad \mathbf{x} = \text{shrink}(\mathbf{x} - \delta\mathbf{g}, \lambda\delta)$
6. **end while**

Example: parallel ISTA for LASSO

distribute blocks of rows to M nodes

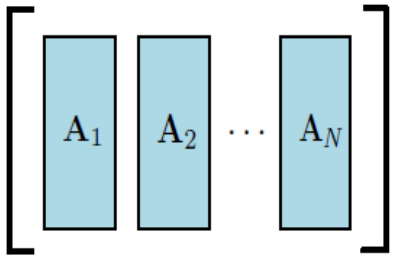


1. initialize $\mathbf{x} = 0$ and δ
2. i=find my processor id
3. processor i load $\mathbf{A}_{(i)}$
4. pre-compute $\mathbf{A}^T\mathbf{b}$
5. **while** not converged **do**
6. processor i computes $\mathbf{c}_i = \mathbf{A}_{(i)}^T\mathbf{A}_{(i)}\mathbf{x}$
7. y=allreduce($\mathbf{c}_i$, SUM)
8. $\mathbf{x} = \text{shrink}(\mathbf{x} - \delta\mathbf{c} + \delta\mathbf{A}^T\mathbf{b}, \lambda\delta)$
9. **end while**

- Assume $\mathbf{A} \in \mathbb{R}^{m\times n}$
- one allreduce per iteration
- speedup $\approx \frac{1}{\rho/M+(1-\rho)+O(\log(M))}$
- P is close to 1
- requires synchronization

Example: parallel ISTA for LASSO

distribute blocks of columns to N nodes



1. initialize $\mathbf{x} = 0$ and δ
2. i=find my processor id
3. processor i load $\mathbf{A}_i$
4. pre-compute $\mathbf{A}^T\mathbf{b}$
5. **while** not converged **do**
6. processor i computes $\mathbf{y}_i = \mathbf{A}_i\mathbf{x}_i$
7. y=allreduce($\mathbf{y}_i$, SUM)
8. processor i computes $\mathbf{z}_i = \mathbf{A}_i^T\mathbf{y}$
9. $\mathbf{x}_i^{k+1} = \text{shrink}(\mathbf{x} - \delta\mathbf{z}_i + \delta\mathbf{A}_i^T\mathbf{b}, \lambda\delta)$
10. **end while**

- one allreduce per iteration
- speedup $\approx \frac{1}{\rho/N+(1-\rho)+O(\frac{m}{n}\log(N)}$
- P is close to 1
- requires synchronization
- if $m \ll n$, this approach is faster

Outline

Unconstrained minimization with coupling variables

$\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N]^T$.
$\mathbf{z}$ is the **coupling/bridging/complicating variable**

$$\min_{\mathbf{x},\mathbf{z}} f(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z})$$

equivalently to separable objective with **coupling constraints**

$$\min_{\mathbf{x},\mathbf{z}} \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}_i), \text{ s.t. } \mathbf{z}_i = \mathbf{z}, \forall i$$

- network utility maximization (NUM), extend to multiple layers
- domain decomposition ($\mathbf{z}$ are overlapping boundary variables)
- system of equations: $\mathbf{A}$ is block-diagonal but with a few dense columns

Primal decomposition (bi-level optimization)

Introduce **easier subproblems**

$$\mathbf{g}_i(\mathbf{z}) = \min_{\mathbf{x}_i} f_i(\mathbf{x}_i, \mathbf{z}), \quad i = 1, \ldots, N$$

then

$$\min_{\mathbf{x},\mathbf{z}} \sum_{i=1}^{N} f_i(\mathbf{x}_i, \mathbf{z}) \Longleftrightarrow \min_{\mathbf{z}} \sum_{i} g_i(\mathbf{z})$$

the RHS is called the **master problem**.
parallel computing: iterate

- fix $\mathbf{z}$, update $\mathbf{x}_i$ and compute $g_i(\mathbf{z})$ in parallel
- update $\mathbf{z}$ using a standard method (typically, subgradient descent)

Notes:

- good for small-dimensional $\mathbf{z}$
- fast if the master problem converges fast

Dual decomposition

Consider

$$\min_{\mathbf{x},\mathbf{z}} \sum_i f_i(\mathbf{x}_i, \mathbf{z}_i)$$

$$\text{s.t.} \sum_i \mathbf{A}_i \mathbf{z}_i = \mathbf{b}$$

Introduce:

$$\begin{aligned} g_i(\mathbf{z}) &= \min_{\mathbf{x}_i} f(\mathbf{x}_i, \mathbf{z}) \\ g_i^*(y) &= \max_{\mathbf{z}} \mathbf{y}^T \mathbf{z} - g_i(z) \quad \text{convex conjugate} \end{aligned}$$

$$\min_{\mathbf{x},\mathbf{z}} \max_{\mathbf{y}} \sum_i f_i(\mathbf{x}_i, \mathbf{z}_i) + \mathbf{y}^T (\mathbf{b} - \sum_i \mathbf{A}_i \mathbf{z}_i)$$

$$\iff \max_{\mathbf{y}_i} \min_{\mathbf{x}_i,\mathbf{z}_i} \left\{ \sum_i (f_i(\mathbf{x}_i, \mathbf{z}_i) - \mathbf{y}^T \mathbf{A}_i \mathbf{z}_i) + \mathbf{y}^T \mathbf{b} \right\}$$

Dual decomposition

$$\min_{\mathbf{x},\mathbf{z}} \max_{\mathbf{y}} \sum_i f_i(\mathbf{x}_i, \mathbf{z}_i) + \mathbf{y}^T(\mathbf{b} - \sum_i \mathbf{A}_i \mathbf{z}_i)$$

$$\iff \max_{\mathbf{y}_i} \min_{\mathbf{x}_i,\mathbf{z}_i} \left\{ \sum_i (f_i(\mathbf{x}_i, \mathbf{z}_i) - \mathbf{y}^T \mathbf{A}_i \mathbf{z}_i) + \mathbf{y}^T \mathbf{b} \right\}$$

$$\iff \max_{\mathbf{y}_i} \left\{ \min_{\mathbf{z}_i} \sum_i (g_i(\mathbf{z}_i) - \mathbf{y}^T \mathbf{A}_i \mathbf{z}_i) + \mathbf{y}^T \mathbf{b} \right\}$$

$$\iff \min_{\mathbf{y}} \sum_i g_i^*(\mathbf{A}_i^T \mathbf{y}) - \mathbf{y}^T b$$

Example: consensus and exchange problems

Consensus problem

$$\min_{\mathbf{z}} \sum_i g_i(\mathbf{z})$$

Reformulate as

$$\min_{\mathbf{z}} \sum_i g_i(\mathbf{z}_i), \text{ s.t. } \mathbf{z}_i = \mathbf{z}, \forall i$$

Its dual problem is the exchange problem

$$\min_{\mathbf{y}} \sum_i g_i^*(\mathbf{y}_i), \text{ s.t. } \sum_i \mathbf{y}_i = 0$$

Primal-dual objective properties

Constrained separable problem

$$\min_{\mathbf{z}} \sum_i g_i(\mathbf{z}) \text{ s.t. } \sum_i \mathbf{A}_i \mathbf{z}_i = \mathbf{b}$$

Dual problem

$$\min_{\mathbf{y}} \sum_i g_i^*(\mathbf{A}_i^T \mathbf{y}) - \mathbf{b}^T \mathbf{y}$$

- If g_i is proper, closed, convex, $g_i^*(\mathbf{a}_i^T \mathbf{y})$ is sub-differentiable
- If g_i is strictly convex, $g_i^*(\mathbf{a}_i^T \mathbf{y})$ is differentiable
- If g_i is strongly convex, $g_i^*(\mathbf{a}_i^T \mathbf{y})$ is differentiable and has Lipschitz gradient
- Obtaining $\mathbf{z}^*$ from $\mathbf{y}^*$ generally requires strict convexity of g_i or strict complementary slackness

Example:

- $g_i(\mathbf{z}) = |\mathbf{z}_i|$ and $g_i^*(\mathbf{a}_i^T\mathbf{y}) = \iota\{-1 \leq \mathbf{a}_i^T\mathbf{y} \leq 1\}$
- $g_i(\mathbf{z}) = \frac{1}{2}|\mathbf{z}_i|^2$ and $g_i^*(\mathbf{a}_i^T\mathbf{y}) = \frac{1}{2}(\mathbf{a}_i^T\mathbf{y})^2$
- $g_i(\mathbf{z}) = \exp(\mathbf{z}_i)$ and $g_i^*(\mathbf{a}_i^T\mathbf{y}) = (\mathbf{a}_i^T\mathbf{y})\ln(\mathbf{a}_i^T\mathbf{y}) - \mathbf{a}_i^T\mathbf{y} + \iota\{\mathbf{a}_i^T\mathbf{y} \geq 0\}$

Example: augmented ℓ_1

Change $|z_i|$ into strongly convex $g_i(z_i) = |z_i| + \frac{1}{2\alpha}|z_i|^2$.
Dual problem:

$$\min_{\mathbf{y}} \sum_i g_i^*(\mathbf{A}_i^T\mathbf{y}) - \mathbf{b}^T\mathbf{y} = \sum_i \frac{\alpha}{2}|\text{shrink}(\mathbf{A}_i^T\mathbf{y})|^2 - \mathbf{b}^T\mathbf{y}$$

Dual gradient iteration (equivalent to linearized Bregman):

$$\mathbf{y}^{k+1} = \mathbf{y}^k - t^k\left(\sum_i \nabla h_i(\mathbf{y}^k) - \mathbf{b}\right)$$

where $\nabla h_i(\mathbf{y}) = \alpha\mathbf{A}_i\text{shrink}(\mathbf{A}_i^T\mathbf{y})$.

- Dual is C^1 and unconstrained. One can apply (accelerated) gradient descent, line search, quasi-Newton method, etc.
- Recover $\mathbf{x}^* = \alpha\text{shrink}(\mathbf{A}_i^T\mathbf{y})$
- Easy to parallelize. Node i computes $\nabla h_i(y)$. One collective communication per iteration.

Alternating direction method of multipliers (ADMM)

- **step 1**: turn problem into the form of

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & f(\mathbf{x}) + g(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{b} \end{aligned}$$

  f and g are **convex**, maybe **nonsmooth**, can **incorporate constraints**

- **step 2**:

$$\begin{aligned} \mathbf{x}^{k+1} &= \arg\min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{y}^k) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}^k - \mathbf{b} - \mathbf{z}^k\|_2^2 \\ \mathbf{y}^{k+1} &= \arg\min_{\mathbf{y}} f(\mathbf{x}^{k+1}) + g(\mathbf{y}) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x}^{k+1} + \mathbf{B}\mathbf{y} - \mathbf{b} - \mathbf{z}^k\|_2^2 \\ \mathbf{z}^{k+1} &= \mathbf{z}^k - \beta(\mathbf{A}x^{k+1} + \mathbf{B}\mathbf{y}^{k+1} - b) \end{aligned}$$

history: dates back to 60s, formalized 80s, parallel versions appeared late 80s, revived very recently, new convergence results recently

Block separable ADMM

Suppose $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N)$ and f is separable, i.e.,

$$f(\mathbf{x}) = f_1(\mathbf{x}_1) + f_2(\mathbf{x}_2) + \cdots + f_N(\mathbf{x}_N).$$

Model

$$\begin{array}{ll} \min\limits_{\mathbf{x},\mathbf{z}} & f(\mathbf{x}) + g(\mathbf{z}) \\ \text{s.t.} & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} = \mathbf{b}. \end{array}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & & \mathbf{0} \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{A}_N \end{bmatrix}$$

Block separable ADMM

The $\mathbf{x}-$update

$$\mathbf{x}^{k+1} \leftarrow \min\, f(\mathbf{x}) + \frac{\beta}{2}\|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}^k - \mathbf{b} - \mathbf{z}^k\|_2^2$$

is separable to N independent subproblems

$$\mathbf{x}_1^{k+1} \leftarrow \min\, f_1(\mathbf{x}_1) + \frac{\beta}{2}\|\mathbf{A}_1\mathbf{x}_1 + (\mathbf{B}\mathbf{y}^k - \mathbf{b} - \mathbf{z}^k)_1\|_2^2,$$

$$\vdots$$

$$\mathbf{x}_N^{k+1} \leftarrow \min\, f_N(\mathbf{x}_N) + \frac{\beta}{2}\|\mathbf{A}_N\mathbf{x}_N + (\mathbf{B}\mathbf{y}^k - \mathbf{b} - \mathbf{z}^k)_N\|_2^2.$$

No coordination is required.

Example: consensus optimization

Model

$$\min \sum_{i=1}^{N} f_i(\mathbf{x})$$

the objective is partially separable.
Introduce N copies $\mathbf{x}_1, \cdots, \mathbf{x}_N$ of $\mathbf{x}$. They have the same dimensions.
New model:

$$\min_{\{\mathbf{x}_i\},\mathbf{z}} \sum_{i=1}^{N} f_i(\mathbf{x}_i), \text{ s.t. } \mathbf{x}_i - \mathbf{z} = \mathbf{0}, \forall i.$$

A more general objective with function g is $\sum_{i=1}^{N} f_i(\mathbf{x}) + g(\mathbf{z})$.
New model:

$$\min_{\{\mathbf{x}_i\},\mathbf{y}} \sum_{i=1}^{N} f_i(\mathbf{x}_i) + g(\mathbf{z}), \text{ s.t. } \begin{bmatrix} I & & \\ & \ddots & \\ & & I \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{bmatrix} - \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \mathbf{z} = \mathbf{0}.$$

Example: consensus optimization

Lagrangian

$$L(\{\mathbf{x}_i\}, \mathbf{z}; \{\mathbf{y}_i\}) = \sum_i (f_i(\mathbf{x}_i) + \frac{\beta}{2}\|\mathbf{x}_i - \mathbf{z} - \mathbf{y}_i\|_2^2))$$

where $\mathbf{y}_i$ is the Lagrange multipliers to $\mathbf{x}_i - \mathbf{z} = 0$.
ADMM

$$\begin{aligned}
\mathbf{x}_i^{k+1} &= \operatorname*{argmin}_{\mathbf{x}_i} f_i(\mathbf{x}_i) + \frac{\beta}{2}\|\mathbf{x}_i - \mathbf{z}^k - \mathbf{y}_i^k\|_2, i = 1, \cdots, N, \\
\mathbf{z}^{k+1} &= \frac{1}{N}\sum_{i=1}^{N}(\mathbf{x}_i^{k+1} - \beta^{-1}\mathbf{y}_i^k), \\
\mathbf{y}_i^{k+1} &= \mathbf{y}_i^k - (\mathbf{x}_i^{k+1} - \mathbf{z}^{k+1}), i = 1, \cdots, N.
\end{aligned}$$

The exchange problem

Model $\mathbf{x}_1, \cdots, \mathbf{x}_N \in \mathbb{R}^n$,

$$\min \sum_{i=1}^{N} f_i(\mathbf{x}_i), \text{ s.t. } \sum_{i=1}^{N} \mathbf{x}_i = \mathbf{0}.$$

- it is the dual of the consensus problem
- exchanging n goods among N parties to minimize a total cost
- our goal: to decouple $\mathbf{x}_i-$updates

An equivalent model

$$\min \sum_{i=1}^{N} f_i(\mathbf{x}_i), \text{ s.t. } \mathbf{x}_i - \mathbf{x}_i^{'} = \mathbf{0}, \forall i, \sum_{i=1}^{N} \mathbf{x}_i^{'} = \mathbf{0}.$$

The exchange problem

ADMM after consolidating the $\mathbf{x}_i'$ update:

$$\begin{aligned}\mathbf{x}_i^{k+1} &= \underset{\mathbf{x}_i}{\operatorname{argmin}} f_i(\mathbf{x}_i) + \frac{\beta}{2}\|\mathbf{x}_i - (\mathbf{x}_i^k - \text{mean}\{\mathbf{x}_i^k\} - \mathbf{u}^k)\|_2^2, \\ \mathbf{u}^{k+1} &= \mathbf{u}^k + \text{mean}\{\mathbf{x}_i^{k+1}\}.\end{aligned}$$

Applications: distributed dynamic energy management

Distributed ADMM I

$$\min_{\{\mathbf{x}_i\},\mathbf{y}} \sum_{i=1}^{N} f_i(\mathbf{x}_i) + g(\mathbf{z}), \text{ s.t. } \begin{bmatrix} I & & \\ & \ddots & \\ & & I \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{bmatrix} - \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix} \mathbf{z} = \mathbf{0}.$$

Consider N computing nodes with MPI (message passing interface).

- $\mathbf{x}_i$ are local variables; $\mathbf{x}_i$ is stored and updated on node i only
- $\mathbf{z}$ is the global variable; computed and communicated by MPI
- $\mathbf{y}_i$ are dual variables, stored and updated on node i only

At each iteration, given $\mathbf{y}^k$ and $\mathbf{z}_i^k$

- each node i computes $\mathbf{x}_i^{k+1}$
- each node i computes $\mathbf{P}_i := (\mathbf{x}_i^{k+1} - \beta^{-1}\mathbf{y}_i^k)$
- MPI gathers $\mathbf{P}_i$ and scatters its mean, $\mathbf{z}^{k+1}$, to all nodes
- each node i computes $\mathbf{y}_i^{k+1}$

Example: distributed LASSO

Model

$$\min \|\mathbf{x}\|_1 + \frac{\beta}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2.$$

Decomposition

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_N \end{bmatrix} \mathbf{x}, \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_N \end{bmatrix}.$$

$\Rightarrow$

$$\frac{\beta}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 = \sum_{i=1}^{N} \frac{\beta}{2}\|\mathbf{A}_i\mathbf{x} - \mathbf{b}_i\|_2^2 =: \sum_{i=1}^{N} f_i(\mathbf{x}).$$

LASSO has the form

$$\min \sum_{i=1}^{N} f_i(\mathbf{x}) + g(\mathbf{x})$$

and thus can be solved by distributed ADMM.

Example: dual of LASSO

LASSO

$$\min \|\mathbf{x}\|_1 + \frac{\beta}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2.$$

Lagrange dual

$$\min_{\mathbf{y}} \{\mathbf{b}^T\mathbf{y} + \frac{\mu}{2}\|\mathbf{y}\|_2^2 : \|\mathbf{A}^T\mathbf{y}\|_\infty \leq 1\}$$

equivalently,

$$\min_{\mathbf{y},\mathbf{z}} \{-\mathbf{b}^T\mathbf{y} + \frac{\mu}{2}\|\mathbf{y}\|_2^2 + l_{\{\|\mathbf{z}\|_\infty \leq 1\}} : \mathbf{A}^T\mathbf{y} + \mathbf{z} = \mathbf{0}\}$$

Standard ADMM:

- primal $\mathbf{x}$ is the multipliers to $\mathbf{A}^T\mathbf{y} + \mathbf{z} = \mathbf{0}$
- $\mathbf{z}-$update is projection to $l_\infty-$ball; easy and separable
- $\mathbf{y}-$update is quadratic

Example: dual of LASSO

- Dual augmented Lagrangian (the scaled form):

$$L(\mathbf{y}, \mathbf{z}; \mathbf{x}) = \mathbf{b}^T\mathbf{y} + \frac{\mu}{2}\|\mathbf{y}\|_2^2 + l_{\|\mathbf{z}\|_\infty \le 1} + \frac{\beta}{2}\|\mathbf{A}^T\mathbf{y} + \mathbf{z} - \mathbf{x}\|_2^2$$

- Dual ADMM iterations:

$$\begin{aligned}
\mathbf{z}^{k+1} &= Proj_{\|\cdot\|_\infty \le 1}(\mathbf{x}^k - \mathbf{A}^T\mathbf{y}^k), \\
\mathbf{y}^{k+1} &= (\mu I + \beta\mathbf{A}\mathbf{A}^T)^{-1}(\beta\mathbf{A}(\mathbf{x}^k - \mathbf{z}^{k+1}) - \mathbf{b}), \\
\mathbf{x}^{k+1} &= \mathbf{z}^k - \gamma(\mathbf{A}^T\mathbf{y}^{k+1} + \mathbf{z}^{k+1}).
\end{aligned}$$

  and upon termination at step K, return primal solution

$$\mathbf{x}^* = \beta\mathbf{x}^K \text{ (de-scaling).}$$

- Computation bottlenecks:
  - $(\mu I + \beta\mathbf{A}\mathbf{A}^T)^{-1}$, unless $\mathbf{A}\mathbf{A}^T = I$ or $\mathbf{A}\mathbf{A}^T \approx I$
  - $\mathbf{A}(\mathbf{x}^k - \mathbf{z}^{k+1})$ and $\mathbf{A}^T\mathbf{y}^k$, unless $\mathbf{A}$ is small or has structures

Example: dual of LASSO

Observe

$$\min_{\mathbf{y},\mathbf{z}}\{\mathbf{b}^T\mathbf{y} + \frac{\mu}{2}\|\mathbf{y}\|_2^2 + l_{\{\|\mathbf{z}_\infty \leq 1\}} : \mathbf{A}^T\mathbf{y} + \mathbf{z} = \mathbf{0}\}$$

- All the objective terms are perfectly separable
- The constraints cause the computation bottlenecks
- We shall try to decouple the blocks of $\mathbf{A}^T$

Distributed ADMM II

A general form with inseparable f and separable g

$$\min_{\mathbf{x},\mathbf{z}} \sum_{l=1}^{L} (f_l(\mathbf{x}) + g_l(\mathbf{z}_l)), \text{ s.t. } \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b}$$

- Make L copies $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_L$ of $\mathbf{x}$
- Decompose

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_L \end{bmatrix}, \mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_L \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_L \end{bmatrix}$$

- Rewrite $\mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{0}$ as

$$\mathbf{A}_l\mathbf{x}_l + \mathbf{z}_l = \mathbf{b}_l, \mathbf{x}_l - \mathbf{x} = \mathbf{0}, l = 1, \cdots, L.$$

Distributed ADMM II

New model:

$$\begin{aligned} \min_{\mathbf{x},\{\mathbf{x}_l\},\mathbf{z}} \quad & \sum_{l=1}^{L}(f_l(\mathbf{x}_l)+g_l(\mathbf{z}_l)) \\ \text{s.t.} \quad & \mathbf{A}_l\mathbf{x}_l+\mathbf{z}_l=\mathbf{b}_l, \mathbf{x}_l-\mathbf{x}=\mathbf{0}, l=1,\cdots,L. \end{aligned}$$

- $\mathbf{x}_l$'s are copies of $\mathbf{x}$
- $\mathbf{z}_l$'s are sub-blocks of $\mathbf{z}$
- Group variables $\{\mathbf{x}_l\}, \mathbf{z}, \mathbf{x}$ into two sets
  - $\{\mathbf{x}_l\}$: given $\mathbf{z}$ and $\mathbf{x}$, the updates of $\mathbf{x}_l$ are separable
  - $(\mathbf{z}, \mathbf{x})$: given $\{\mathbf{x}_l\}$, the updates of $\mathbf{z}_l$ and $\mathbf{x}$ are separable
    Therefore, standard (2-block) ADMM applies.
- One can also add a simple regularizer $h(\mathbf{x})$

Distributed ADMM II

Consider L computing nodes with MPI.

- $\mathbf{A}_l$ is local data store on node l only
- $\mathbf{x}_l, \mathbf{z}_l$ are local variables; $\mathbf{x}_l$ is stored and updated on node l only
- $\mathbf{x}$ is the global variable; computed and dispatched by MPI
- $\mathbf{y}_l, \bar{\mathbf{y}}_l$ are Lagrange multipliers to $\mathbf{A}_l\mathbf{x}_l + \mathbf{z}_l = \mathbf{b}_l$ and $\mathbf{x}_l - \mathbf{x} = \mathbf{0}$,respectively, stored and updated on node l only

At each iteration,

- each node l computes $\mathbf{x}_l^{k+1}$, using data $\mathbf{A}_l$
- each node l computes $\mathbf{z}_l^{k+1}$, prepares $\mathbf{P}_l = (\cdots)$
- MPI gathers $\mathbf{P}_l$ and scatters its mean, $\mathbf{x}^{k+1}$, to all nodes l
- each node l computes $\mathbf{y}_l^{k+1}, \bar{\mathbf{y}}_l^{k+1}$

Example: distributed dual LASSO

Recall

$$\min_{\mathbf{y},\mathbf{z}}\{\mathbf{b}^T\mathbf{y} + \frac{\mu}{2}\|\mathbf{y}\|_2^2 + l_{\{\|\mathbf{z}\|_\infty \le 1\}} : \mathbf{A}^T\mathbf{y} + \mathbf{z} = \mathbf{0}\}$$

Apply distributed ADMM II

- decompose $\mathbf{A}^T$ to row blocks, equivalently, $\mathbf{A}$ to column blocks.
- make copies of $\mathbf{y}$
- parallel computing $+$ MPI(gathering and scatting vectors of size dim($\mathbf{y}$))

Recall distribute ADMM I

- decompose $\mathbf{A}$ to row blocks.
- make copies of $\mathbf{x}$
- parallel computing $+$ MPI (gathering and scatting vectors of size dim($\mathbf{x}$))

Between I and II, which is better?

- If $\mathbf{A}$ is fat
  - column decomposition in approach II is more efficient
  - the global variable of approach II is smaller
- If $\mathbf{A}$ is tall
  - row decomposition in approach I is more efficient
  - the global variable of approach I is smaller

Distributed ADMM II

A formulation with separable f and separable g

$$\min \sum_{j=1}^{N} f_j(\mathbf{x}_j) + \sum_{i=1}^{M} g_i(\mathbf{z}_i), \text{ s.t. } \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b},$$

where

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N), \mathbf{z} = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_M).$$

Decompose $\mathbf{A}$ in both directions as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1N} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2N} \\ & & \cdots & \\ \mathbf{A}_{M1} & \mathbf{A}_{M2} & \cdots & \mathbf{A}_{MN} \end{bmatrix}, also\ \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_M \end{bmatrix}.$$

Same model:

$$\min \sum_{j=1}^{N} f_j(\mathbf{x}_j) + \sum_{i=1}^{M} g_i(\mathbf{z}_i), \text{ s.t. } \sum_{j=1}^{N} \mathbf{A}_{ij}\mathbf{x}_j + \mathbf{z}_i = \mathbf{b}_i, i = 1, \cdots, M.$$

Distributed ADMM III

$\mathbf{A}_{ij}\mathbf{x}_j'$s are coupled in the constraints. Standard treatment:

$$\mathbf{p}_{ij} = \mathbf{A}_{ij}\mathbf{x}_j.$$

New model:

$$\min \sum_{j=1}^{N} f_j(\mathbf{x}_j) + \sum_{i=1}^{M} g_i(\mathbf{z}_i), \text{ s.t. } \begin{array}{l} \sum_{j=1}^{N} \mathbf{p}_{ij} + \mathbf{z}_i = \mathbf{b}_i, \forall i, \\ \mathbf{p}_{ij} - \mathbf{A}_{ij}\mathbf{x}_j = 0, \forall i, j. \end{array}$$

ADMM

- alternate between $\{\mathbf{p}_{ij}\}$ and $(\{\mathbf{x}_j\}, \{\mathbf{z}_i\})$
- $\mathbf{p}_{ij}-$subproblems have closed-form solutions
- $(\{\mathbf{x}_j\}, \{\mathbf{z}_i\})$-subproblem are separable over all $\mathbf{x}_j$ and $\mathbf{z}_i$
  - $\mathbf{x}_j-$update involves f_j and $\mathbf{A}_{1j}^T\mathbf{A}_{1j}, \cdots, \mathbf{A}_{Mj}^T\mathbf{A}_{Mj}$;
  - $\mathbf{z}_i-$update involves g_i.
- ready for distributed implementation

Question: how to further decouple f_j and $\mathbf{A}_{1j}^T\mathbf{A}_{1j}, \cdots, \mathbf{A}_{Mj}^T\mathbf{A}_{Mj}$?

Distributed ADMM IV

For each $\mathbf{x}_j$, make M identical copies:$\mathbf{x}_{1j}, \mathbf{x}_{2j}, \cdots, \mathbf{x}_{Mj}$.
New model:

$$\min \sum_{j=1}^{N} f_j(\mathbf{x}_j) + \sum_{i=1}^{M} g_i(\mathbf{z}_i), \text{ s.t. } \begin{array}{ll} \sum_{j=1}^{N} \mathbf{p}_{ij} + \mathbf{z}_i = \mathbf{b}_i, & \forall i, \\ \mathbf{p}_{ij} - \mathbf{A}_{ij}\mathbf{x}_{ij} = \mathbf{0}, & \forall i, j, \\ \mathbf{x}_j - \mathbf{x}_{ij} = \mathbf{0}, & \forall i, j. \end{array}$$

ADMM

- alternate between $(\{\mathbf{x}_j\}, \{\mathbf{p}_{ij}\})$ and $(\{\mathbf{x}_j\}, \{\mathbf{z}_i\})$
- $(\{\mathbf{x}_j\}, \{\mathbf{p}_{ij}\})$-subproblem are separable
  - $\mathbf{x}_j-$update involves f_j only; computes prox_{f_j}
  - $\mathbf{p}_{ij}-$update is in closed form
- $(\{\mathbf{x}_{ij}\}, \{\mathbf{z}_i\})$-subproblem are separable
  - $\mathbf{x}_{ij}-$update involves $(\alpha I + \beta \mathbf{A}_{ij}^T \mathbf{A}_{ij})$;
  - $\mathbf{y}_i-$update involves g_i only; computes prox_{g_i}.
- ready for distributed implementation

Outline

Stochastic Gradient Descent

Consider

$$\min_{x \in \mathbb{R}^n} f(x) = \sum_{e \in E} f_e(x_e)$$

- e denotes a small subset of $\{1, 2, \ldots, n\}$.
- In many of machine learning problems, n and $|E|$ are both large. **Note**: the 'set' E can contain a few copies of the same element e.
- f_e only acts on a few components of x, say x_e.
- f_e can easily be regarded as a function over $\mathbb{R}^n$, just by ignoring the components not in the subset e.

Problem Definition

Example: machine learning applications

- Minimize the empirical risk

$$\min_x \frac{1}{n} \sum_{i=1}^{n} l_i(a_i^T x)$$

- a_i represents the ith data point, x is the model. l_i is a loss function.
- Logistic regression, least squares, SVM ...
- If each a_i is sparse, then it becomes our problem today.

Problem Definition

Example: generic minimization problem

- Minimize the following function

$$\min_{x_1,\ldots,x_{m_1}} \min_{y_1,\ldots,y_{m_2}} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \phi_{i,j}(x_i, y_j)$$

- $\phi_{i,j}$ is a convex scalar function. x_i and y_j are vectors.
- For matrix completion and matrix factorization:

$$\phi_{i,j} = (A_{i,j} - x_i^T y_j)^2$$

- $n = m_1 m_2$ functions, each of which interacts with only two variables.
- A variable is shared among at most $m_1 + m_2$ functions.

Stochastic Gradient Descent

- Gradient method is given by

$$x_{k+1} = x_k - \gamma_k \nabla f(x_k)$$

  where $\nabla f(x_k) = \sum_{e \in E} \nabla f_e(x_k)$.

- Stochastic Gradient Descent(SGD)

$$x_{k+1} = x_k - \gamma_k \nabla f_{s_k}(x_k)$$

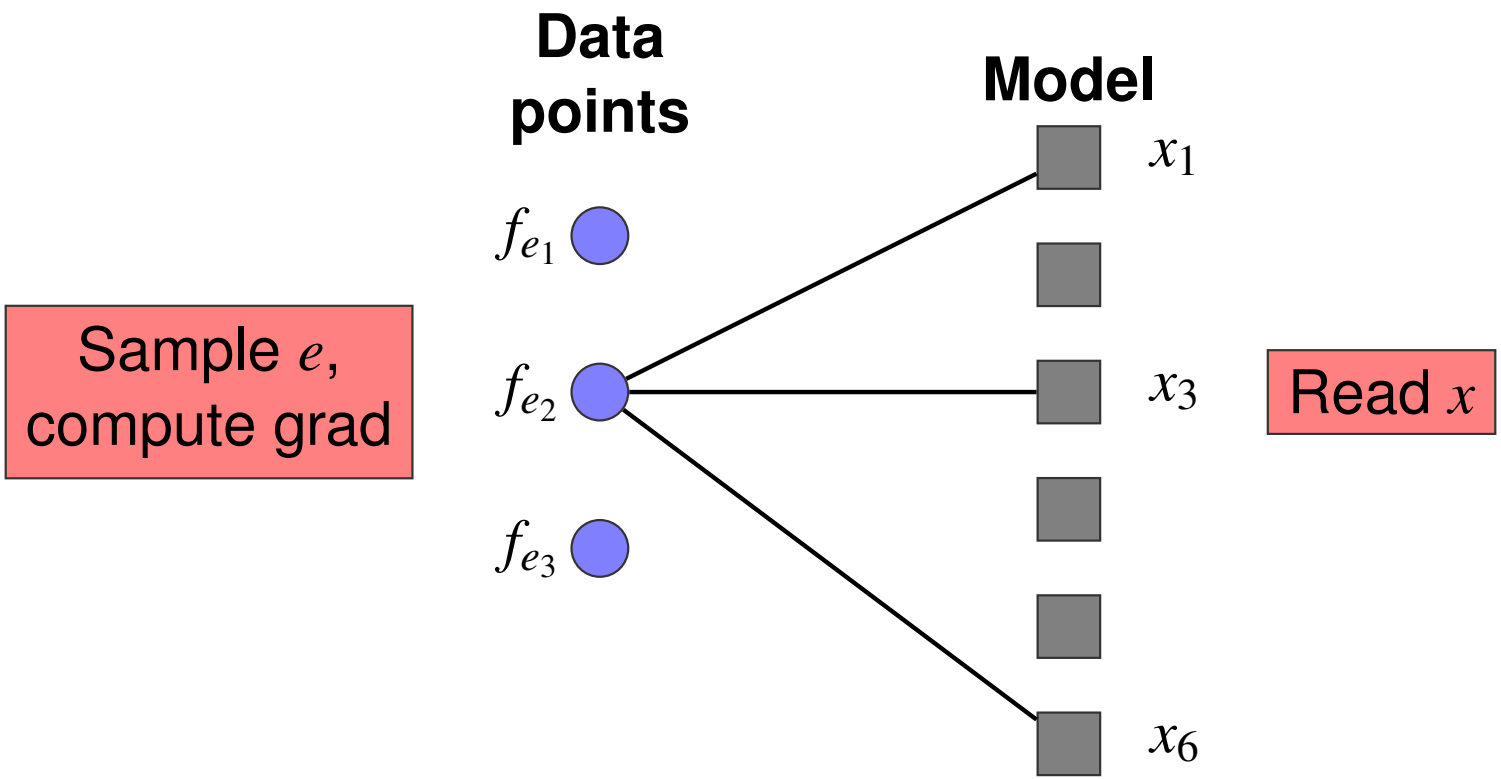
  where s_k is randomly sampled from E.

- γ_k is the step size(or learning rate). Can be a constant or shrinking.

- Idea of SGD: computing the entire $\nabla f(x)$ may be too costly for large $|E|$ and n. Can we compute just a small proportion of $\nabla f(x)$ and ensure the convergence?

Stochastic Gradient Descent

One step of SGD:



SGD: Advantages

SGD has been a round for a while, for good reasons:

- Less computation than classic GD.
- Robust to noise.
- Simple to implement.
- Near-optimal learning performance.
- Small computational foot-print.
- Stable.

Parallel SGD

- It may takes a **HUGE** number of updates for SGD on large datasets.
- **Goal**: a parallel version of SGD.
- **How to parallelize SGD?**
  - Parallelize one update – computing $\nabla f_{s_k}(x_k)$ is cheap, even for deep nets. Thus it may be not worth working out parallel codes on a cheap subroutine.
  - Parallelize the updates – SGD is sequential, making it nearly impossible for the parallel stuff.
  - Can we parallelize a sequential algorithm?

Parallel SGD

Can we parallelize a sequential algorithm?

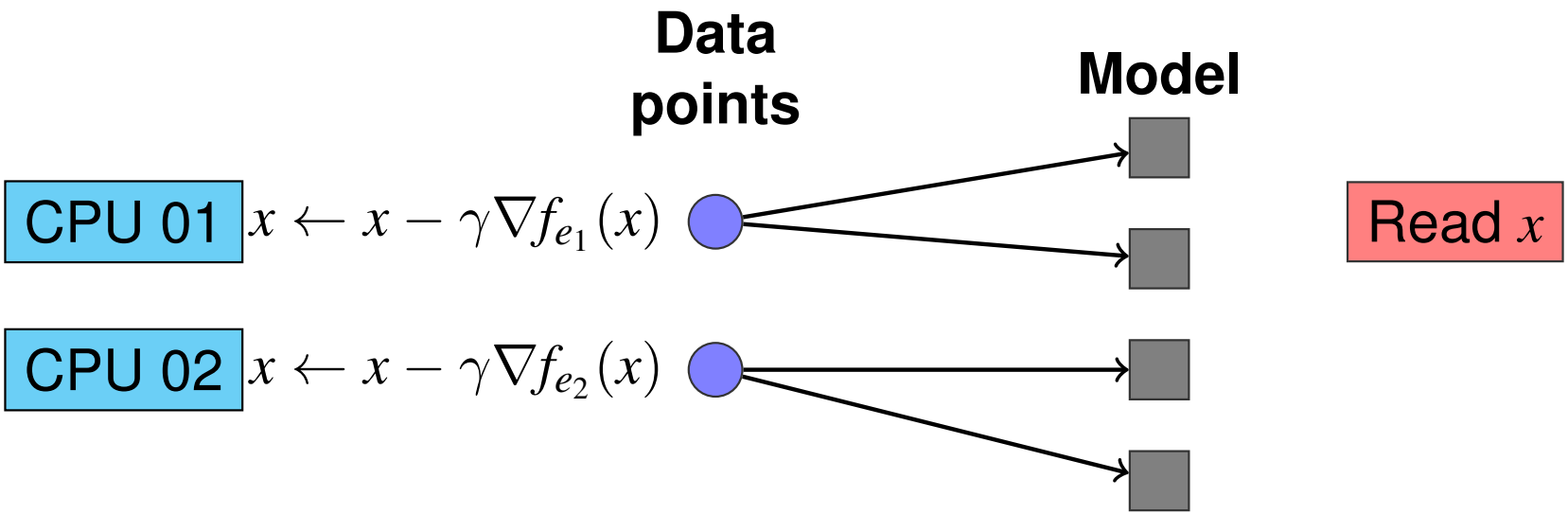
- No – for most cases.
- Almost yes – for problems with structures of sparsity.
- For our problem:

$$\min_{x \in \mathbb{R}^n} f(x) = \sum_{e \in E} f_e(x_e)$$

  If most of f_e's don't share the same component of x, may be we can exploit the sparsity to parallelize SGD.

Parallel SGD

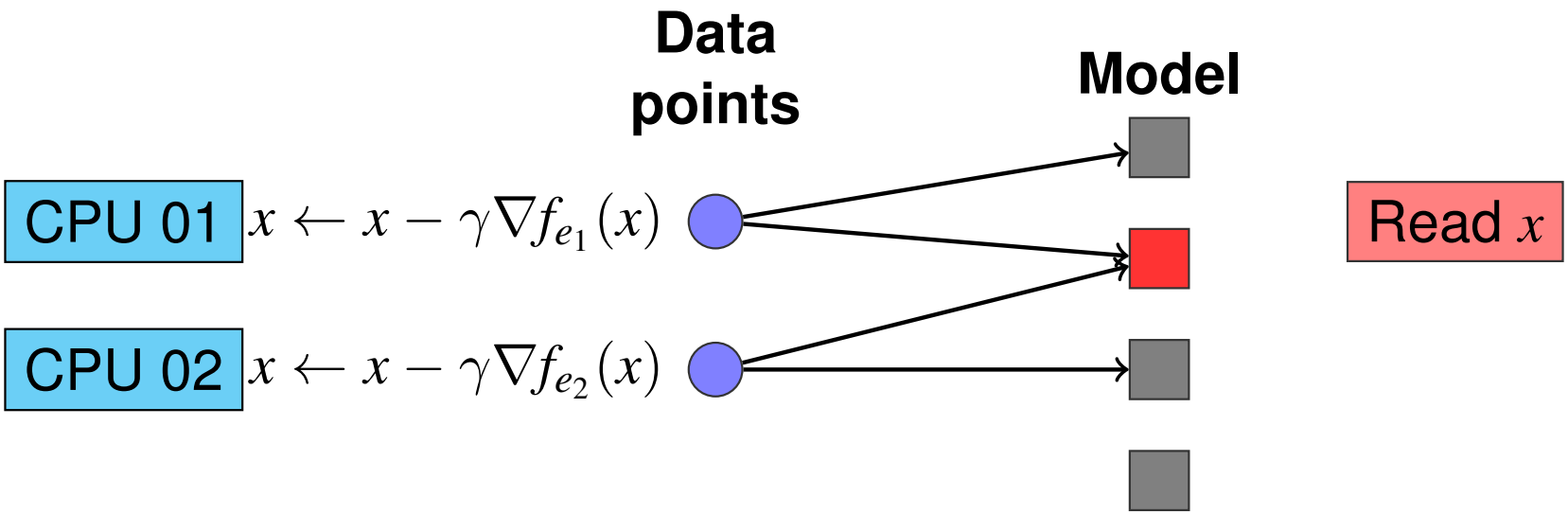
Good Case:



- The components of f_{e_1} and f_{e_2} don't overlap.
- Two parallel updates means two serial updates.
- No conflicts means speed up!

Parallel SGD

Bad Case:



- The components of f_{e_1} and f_{e_2} overlap at x_2.
- Conflicts mean less parallelization.
- What should be done to handle the conflicts?

Parallel SGD

Why conflicts affect so much?

- CPUs don't have direct access to the memory.
- Computations can only be done on CPU cache.
- Data are read from the memory to CPU cache. After the computations are finished, results are pushed back to the memory.
- A CPU has no idea whether its 'colleagues' have local updates which have not been pushed to the memory.

Parallel SGD

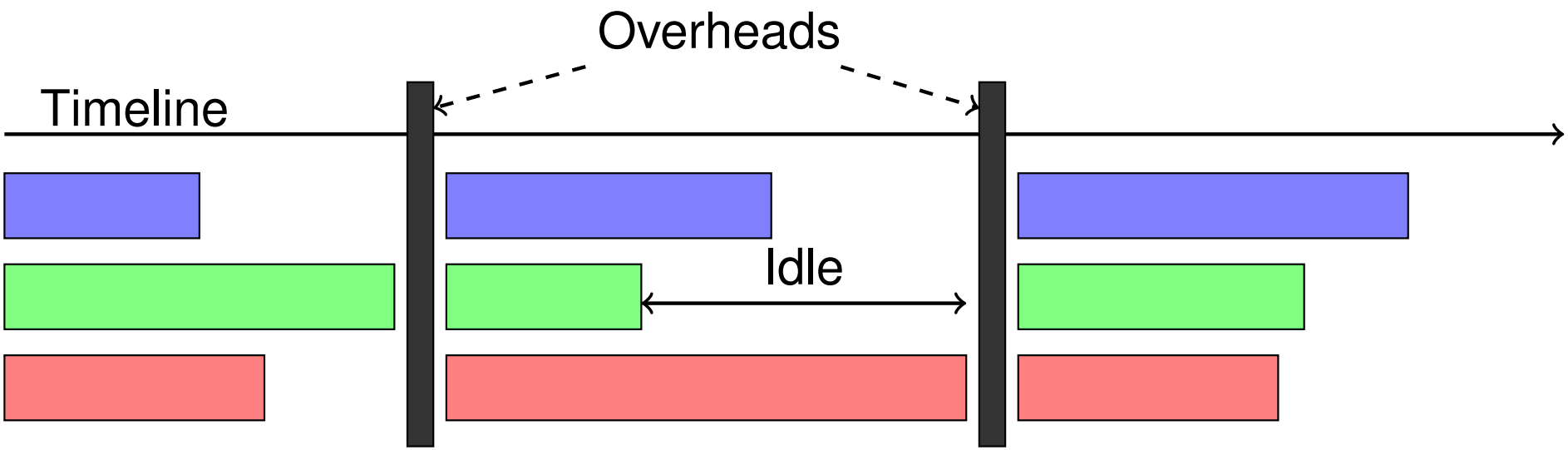
How to deal with the conflicts?

- By ‘coordinate’ or ‘memory lock’ approach.
- **Lock type**: exclusive lock.
- Tell others: I’m updating the variable NOW. Please don’t make any changes until I have finished.
- Ensure the correctness when performing parallel SGD.
- Provide only limited speedup for making parallel SGD as a ‘sequential’ program. Things are even worse when conflicts occur too often – even slower than the sequential version of SGD!

Asynchronous Updates

Q: How to deal with the conflicts?
A: Asynchronous programming tells us to just ignore it.

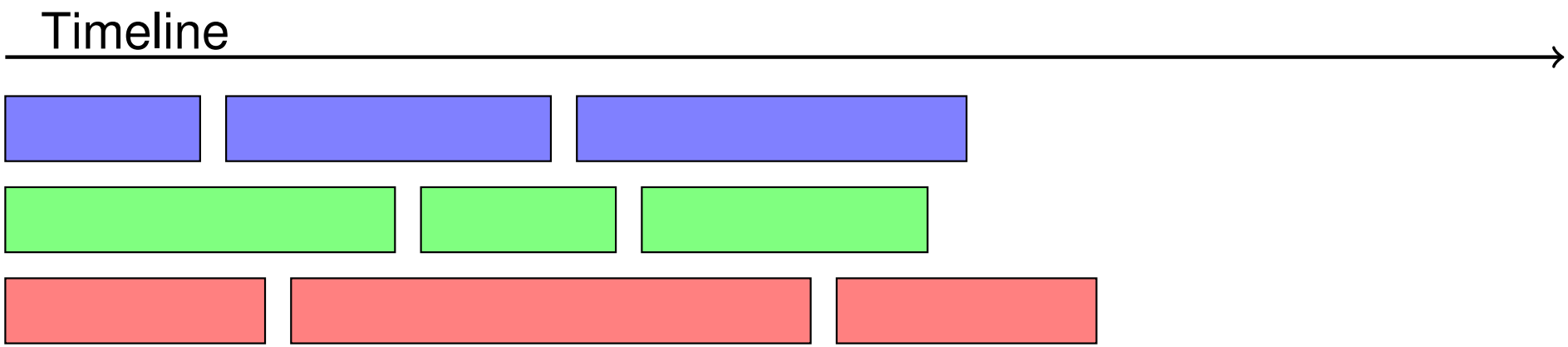
The synchronous world:



- Load imbalance causes the idle.
- Correct but slow.

Asynchronous Updates

The asynchronous world:



- No synchronizations among the workers.
- No idle time – every worker is kept busy.
- High scalability.
- Noisy but fast.

Hogwild! the Asynchronous SGD

If we remove all locking and synchronizing processes in parallel SGD. Then we obtain the Hogwild! algorithm.

Algorithm 1 Hogwild! the Asynchronous SGD

Each processor asynchronously do
loop
    Sample e randomly from E.
    Read the current point x_e from the memory.
    Compute $G_e(x) := |E|\nabla f_e(x)$.
    for $v \in e$ **do**
        $x_v \leftarrow x_v - \gamma b_v^T G_e(x)$.
    end for
end loop

Hogwild! the Asynchronous SGD

A variant of Hogwild!:

Algorithm 2 Hogwild! the Asynchronous SGD(Variant 1)

1: Each processor asynchronously do
2: **loop**
3: Sample e randomly from E.
4: Read the current point x_e from the memory.
5: Compute $G_e(x) := |E|\nabla f_e(x)$.
6: Sample $v \in e$, then $x_v \leftarrow x_v - \gamma |e| b_v^T G_e(x)$.
7: **end loop**

Note:

- The entire gradient is computed. Only one component is updated.
- The $|e|$ factor ensures $\mathbb{E}[|e| b_v^T G_e(x)] = \nabla f(x)$.
- Seems wasteful, but easy to analyze.

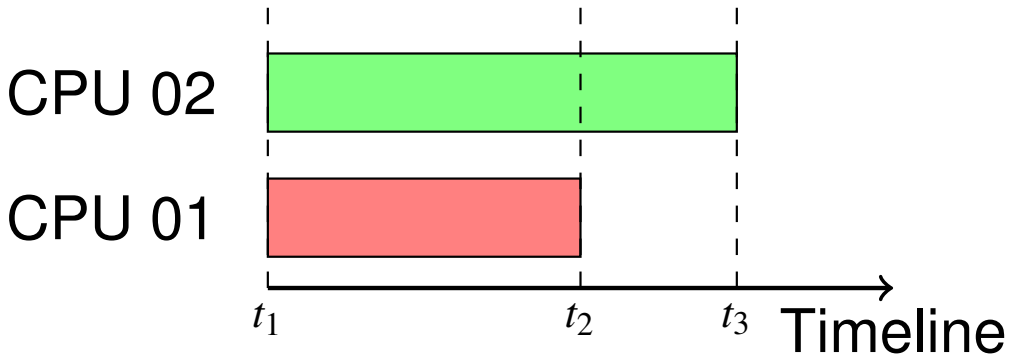
Issues of the Asynchronous Updates

- The inconsistency between the CPU cache and the memory makes the results incorrect.
- Older updates can be overwritten by newer ones.
- Suppose we want to perform two updates with two threads

$$\begin{aligned} x_3 &= x_2 - \gamma \nabla f_{e_2}(x_2) \\ &= x_1 - \gamma(\nabla f_{e_2}(x_2) + \nabla f_{e_1}(x_1)) \end{aligned}$$

- The computation of $\nabla f_{e_1}(x_1)$ and $\nabla f_{e_2}(x_2)$ is assigned to CPU1 and CPU2 respectively.

Issues of the Asynchronous Updates



time	x in ∇f		x in mem	∇f in cache		what	
t_1	x_1	x_1	x_1	—	—	read x_1 from memory	
t_2	x_2	x_1	$x_1 - \gamma \nabla f_{e_1}(x_1)$	$\nabla f_{e_1}(x_1)$	—	Update mem.	Computing
t_3	x_2	x_2	$x_2 - \gamma \nabla f_{e_2}(x_1)$	$\nabla f_{e_1}(x_1)$	$\nabla f_{e_2}(x_1)$	Idle	Update mem.

- What we should get: $x_1 - \gamma(\nabla f_{e_1}(x_1) + \nabla f_{e_2}(x_2))$.
- What we actually get: $x_1 - \gamma(\nabla f_{e_1}(x_1) + \nabla f_{e_2}(x_1))$.

Analysis of Hogwild!

Assumptions

- ∇f is Lipschitz continuous.

$$\|\nabla f(x) - \nabla f(y)\| \leqslant L\|x - y\|$$

- f is strongly convex with modulus c. Each f_e is convex.

$$f(y) \geqslant f(x) + (y - x)^T \nabla f(x) + \frac{c}{2}\|y - x\|^2$$

- The gradient of f_e is bounded. Recall that $G_e(x) = |E|\nabla f_e(x)$.

$$\|G_e(x)\| \leqslant M, \ \forall \ x$$

- $\gamma c < 1$, otherwise even gradient descent alg will fail.
- Based on the Hogwild!(Variant 1) algorithm.

Proposition 4.1 (Main result).

Suppose the lag between when a gradient is computed and when it is used in step j – namely $j - k(j)$ – is always less or equal than τ, and γ is defined to be

$$\gamma = \frac{\theta \varepsilon c}{2LM^2\Omega(1 + 6\rho\tau + 4\tau^2\Omega\Delta^{1/2})} \tag{1}$$

for some $\varepsilon > 0$ and $\theta \in (0, 1)$. Define $D_0 = \|x_0 - x_\star\|^2$ and let k be an integer satisfying

$$k \geqslant \frac{2LM^2\Omega(1 + 6\tau\rho + 6\tau^2\Omega\Delta^{1/2}) \log(LD_0/\varepsilon)}{c^2\theta\varepsilon} \tag{2}$$

Then after k component updates of x, we have $\mathbb{E}[f(x_k) - f_\star] \leqslant \varepsilon$.

Remarks on Prop 4.1

- Consider the Hogwild! algorithm as an SGD with lags.

$$x_{j+1} \leftarrow x_j - \gamma |e| \mathcal{P}_v G_e(x_{k(j)})$$

- The lags should be bounded by τ. That is, at the jth update, the point used to compute the gradient is within the last τ steps.
- τ is proportional to the number of threads.
- The step size γ should be $\mathcal{O}(\varepsilon)$ to ensure the convergence.
- To obtain an accuracy of ε, we need at least $\mathcal{O}(1/\varepsilon)$ updates, modulo the $\log(1/\varepsilon)$ factor.

Outline

Problem Definition

Consider

$$\min_{x \in \mathbb{R}^d} f(x) = \sum_{e \in E} f_e(x_e)$$

- e denotes a small subset of $\{1, 2, \ldots, d\}$.
- In many of machine learning problems, d and $n = |E|$ are both large.
- f_e only acts on a few components of x, say x_e.
- f_e can easily be regarded as a function over $\mathbb{R}^d$, just by ignoring the components not in the subset e.

Algorithms to Solve the Problem

- HOGWILD! (Asynchronous SGD):

$$x_{k+1} \leftarrow x_k - \gamma_k \nabla f_{s_k}(x_k)$$

- All cores perform the updates asynchronously with no memory locking.
- Ignores the conflict variables.
- No convergence in some cases. (Lose some of the properties of SGD)
- Complicated theoretical analysis |>_<|

Another Way?

To deal with conflicts:

- Traditional parallelizations
  - Lock the conflict variables when updating.
  - Ensure the correctness, but only limited speedup.
- HOGWILD!
  - Ignore the conflicts when updating.
  - Better speedup, but higher risk of not converging.
- CYCLADES
  - Avoid the conflicts by rearranging the updates.
  - Better speedup.
  - Preserve the property of SGD in high probability.

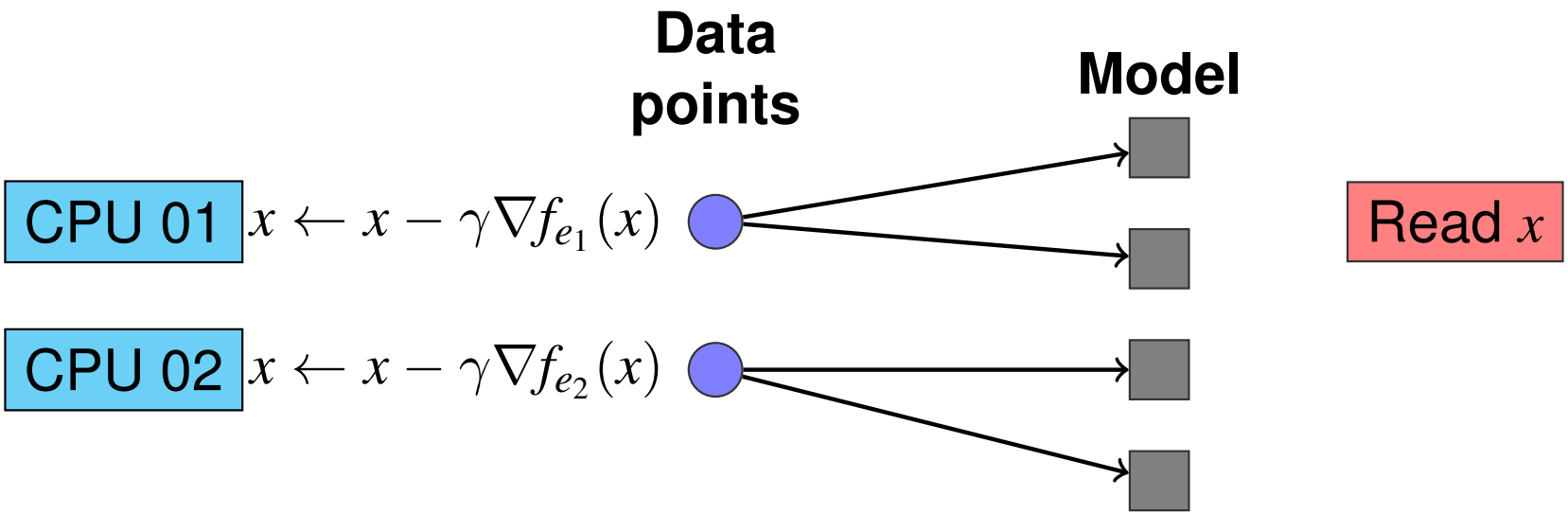
About the Conflicts

Why conflicts affect so much?

- CPUs don't have direct access to the memory.
- Computations can only be done on CPU cache.
- Data are read from the memory to CPU cache. After the computations are finished, results are pushed back to the memory.
- A CPU has no idea whether its 'colleagues' have local updates which have not been pushed to the memory.

About the Conflicts

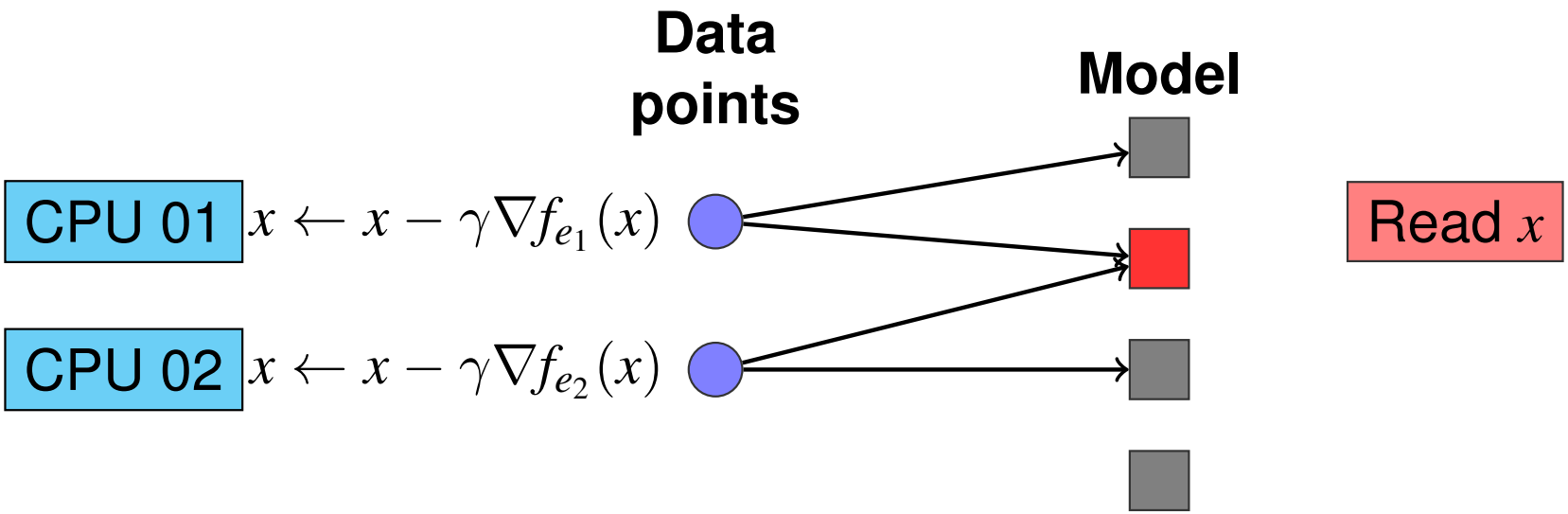
Good Case:



- The components of f_{e_1} and f_{e_2} don't overlap.
- Two parallel updates means two serial updates.
- No conflicts means speed up!

About the Conflicts

Bad Case:



- The components of f_{e_1} and f_{e_2} overlap at x_2.
- Conflicts mean less parallelization.
- What should be done to handle the conflicts? HOGWILD! tells us to ignore it. CYCLADES tells us to avoid it.

The Updates Conflict Graph

Definition 1 (Bipartite update-variable graph).

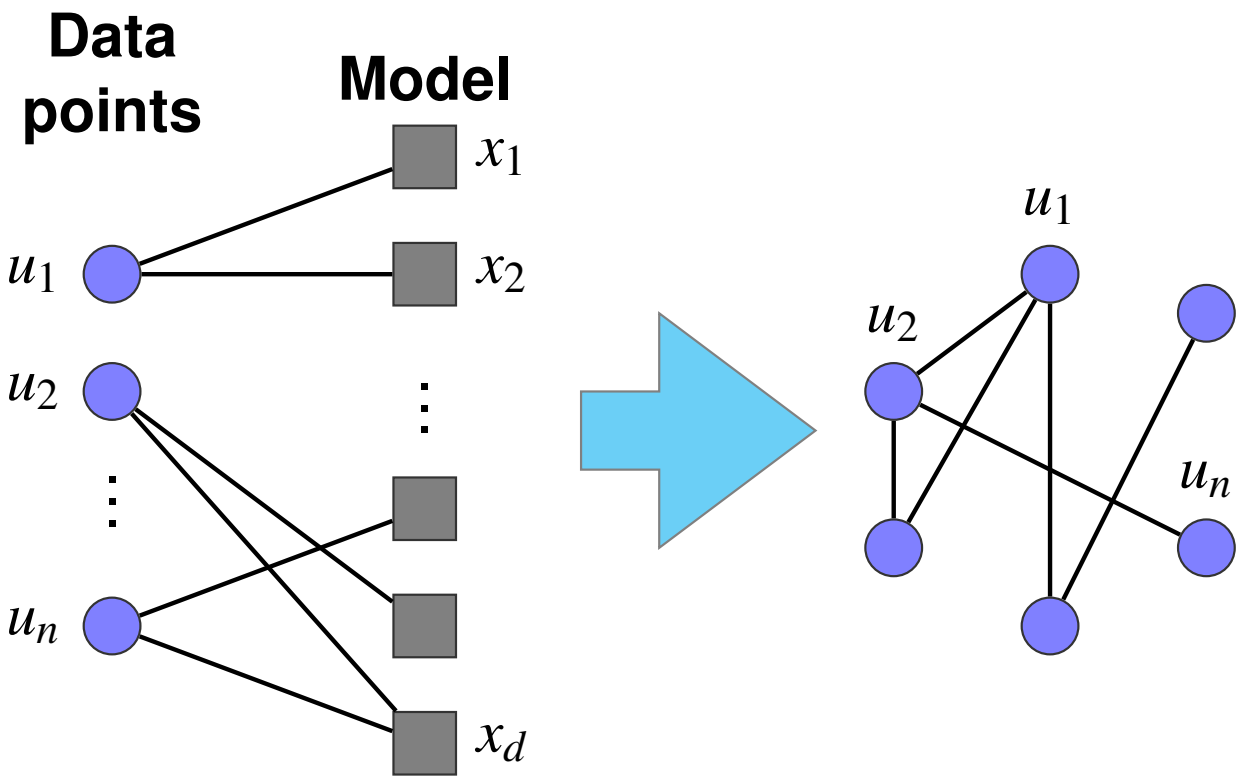
We denote as G_u the bipartite update-variable graph between the updates $u_1, \ldots, u_n$ and the d variables. In G_u an update u_i is linked to a variable x_j, if u_i requires to read and write x_j. E_u denotes the number of edges in the bipartite graph. Δ_L denotes the left max vertex degree of G_u. $\bar{\Delta}_L$ denotes its average left degree.

Definition 2 (Conflict Graph).

We denot by G_c a conflict graph on n vertices, each corresponding to an update u_i. Two vertices of G_c are linked with an edge, if and only if the corresponding updates share at least one variable in the bipartite-update graph G_u. Δ is denoted as the max vertex degree of G_c.

The Updates Conflict Graph

From a bipartite graph to a conflict graph



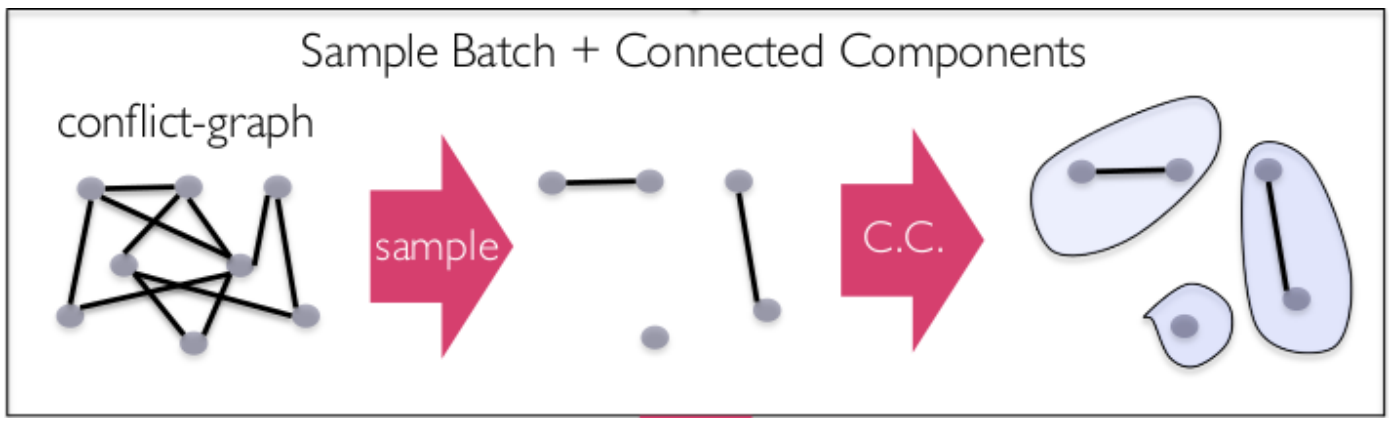
The Updates Conflict Graph

Remarks

- In the conflict graph, if two vertices are linked with an edge, then their corresponding updates conflict with each other.
- G_c is introduced to analyze the CYCLADES algorithm. In practice we never construct it.
- Suppose the number of connected components(CC) is N_{cc}, then we can employ N_{cc} workers to process the updates asynchronously. No conflicts occur. **However,** N_{cc} **equals to 1 for most problems!**

The Idea of CYCLADES

- We need more CCs to perform parallelization.
- Just sample part of the updates.



How many CCs we can get if we perform the sampling?

The Number of CCs

Theorem 1.

Let G be a graph on n vertices, with maximum vertex degree Δ. Let us sample each vertex independently with probability $p = \frac{1-\varepsilon}{\Delta}$ and define G' as the induced subgraph on sampled vertices. Then, the largest CC of G' has the size at most $\frac{4}{\varepsilon^2} \log n$, with high probability.

Theorem 2.

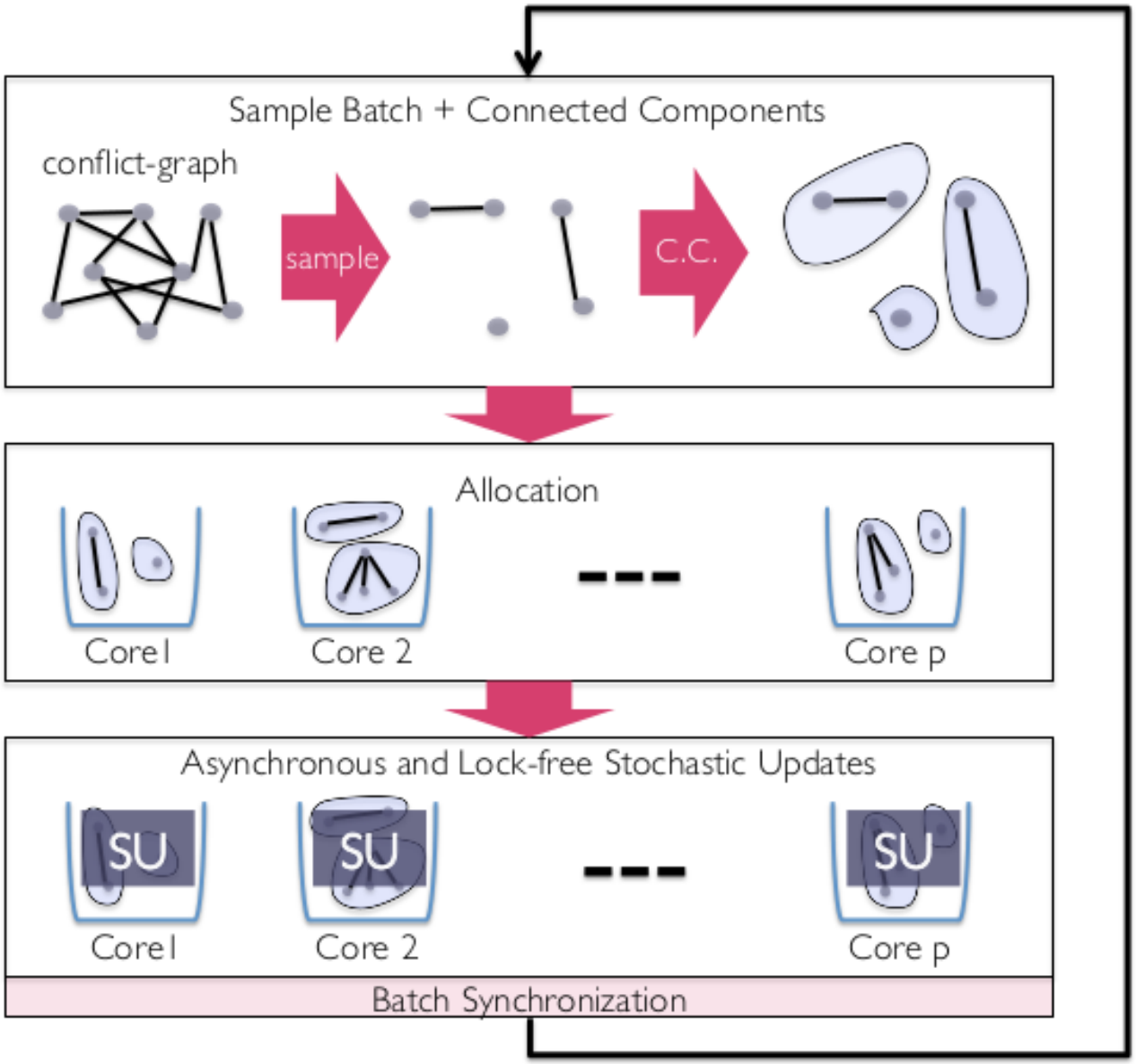
Let G be a graph on n vertices, with maximum vertex degree Δ. Let us sample $B = (1-\varepsilon)\frac{n}{\Delta}$, with or without replacement, and define G' as the induced subgraph on sampled vertices. Then, the largest CC of G' has the size at most $\mathcal{O}\left(\frac{\log n}{\varepsilon^2}\right)$, with high probability.

The Number of CCs

Remarks

- Theorem 2 can be regard as a corollary of Theorem 1(Not obvious!). We'll give a brief proof of Theorem 1 later.
- From Theorem 2, if one samples $B = (1-\varepsilon)\frac{n}{\Delta}$ vertices, then there will be at least $\mathcal{O}(\varepsilon^2 B/\log n)$ CCs, each of size at most $\mathcal{O}(\log n/\varepsilon^2)$.
- The number of CCs is increased a lot – OK. Good for parallelization.

The Idea of CYCLADES



The Idea of CYCLADES

- After sampling, use certain algorithm to find all CCs in a parallel way(to be continued).
- When all CCs are determined, allocate them to the workers(consider the load balance).
- Each worker performs stochastic updates asynchronously and individually, with no conflicts and false sharing issues.
- Repeat the three phases until finished.

CYCLADES Outline

Algorithm 3 CYCLADES

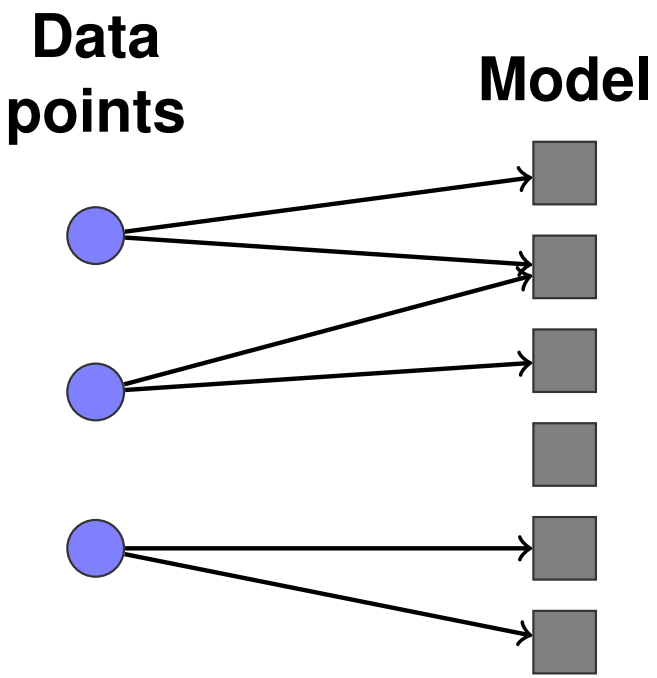
1: **Input**: G_u, T, B.
2: Sample $n_b = T/B$ subgraphs from G_u.
3: **for** batch $i = 1 : n_b$ **do**
4: Allocate CCs to P cores.
5: **for** each core asynchronously **do**
6: Perform local updates in allocated CCs.
7: **end for**
8: **end for**

- T is the total updates you'd like to run.
- One synchronization at the end of each batch.
- G_u is determined by the structure of problem.

Compute CCs of All Batches in Parallel

- If we use the conflict graph G_c to compute the CCs...
- The cost is the order of sampled edges.
- However, constructing G_c requires n^2 time. A high cost – no, thanks.
- Compute CCs using the bipartite graph only!

Compute CCs of All Batches in Parallel



Simple message passing idea

- Data points send their label.
- Variables compute min and send back.
- Data points compute min.
- Iterate until you're done.

Compute CCs of All Batches in Parallel

Remarks

- The iterations needed are bounded with the length of the longest-shortest path(i.e. the diameter of G_c).
- Overall complexity: $\mathcal{O}(E_u \log n/P)$, P is the number of cores. Note: useful when P is large.
- Only need the bipartite graph G_u, not the conflict graph G_c.

Allocating the Conflict Groups

Once we obtain the CCs of the batch, we have to allocate these CCs to P cores.

- w_i is the cost of updating with the ith data point.
- $W_{C(i)} = \sum_{j \in C(i)} w_j$ is the cost of updating the ith CC.
- For each batch update, we need to

$$\min \max W_{C(i)}$$

  to get the best load balance.
- An NP hard problem. We can utilize an approximation algorithm to obtain a sub-optimal plan.

Allocating the Conflict Groups

Algorithm 4 Greedy allocation

1: Estimate $W_1, \ldots, W_m$, the cost of each CC.
2: Sort the W_i's in the descending order.
3: **for** $i = 1 : m$ **do**
4: Choose the currently largest W_i.
5: Add W_i to the cores with least sum of cost currently.
6: **end for**

- A $4/3$-approximation algorithm.
- w_i is proportional to the out-degree of that update.

Main Results

Theorem 3 (Main result).

Let us assume any given update-variable graph G_u with average, and max left degree $\bar{\Delta}_L$ and Δ_L, such that $\Delta_L/\bar{\Delta}_L \leqslant \sqrt{n}$, and with induced max conflict degree Δ. Then, CYCLADES on $P = \mathcal{O}(n/\Delta\Delta_L)$ cores with batch sizes $B = (1-\varepsilon)\frac{n}{\Delta}$ can execute $T = cn$ updates, for any $c > 1$, in time

$$\mathcal{O}(\frac{E_u \cdot \kappa}{P} \log^2 n)$$

with high probability.

- κ is the cost to update one edge in E_u.
- CYCLADES can achieve the same result as the serial algorithms with high probability. Thus it requires similar properties of the objective function f and its components f_e.

Main Results

Remarks of Thm 3

- The sampling of each batch can be with or without replacement.
- The number of cores is bounded, for it's hard to allocate the work of computing CCs evenly if P is too large.
- The batch size B should be bounded. This is very important. Small B will induce a subgraph with many CCs.
- Theorem 1 is the most important foundation to develop the main result.