Homework 3 for “Algorithms for Big-Data Analysis”

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1. For each of the following function on \( \mathbb{R}^n \), explain how to calculate a subgradient at a given \( x \).
   A reference on subgradients is http://bicmr.pku.edu.cn/~wenzw/bigdata/subgradients.pdf
   
   (a) \( f(x) = \|Ax - b\|_2 + \|x\|_2 \) where \( A \in \mathbb{R}^{m \times n} \) and \( x \in \mathbb{R}^n \).
   
   (b) \( f(x) = \inf_y \|Ay - x\|_\infty \) where \( A \in \mathbb{R}^{n \times m} \) and \( x \in \mathbb{R}^n \).

2. Give a formula or simple algorithm for evaluating the proximal operator
   \( \text{prox}_f(x) = \arg \min_u \left( f(u) + \frac{1}{2} \|u - x\|_F^2 \right) \).
   
   (a) \( f(x) = \|x\|_1 \) with domain \( \text{dom}(f) = \{ x \mid \|x\|_\infty \leq 1 \} \)
   
   (b) \( f(x) = \max_k x_k \)
   
   (c) \( f(x) = \|Ax - b\|_1 \) where \( AA^T = D \) and \( D \) is a diagonal matrix whose diagonal elements are positive.

3. Given \( w \in \mathbb{R}^n, \alpha, \sigma > 0 \), write down an algorithm for solving the problem
   \[
   \min_{t,y} \phi(t,y),
   \]
   where
   \[
   \phi(t,y) := t + \frac{1}{(1 - \alpha)n} \sum_{i=1}^{n} (y_i - t)_+ + \frac{\sigma}{2} \|y - w\|_2^2,
   \]
   where \( x_+ := \max(x, 0) \).

4. Let \( S^n = \{ X \in \mathbb{R}^{n \times n} \mid X^T = X \} \) and \( S^++_n = \{ X \in \mathbb{R}^{n \times n} \mid X^T = X, X \text{ is positive definite} \} \). Find the proximal operator of the function \( f(X) = -\log \det X \) where \( X \in S^n \) and \( \text{dom} f = S^++_n \). Here, the proximal operator is defined as
   \[
   \text{prox}_f(X) = \arg \min_U \left( f(U) + \frac{1}{2} \|U - X\|_F^2 \right),
   \]
   where \( \| \cdot \|_F \) is the Frobenius norm.