Lectures 4-5 of Berger's IHP notes

Lecture 4 Higher ramification theory

- K CDVF of residue field k.

  Assume $\mathrm{char}\, k = p > 0$ & k is perfect
- For L a finite ext'n of K, we normalize v_L so that $v_L(L^\times) = \mathbb{Z}$

  ϖ_L uniformizer $v_L(\varpi_L) = 1$.

<u>Def'n</u> Let L/K be a finite Galois ext'n. We define the <u>lower ramification filtration</u> as follows

$$G_{L/K,u} := \{ g \in G_{L/K} \text{ such that } \forall x \in \mathcal{O}_L,\ v_L(gx - x) \geq u+1 \}$$

Enough to test this on a generator α s.t. $\mathcal{O}_L = \mathcal{O}_K[\alpha]$

wild inertia

In particular, $G_{L/K} = \underbrace{G_{L/K,-1} \supseteq I_{L/K}}_{G_{\ell/k}} = \underbrace{G_{L/K,0} \supseteq P_{L/K}}_{\text{quot of } \prod_{\ell \neq p} \mathbb{Z}_\ell(1)} = \underbrace{G_{L/K,1} \supseteq G_{L/K,2} \supseteq}_{\text{all } p\text{-groups}} \cdots$

For $n \geq 1$
$$\mathrm{gr}_n G_{L/K} \hookrightarrow \frac{(1+\varpi_L^n \mathcal{O}_L)^\times}{(1+\varpi_L^{n+1}\mathcal{O}_L)^\times} \simeq \varpi_L^n \cdot \ell$$

$$g \longmapsto g\varpi_L / \varpi_L \qquad \text{indep. of choice of } \varpi_L$$

$\Rightarrow$ each $\mathrm{gr}_n G_{L/K}$ is an abelian group killed by p.

<u>Key fact</u>: If $L \,|\, F \,|\, K$ (Galois)
$$G_{L/F,u} = G_{L/K,u} \cap G_{L/F}$$

<u>But</u>: $G_{F/K,u}$ & $G_{L/K,u}$ are not directly related

<u>Definition</u>. For L/K above, define the <u>Herbrand function</u> as follows:

for $u \geq -1$, $$\varphi_{L/K}(u) := \int_0^u \frac{dt}{[G_{L/K,0} : G_{L/K,t}]}$$

This is a continuous piecewise linear, increasing, concave function

Let $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be its inverse.

& define $G^v_{L/K} := G_{L/K, \psi_{L/K}(v)}$.

or equivalently, $G^{\varphi_{L/K}(u)}_{L/K} = G_{L/K,u}$

E.g. $L/K = \mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ $\quad n \geq 2$. $\quad G = G_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p}$

$L = \mathbb{Q}_p(\zeta_{p^n})$

$|$

$\vdots$

$|$

$\mathbb{Q}_p(\zeta_{p^2}) \qquad (1+p^2\mathbb{Z}_p/p^n\mathbb{Z}_p) = G_{p^2-1} = \cdots = G_p$ $\quad$ sends ζ_{p^n} to $\zeta_{p^n}^{1+p^2}$

$v_L(\zeta_{p^n} - \zeta_{p^n}^{1+p^2}) = v_L(1-\zeta_{p^{n-2}}) = p^2$

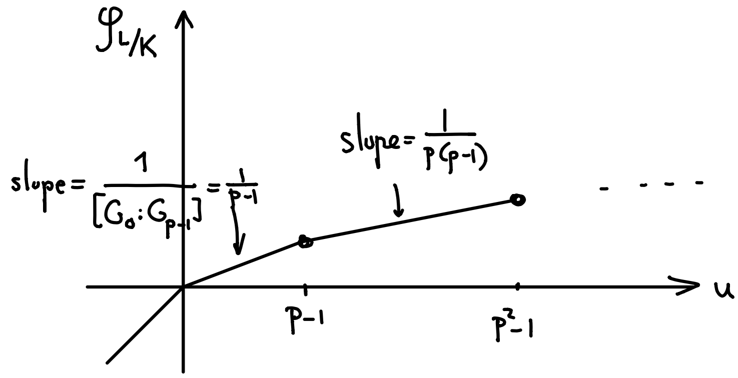
$| \qquad\qquad \cap$

$\mathbb{Q}_p(\zeta_p) \qquad (1+p\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times = G_{p-1} = G_{p-2} = \cdots = G_1$ $\quad$ sends ζ_{p^n} to $\zeta_{p^n}^{1+p}$

$v_L(\zeta_{p^n} - \zeta_{p^n}^{1+p}) = v_L(1-\zeta_{p^{n-1}})$
$= p$

$| \qquad\qquad \cap$

$\mathbb{Q}_p \qquad (\mathbb{Z}/p^n\mathbb{Z})^\times = G_0 = G_{-1}$



$$\Rightarrow \quad G^0 = G_0 \;\cup\; G^1 = G_{p-1} \;\cup\; G^2 = G_{p^2-1} \;\cdots$$

<u>Key Theorem</u> If $L \,|\, F \,|\, K$ is a tower of Galois representation, then the image of $G^v_{L/K}$ in $G_{F/K}$ is $G^v_{F/K}$.

In other words, $G^v_{F/K} = G^v_{L/K} G_{L/F} / G_{L/F}$.

<u>Cor</u>: For $v \geq -1$, can define $G^v_K := \varprojlim_{L/K} G^v_{L/K}$

and $G^v_{L/K} = G^v_K G_L / G_L$.

Also define $G^{v+}_K := \overline{\bigcup_{w > v} G^w_K}$ ← closure

$\rightsquigarrow$ Artin & Swan conductors:

For a finite image rep'n $\rho : G_K \to GL(V)$ $\quad V$ / char 0 field.

$$\mathrm{Art}(\rho) = \sum_{v \geq 0} v \cdot \dim \left(V^{G_K^v} \big/ V^{G_K^{v+}} \right)$$

$$\mathrm{Swan}(\rho) = \sum_{v \geq 1} (v-1) \cdot \dim \left(V^{G_K^v} \big/ V^{G_K^{v+}} \right)$$

Explicitly, if V is irreducible, $\rho : G_K \twoheadrightarrow G_{L/K} \hookrightarrow GL(V)$

$$\mathrm{Art}(\rho) = \dim V \cdot \underbrace{b(L/K)}$$

upper numbering ramification break

$= \inf \{ u \,;\, G_{L/K}^u = 1 \}$

<u>Theorem</u> (Hasse-Arf) $\quad \mathrm{Art}(\rho), \mathrm{Swan}(\rho) \in \mathbb{Z}$.

<u>Cor</u>: If L/K cyclic abel. ext'n, $\exists$ faithful char $\chi : G_{L/K} \hookrightarrow \mathbb{C}^\times$

then $\mathrm{Art}(\chi) = b(L/K) \in \mathbb{Z}$.

$\Rightarrow$ All "upper ramification breaks" of abelian ext'ns are integers.

<u>Traces</u>: L/K. $\quad \mathrm{Tr} : L \times L \longrightarrow K$

$$a, b \longmapsto \mathrm{Tr}_{L/K}(a, b)$$

The dual of $\mathcal{O}_L$ is $\delta_{L/K}^{-1}$, $\quad \delta_{L/K}$ = different ideal.

If $\mathcal{O}_L = \mathcal{O}_K[\alpha] = \mathcal{O}_K[x]/P(x)$, then

$\delta_{L/K}$ is generated by $P'(\alpha)$.

$\Rightarrow$ When L/K is Galois,

$$\mathrm{val}_L(\delta_{L/K}) = \mathrm{val}_L \Big(\prod_{\substack{g \neq 1 \\ g \in G_{L/K}}} (g(\alpha) - \alpha) \Big) = \sum_{t=0}^{\infty} \left| G_{L/K,t} - \{1\} \right|$$

$$= \int_{-1}^{\infty} \left(|G_{L/K,t}| - 1 \right).$$

For $b \in \mathbb{R}_{\geq -1}$ & L a fin. ext'n of K,

set $L^b := L^{G^b_{L/K}}$

<u>Example</u> If $K = \mathbb{Q}_p$, $L_n := \mathbb{Q}_p(\zeta_{p^n})$

then $L_n^u \subseteq L_{\lceil u \rceil}$

(b/c $G^u_{L_n/K} = \left(1 + p^{\lceil u \rceil}\mathbb{Z}_p / p^n \mathbb{Z}_p\right)^\times$

Then
$$\mathrm{val}_K(\delta_{L/K}) = \frac{1}{e_{L/K}} \cdot \int_{-1}^{\infty} \left(|G_{L/K,t}| - 1 \right)$$
$$= \int_{-1}^{\infty} \left(\frac{|G_{L/K}^{\varphi_{L/K}(t)}|}{e_{L/K}} - \frac{1}{e_{L/K}} \right)$$
$$\overset{\substack{u = \varphi_{L/K}(t) \\ t = \psi_{L/K}(u)}}{=} \int_{-1}^{\infty} \left(\left(\frac{|G^u_{L/K}|}{e_{L/K}} \right) - \frac{1}{e_{L/K}} \right) \cdot [G^0_{L/K} : G^u_{L/K}]$$
$$= \int_{-1}^{\infty} \left(1 - \frac{1}{|G^u_{L/K}|} \right) = \int_{-1}^{\infty} \left(1 - \frac{1}{[L : L^u]} \right)$$

<u>Lemma 4.2</u> If L/K fin. ext'n and if $I \subseteq \mathcal{O}_L$ is an ideal, then
$$v_K\left(\mathrm{Tr}_{L/K}(I)\right) = \lfloor v_K(I \cdot \delta_{L/K}) \rfloor$$

Proof: Note: $\mathrm{Tr}_{L/K}(x \cdot \mathcal{O}_L) \subseteq \mathcal{O}_K \iff x \in \delta_{L/K}^{-1}$

$\Rightarrow$ for $I \subseteq \mathcal{O}_L$, $J \subseteq \mathcal{O}_K$,
$$\mathrm{Tr}_{L/K}(I) \subseteq J \iff I \subseteq \delta_{L/K}^{-1} \cdot J \iff I \cdot \delta_{L/K} \subseteq J$$

So $\mathrm{Tr}_{L/K}(I)$ is the smallest ideal $J \subseteq \mathcal{O}_K$ containing s.t. $J \cdot \mathcal{O}_L$ contains $I \cdot \delta_{L/K}$

Lecture 5 Ax and Ax-Sen-Tate's theorem

Theorem 5.1. If F is a complete valued field, then $\widehat{F^{alg}}$ is algebraically closed.

Proof: Prove by induction on degree that every polynomial $P(x) \in \widehat{F^{alg}}[x]$ of degree ≥ 1 has a root.

- May assume that $P(X) \in \mathcal{O}_{\widehat{F^{alg}}}[x]$ is monic

(o/w can use $NP(P)$ to factor it)

Write $P(X) = \lim P_n(x)$ for monic $P_n(x) \in \mathcal{O}_{F^{alg}}[x]$

Pick one root α_n of P_n so $P(\alpha_n) \to 0$

* If $P'(\alpha_n) \not\to 0$, then $\exists\, n$ s.t. $P(\alpha_n)$ small but $P'(\alpha_n)$ not too small

Hensel's lemma $\Rightarrow \exists\, \alpha \in \mathcal{O}_{\widehat{F^{alg}}}$ a zero of P

* If $P'(\alpha_n) \to 0$, then inductive hypo $\Rightarrow P'(x)$ decomposes

$\Rightarrow \alpha_n \to$ a zero of $P'(x)$, say α

Then $P(\alpha)$ is the limit of $P(\alpha_n)$, which is also 0. ✓

Theorem 5.2 If F is a valued field of char $p>0$, then F^{sep} is dense in F^{alg}.

Proof: If $y \in F^{alg}$, then $y^{p^n} = \alpha \in F^{sep}$ for some n.

WLOG $v(\alpha) \geq 0$.

Take $\pi \in F$ with $v(\pi) > 0$

seperable equation

Let y_i be a root of $Y^{p^n} - \pi^i Y - \alpha = 0 \quad \Rightarrow y_i \in F^{sep}$ & $v(y_i) \geq 0$

$$\Rightarrow \left.\begin{array}{l} y^{p^n} = \alpha \\ y_i^{p^n} = \alpha + \pi^i y_i \end{array}\right\} \Rightarrow (y - y_i)^{p^n} = \pi^i y_i$$

$\Rightarrow y_i \to y$ as $i \to \infty$

Theorem 5.3 (Ax-Sen-Tate)

If F is a complete p-adic field, and if $F \subset K \subset F^{alg}$ (K a possibly infinite ext'n of F)

then $\left(\widehat{F^{alg}}\right)^{Gal(F^{alg}/K)} = \widehat{K}$

$$(\widehat{F^{alg}})^{G_K} = \hat{K}$$

Proof: Clearly, $\hat{K} \subseteq \widehat{F^{alg}}^{\,G_K}$

Why do we care?

<u>Answer</u>: Making sure that take completion is okay with Galois theory.

$$\begin{array}{ccccc}
\mathbb{Q}_p^{alg} & & F^{alg} & \rightsquigarrow & \widehat{F^{alg}} \\
| & & |\,\big)\, G_K & & |\,\big)\, \text{“}G_K\text{”} \\
\mathbb{Q}_p(\mu_{p^\infty}) & & K & \rightsquigarrow & \hat{K} \\
| & & | & & |\,\big)\, \text{“}\mathrm{Gal}(\hat{K}/F)\text{”} \\
\text{e.g. } \mathbb{Q}_p & & F & & F
\end{array}$$

Now show the opposite inclusion

If $\alpha \in F^{alg}$, let $\Delta_K(\alpha) := \inf\limits_{g \in G_K} \mathrm{val}_p(g(\alpha) - \alpha)$ $\qquad \mathrm{val}_p(p) = 1$.

(so if $\alpha \in K$, then $\Delta_K(\alpha) = \infty$.)

Let $\alpha \in \widehat{F^{alg}}^{\,G_K}$. Write $\alpha = \lim \alpha_n$ with $\alpha_n \in F^{alg}$

Then $\Delta_K(\alpha_n) \geq \mathrm{val}_p(\alpha - \alpha_n) \longrightarrow \infty$

<u>Lemma 5.4.</u> If $\alpha \in F^{alg}$, then $\exists\, \delta \in K$ s.t.

$$\mathrm{val}_p(\alpha - \delta) \geq \Delta_K(\alpha) - \frac{p}{(p-1)^2}$$

$\Rightarrow$ $\exists\, \delta_n \in K$ s.t. $\mathrm{val}_p(\alpha_n - \delta_n) \to \infty$ $\Rightarrow \alpha = \lim \delta_n$ ✓.

Proof: Induction on $n := [K(\alpha) : K]$ s.t. we can find such a δ

with $\mathrm{val}_p(\alpha - \delta) \geq \Delta_K(\alpha) - \sum\limits_{k=0}^{m} \dfrac{1}{p^k(p-1)}$ $\quad$ where $p^{m+1} \leq n < p^{m+2}$

Basically, this says that if $\Delta_K(\alpha)$ is large, can find $\delta \in K$ very close to α.

assume this

Write $Q(x) = \mathrm{minpoly}(\alpha)$ over K

When $K(\alpha)$ is Galois over K,

$$Q(x) = \prod_{g \in \mathrm{Gal}(K(\alpha)/K)} (x - g(\alpha)) \approx (x-\alpha)^n \quad \text{has coeffs in } K.$$

So if $p \nmid n$. $\quad \alpha \approx -\dfrac{a_{n-1}}{n}$ ✓

if $n = p^k \cdot r$, then we consider a_{n-p^k}.

Say degree $Q(x) = n$. $Q(x) = x^n + a_{n-1}x^{n-1} + \cdots \in K[x]$

Case 1: $p \nmid n$, then $val_p(a_{n-1} - n\alpha) \geq \Delta_K(\alpha)$

We are done!

$$\begin{array}{l} K(\alpha) \\ \Big| \text{ degree prime-to-}p \\ K_r \\ \Big| \text{ p-power} \\ K \end{array}$$

Apply Case 1, $\exists\, \alpha' \in K_r$

s.t. $val_p(\alpha - \alpha') \geq \Delta_K(\alpha)$

$\Rightarrow \Delta_K(\alpha') \geq \Delta_K(\alpha)$

Reduces question to α'.

Case 2. $n = p^k$.

Reduce α zero of $Q(x)$

to β zero of $Q^{(p^{k-1})}(x)$) by our earlier philosophy $\alpha \approx \beta$.

Need to find a β s.t. $val_p(\alpha - \delta)$ is "large" $\geq \Delta_K(\alpha) - \frac{1}{p^{k-1}(p-1)}$

Let $P(x) = Q(x+\alpha) \in F^{alg}[x]$

Known: All zeros of $P(x)$ have val'n $\geq \Delta_K(\alpha)$

WTS: $P^{(p^{k-1})}(x) = Q^{(p^{k-1})}(x+\alpha)$ has a zero "$\beta - \alpha$" whose val'n $\geq \Delta_K(\alpha) - \frac{1}{p^{k-1}(p-1)}$

Write $P(x) = x^{p^k} + \cdots + a_{p^k}$

$\Rightarrow val_p(a_{p^k - p^{k-1}}) \geq (p^k - p^{k-1}) \cdot \Delta_K(\alpha)$

$$\frac{P^{(p^{k-1})}(x)}{p^k \cdot \cdots \cdot (p^k - p^{k-1} + 1)} = x^{p^k - p^{k-1}} + \cdots + \frac{a_{p^k - p^{k-1}}}{\binom{p^k}{p^{k-1}}}$$

So at least one zero of $P^{(p^{k-1})}(x)$ has val'n

$$\geq \frac{1}{p^k - p^{k-1}} \left(val_p(a_{p^k - p^{k-1}}) - \underbrace{val_p\binom{p^k}{p^{k-1}}}_{=1} \right)$$

$$\geq \Delta_K(\alpha) - \frac{1}{p^k - p^{k-1}}$$

So question about α $\rightsquigarrow$ question about β $\longrightarrow$ . . . .

$[K(\alpha):K] = p^k$ $\quad$ $[K(\beta):K] < p^k$ $\quad$ degree p-power part of $[K(\beta):K]$