Lectures 2–5 of Berger's IHP notes

Lecture 2 Hensel's lemma & Newton polygon.

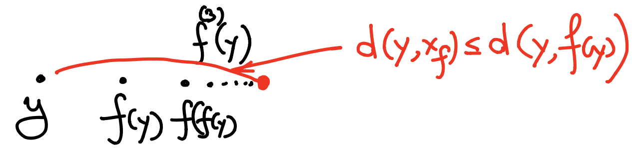
Theorem 2.1 : (Fix point theorem)

Let X be a complete metric space & $f: X \to X$ satisfies
$$d(f(x), f(y)) \le \lambda \cdot d(x,y) \text{ for some } 0 \le \lambda < 1.$$

Then f admits a unique fixed pt x_f.

(proof: Start w/ any x, $x, f(x), f(f(x)), \cdots$ } same limit)
if $y \neq x$ $y, f(y), f(f(y)), \cdots$

Moreover, if the metric is an ultrametric (satisfies strong triangle inequality)
then for every $x \in X$, $d(y, x_f) = d(y, f(y))$



Theorem 2.2 (Hensel's lemma ; Newton method form)

- A domain, complete for a valuation $\mathrm{val}(\cdot)$, which is ≥ 0 on A.

  e.g. $\mathbb{Z}_p$, $\mathbb{Z}_p[x]^\wedge \leftarrow \mathrm{val}\left(\sum_{n\geq 0} a_n x^n\right) = \min_n \mathrm{val}(a_n)$

  K CDVF with residual field $\mathbb{F}_p(x_1, \cdots, x_n)$, $A = (\text{subring of } \mathcal{O}_K)^\wedge$.

- If $P(x) \in A[x]$ & $\alpha_0 \in A$, s.t.

  ① $P(\alpha_0)/P'(\alpha_0)^2 \in A$

  ② $\mathrm{val}\left(P(\alpha_0)/P'(\alpha_0)^2\right) = \varepsilon > 0$,

  } e.g. $P(x) \in \mathcal{O}_K[x]$ & $\begin{cases} P(\alpha_0) \bmod m_K = 0 \\ P'(\alpha_0) \bmod m_K \neq 0 \end{cases}$

Then $\exists\, \alpha \in A$ s.t. $P(\alpha) = 0$ and $\mathrm{val}(\alpha - \alpha_0) \geq \varepsilon + \mathrm{val}(P'(\alpha_0))$

i.e. if α_0 is so that $P(\alpha_0)$ is small, then α_0 is very close to a zero of $P(x)$

Proof is in Berger's notes, instead, we explain the proof of its generalization
"Improving a factorization $P \approx QR$ to $P = \tilde{Q}\tilde{R}$" when $A = \mathcal{O}_K$.

- Gauss norm on $K[x]$, $\|a_0+a_1x+\dots+a_nx^n\|_G := \max|a_i|$

Set $W_d := K[x]^{\deg\le d-1}$

- If $Q(x)$ & $R(x)$ are polynomials of deg $\le m$ & $\le n$, resp.

consider
$$\begin{aligned}\theta_{Q,R}: W_n \times W_m &\longrightarrow W_{m+n-1}\\ (U, V) &\longmapsto QU+RV\end{aligned}$$

$$\mathrm{res}(Q,R) := \det(\theta_{Q,R})_{\text{basis } 1,x,x^2,\dots}$$

Rmk: When $P(x) = a_0+a_1x+\dots+a_nx^n$,

$Q(x) = x-\alpha_0$, $\rightsquigarrow P(x) = (x-\alpha_0)\cdot \underbrace{R(x)}_{\deg n-1} + P(\alpha_0)$

$$\begin{aligned}\theta_{Q,R}: K \times W_{n-1} &\longrightarrow W_n\\ (a, V) &\longmapsto a\cdot R(x) + (x-\alpha_0)\cdot V\end{aligned}$$

To compute $\det\theta_{Q,R}$, may change variable $x \mapsto x+\alpha_0$,

& assume $\alpha_0 = 0$ $\Rightarrow R(x) = a_1+a_2x+\dots+a_{n-1}x^{n-1}$

$$\mathrm{res}(\theta_{Q,R}) = \det\begin{pmatrix} a_1 & & & 0\\ a_2 & 1 & & \\ \vdots & & 1 & \\ a_{n-1} & 0 & & \ddots\ 1\end{pmatrix} = a_1 = P'(\alpha_0)$$

Theorem 2.3. (Improve a factorization $P\approx QR$ into $P=\tilde{Q}\tilde{R}$)

* $P, Q, R \in \mathcal{O}_K[x]$ $\deg P = m+n$, $\deg Q = m$, $\deg R = n$
* $0 \le \lambda < 1$.

Assume
- $\deg(P-QR) \le m+n-1$
- $\|P-QR\|_G \le \lambda\cdot|\mathrm{res}(Q,R)|^2$

$\Longleftrightarrow$ $\left|\dfrac{P(\alpha_0)}{P'(\alpha_0)^2}\right| \le \lambda.$ when $Q = x-\alpha_0$, $P = QR+P(\alpha_0)$

Then there exist polynomials $\tilde{Q}$ and $\tilde{R}$ such that

(1) $\deg(\tilde{Q}-Q) \le m-1$ and $\|Q-\tilde{Q}\|_G \le \lambda\cdot|\mathrm{res}(Q,R)|$

(2) $\deg(\tilde{R}-R) \le n-1$ and $\|R-\tilde{R}\|_G \le \lambda\cdot|\mathrm{res}(Q,R)|$

(3) $P = \tilde{Q}\tilde{R}$.

Proof: If $\mathrm{res}(Q,R) = 0$, we are done as $P = QR$.

Assume $\rho := |\mathrm{res}(Q,R)| \neq 0$. & $\rho < 1$.

Then $\theta_{Q,R} : W_n \times W_m \longrightarrow W_{m+n}$ is invertible.

Hope: $P = (Q+V)(R+U)$

$$\iff \underbrace{QU + RV}_{\theta_{Q,R}(U,V)} = QR - P + UV$$

Consider $\varphi : W_n \times W_m \longrightarrow W_n \times W_m$

$$(U,V) \longmapsto \theta_{Q,R}^{-1}(QR - P + UV)$$

so that $\varphi(U,V) = (U,V)$

$$\iff \theta_{Q,R}(U,V) = QR - P + UV \quad \checkmark$$

Want to use fixed point theorem on

$$\varphi \circlearrowright B(0,\lambda\rho) \subseteq W_n \times W_m$$

$$B(0,\lambda\rho) = \{(U,V) \in W_n \times W_m \,,\ \|U\|_G \le \lambda\rho,\ \|V\| \le \lambda\rho\}$$

Need to check: ① $\|U\|_G \le \lambda\rho,\ \|V\|_G \le \lambda\rho \Rightarrow \|\varphi(U,V)\|_G \le \lambda\rho$

$$\|\theta_{Q,R}^{-1}(QR-P+UV)\|_G \le \rho^{-1}\|QR-P+UV\|_G$$

$$\le \rho^{-1}\min\{\lambda\rho^2, (\lambda\rho)^2\} \le \lambda\rho$$

② If $(U,V), (U',V') \in B(0,\lambda\rho)$, then

$$\|\varphi(U,V) - \varphi(U',V')\|_G \overset{?}{\le} \lambda \cdot \|(U-U', V-V')\|_G$$

$$= \|\theta_{Q,R}^{-1}(UV - U'V')\|_G$$

$$\le \rho^{-1}\|UV - U'V'\|_G = \rho^{-1}\|U(V-V') + V(U-U')\|_G$$

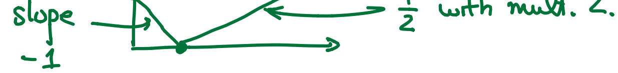
$$\le \rho^{-1} \cdot \lambda\rho \cdot \|(U-U', V-V')\|_G . \quad \checkmark \qquad \square$$

Newton polygon K completely valued field, $v_K(-)$

$$P(x) = a_0 + a_1 x + \cdots + a_d x^d \in K[x]$$

$NP(P)$ = lower convex hull of $(0, v_K(a_0)), (1, v_K(a_1)), \dots, (d, v_K(a_d))$

E.g. $P(x) = p + x + p^3x^2 + px^2$



Theorem 2.4. If all the slopes of $NP(P)$ are $\lambda_1 < \cdots < \lambda_r$ with mult. $m_1, \ldots, m_r$ then $P(x)$ has exactly m_i zeros with valuation $-\lambda_i$

Cor: $NP(P\cdot Q) = NP(P) \# NP(Q)$

This is to union all slopes of $NP(P)$ & $NP(Q)$ with multiplicity & then reorder them to redraw a Newton polygon.

(b/c $\{$zeros of $P\cdot Q\} = \{$zeros of $P\} \cup \{$zeros of $Q\}$)

Proof: Equivalent to show the inverse statement:

Assume that $P(x)$ has exactly m_i zeros with valuation $-\lambda_i$: $\alpha_i^{(1)}, \cdots \alpha_i^{(m_i)}$

May assume $P(0)=1$

Let NP be the convex polygon starts at $(0,0)$ with slopes λ_i (mult. m_i)

$$\Rightarrow P(x) = \prod_{i,j} \left(1 - (\alpha_i^{(j)})^{-1} x\right) = a_0 + a_1 x + \cdots + \underset{1}{\overset{\shortparallel}{a_d}} x^d$$

$a_n = \pm \sum$ product of n numbers in $(\alpha_i^{(j)})^{-1}$'s.

$\Rightarrow v_K(a_n) \geq NP$ at n

Equality is achieved precisely at

$n = m_1 + \cdots + m_k$

b/c in this case, the leading term is

$$\left(\prod_{1\leq i\leq k} \alpha_i^{(j)}\right)^{-1} \qquad \square$$

E.g. α_1, β_1 val -1

α_2 val -2

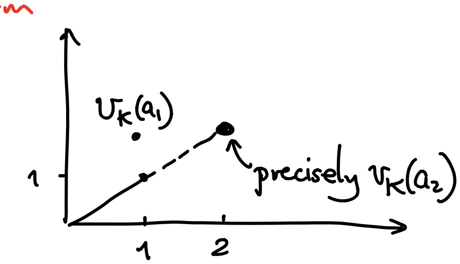
α_3 val -3

$\Rightarrow -a_1 = \alpha_1^{-1} + \beta_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1}$

$\Rightarrow v_K(a_1) \geq 1$

$a_2 = \alpha_1^{-1}\beta_1^{-1} + \alpha_1^{-1}\alpha_2^{-1} + \alpha_1^{-1}\alpha_3^{-1} + \beta_1^{-1}\alpha_2^{-1} + \beta_1^{-1}\alpha_3^{-1} + \alpha_2^{-1}\alpha_3^{-1}$

leading term



Cor: $P(x) = \prod P_i(x)$ with $P_i(x)$ having exactly the zeros with valn $-\lambda_i$.

Lecture 3 Holomorphic functions

- $E \subseteq \mathbb{C}_p$ closed subfield.
- $f(x) = \sum_{n\in\mathbb{Z}} a_n x^n$ formal Laurent series
- $I \subset \mathbb{R} \cup \{\infty\}$ interval.

$\rightsquigarrow A(I) := \{ z \in \mathbb{C}_p,\ \mathrm{val}(z) \in I \}$

Definition: We say $f(x)$ is holomorphic on the annulus $A(I)$ if the following equivalent conditions hold:

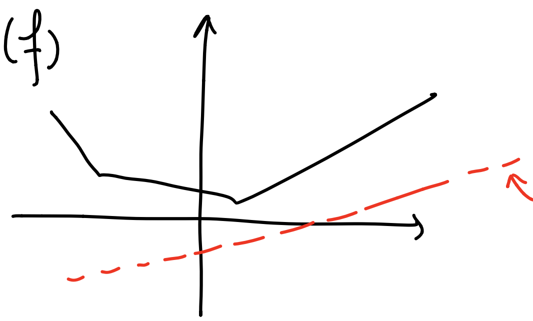
(1) $f(x)$ converges at each $z \in A(I)$

(2) $\underbrace{\mathrm{val}_p(a_n) + n\mu}_{\mathrm{val}_p(a_n z^n) \text{ with } \mathrm{val}_p(z) = \mu} \to \infty$ as $n \to \pm\infty$ for every $\mu \in I$.

$\to$ e.g. $I = [r, s]$ just need $\mathrm{val}_p(a_n) + nr \to \infty$ as $n \to +\infty$

& $\mathrm{val}_p(a_n) + ns \to \infty$ as $n \to -\infty$

(3) $NP(f)$



consider lines with slope $-\mu$ with $\mu \in I$.

$\forall \mu \in I$, $NP(f)$ lies above a line with slope $-\mu$.

- $\mathcal{H}(I) := \{ \text{holomorphic functions on } A(I) \}$ $\leftarrow$ $\forall \mu \in I$ valuation $V(f;\mu) := \min_{n\in\mathbb{Z}} (\mathrm{val}_p(a_n) + n\mu)$

$\cup\!\!|$

$\mathcal{B}(I) := \{ \text{bounded holomorphic functions on } A(I) \}$

$\uparrow$ meaning $\{ V(f;\mu) ;\ \mu \in I \}$ is bounded below

when $\infty \in I$, only consider power series.

* $\mathcal{H}(I) =$ Fréchet complete for $(V(\cdot\,;\mu))_{\mu \in I}$.

i.e. $f_i \to f$ if $\forall \mu \in I$, $V(f_i - f ; \mu) \to \infty$

* If I is closed, $\mathcal{H}(I) = \mathcal{B}(I)$.

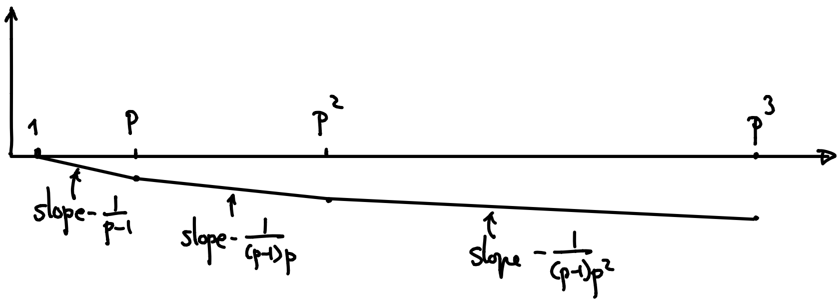
In general, $\mathcal{H}(\bar{I}) \subseteq \mathcal{B}(I) \subseteq \mathcal{H}(I)$

closure of I.

Examples: $I = (0, +\infty]$, i.e. $A(I)$ = open disk.

① $f(x) = \log(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{n-1}\frac{x^n}{n}$.

$NP(f) =$



$\Updownarrow$

zeros of $\log(1+x)$ are precisely $\zeta_{p^n}^{j} - 1$'s. note: $v_p(\zeta_{p^n} - 1) = \frac{1}{p^{n-1}(p-1)}$.

$f(x) \in \mathcal{H}(I)$ but not bounded.

Remark: $\Phi_n = n^{th}$ cyclotomic polynomial $= \frac{(1+x)^{p^n} - 1}{(1+x)^{p^{n-1}} - 1}$

Then $\log(1+x) = x \cdot \prod_{n \geq 1} \frac{\Phi_n(x)}{p}$

② $f(x) = 1 + x + x^2 + \cdots$. $f(x) \in \mathcal{B}(I)$ is bounded

but $f(x) \notin \mathcal{H}([0, +\infty])$

③ $f(x) = \sum p^n x^{n^2}$, $f(x) \in \mathcal{H}([0, +\infty])$

Theorem 3.2 If $f(x) \in \mathcal{H}(I)$, $s \in I$, then

width of slope $-s$ segment in $NP(f)$ = # of zeros of $f(x)$ with val'n s.

We can factor $f(x) = g_s(x) P(x)$ unique up to scalar multiple

s.t. $g_s(x)$ has all zeros with val'n s

$P(x)$ has no zeros with val'n s.

Cor. If $f(x) \in \mathcal{H}(I)$, then $f(x)$ has finitely many zeros in $A(I) \Rightarrow f(x) \in \mathcal{B}(I)$.

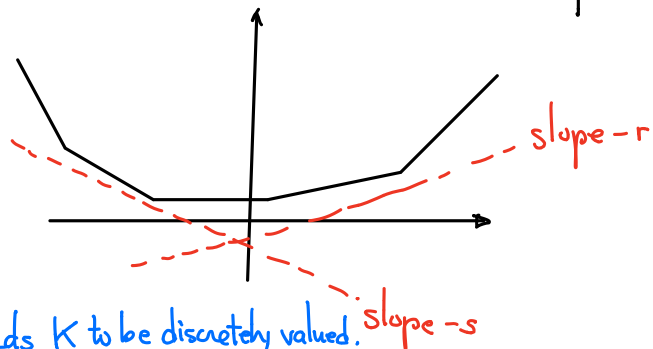
$\Leftarrow$ if K has discrete valuation

Proof: Say $r = \inf(I)$ & $s = \inf(I)$

Then $f(x) \in \mathcal{H}(I)$ is bounded $\Leftrightarrow$ $val_p(a_n) + nr$ is bounded below when $n \to +\infty$

$\mathrm{val}_p(a_n)+ns$ is bounded below when $n\to-\infty$

$\iff$ NP(f) is above two lines with slopes $-s$ & $-r$, respectively.



$\Rightarrow$ needs K to be discretely valued.

$\Longleftarrow$ NP(f) has only finitely many segments with slopes $\in(s,r)$

Counterexample when K is not discretely valued. say $K=\mathbb{C}_p$

$$f(x)=\sum_{n\geq 1}\frac{x^n}{p^{1-\frac{1}{n}}}$$ is bounded on $A\big((0,+\infty]\big)$

but its zeros have p-adic val'n $\frac{1}{n(n+1)}$ $n\geq 1$. & ∞

"Typical factorization"

- $f(x)\in\mathcal{H}\big([r,s]\big)$ or $\mathcal{B}\big((r,s)\big)$ $\quad f(x)=P(x)\cdot u(x)$ for $P(x)$ polynomial with zeros in $A\big([r,s]\big)$, $u(x)\in\mathcal{H}\big([r,s]\big)^\times$ or $\mathcal{B}\big((r,s)\big)$
- $f(x)\in\mathcal{H}\big((r,s)\big)$. Write $(r,s)=\bigcup_{n\geq1}[r_n,s_n]$ increasing union

then $$f(x)=u(x)\cdot\prod_{n\geq1}P_n(x)$$

polynomial w/ zeros in $[r_n,s_n]\setminus[r_{n-1},s_{n-1}]$ & $P_n(0)=1$.

Corollary 3.4 If $f(x)$ has no zeroes in $A(I)$, then $f(x)\in\mathcal{B}(I)$

$\Rightarrow \mathcal{H}(I)^\times\subset\mathcal{B}(I)$.

Corollary 3.5 The ring $\mathcal{B}(I)$ is a PID.

Proof: $J\subseteq\mathcal{B}(I)$, Each $f(x)=P_f(x)\cdot u_f(x)$ for $P_f(x)\in K[x]$ polynomial, $u_f(x)$ unit in $\mathcal{B}(I)$

$(f_J)=\big(P_f(x)\,;\,f\in J\big)$ is an ideal in $E[x]$

& $J=(f_J)$.

- Assume K is CDVF.

A divisor D on $A(I)$ is a formal sum $\sum_{z\in A(I)} n_z\cdot[z]$ $\quad n_z\in\mathbb{Z}$

s.t. for each $J\subseteq I$ closed, $\sum_{z\in A(J)} |n_z|$ is finite

Then each $f(x)\in\mathcal{H}(I) \rightsquigarrow \mathrm{div}(f)=\sum$ zeros of f is a divisor on $\mathcal{H}(I)$.

Theorem 3.7 (Chinese remainder theorem)

Suppose we are given
- an effective divisor $D=\sum_{z\in A(I)} n_z\cdot[z]$ on $A(I)$
- polynomials Q_s for each $s\in I$ s.t. $\deg Q_s < \sum_{z\in A([s,s])} n_z$.

Then $\exists f\in\mathcal{H}(I)$ s.t. $f\equiv Q_s \mod \prod_{z\in A([s,s])}(x-z)^{n_z}$

Corollary 3.8. The ring $\mathcal{H}(I)$ is a Bézout domain,

i.e. every finitely generated ideals are principal.

Example: $\mathcal{H}((0,+\infty])$ is not a PID

Recall $\log(1+x) = x\cdot\prod_{n\geq 1}\frac{\Phi_n(x)}{p}$

$$I=\left(\prod_{n\geq n_0}\frac{\Phi_n(x)}{p}\ ;\ \text{for all } n_0\geq 1\right)$$

Lecture 5 Ax and Ax-Sen-Tate's theorem

Theorem 5.1. If F is a complete valued field, then $\widehat{F^{alg}}$ is algebraically closed.

Proof: Prove by induction on degree that every polynomial $P(x)\in\widehat{F^{alg}}[x]$ of degree ≥ 1 has a root.

- May assume that $P(x)\in\mathcal{O}_{\widehat{F^{alg}}}[x]$ is monic

(o/w can use $NP(P)$ to factor it)

Write $P(x)=\lim P_n(x)$ for monic $P_n(x)\in\mathcal{O}_{F^{alg}}[x]$

Pick one root α_n of P_n so $P(\alpha_n)\to 0$

* If $P'(\alpha_n)\not\to 0$, then $\exists\, n$ s.t. $P(\alpha_n)$ small but $P'(\alpha_n)$ not too small

Hensel's lemma $\Rightarrow \exists\,\alpha\in\mathcal{O}_{\widehat{F^{alg}}}$ a zero of P

* If $P'(\alpha_n)\to 0$, then inductive hypo $\Rightarrow P'(x)$ decomposes

$\Rightarrow \alpha_n \to$ a zero of $P'(x)$, say α

Then $P(\alpha)$ is the limit of $P(\alpha_n)$, which is also 0. ✓

Theorem 5.2 If F is a valued field of char $p>0$, then F^{sep} is dense in F^{alg}.

Proof: If $y \in F^{alg}$, then $y^{p^n} = \alpha \in F^{sep}$ for some n.

WLOG $v(\alpha) \geq 0$.

Take $\pi \in F$ with $v(\pi) > 0$

seperable equation

Let y_i be a root of $Y^{p^n} - \pi^i Y - \alpha = 0 \quad \Rightarrow y_i \in F^{sep}$ & $v(y_i) \geq 0$

$$\Rightarrow \left.\begin{aligned} y^{p^n} &= \alpha \\ y_i^{p^n} &= \alpha + \pi^i y_i \end{aligned}\right\} \Rightarrow (y - y_i)^{p^n} = \pi^i y_i$$

$\Rightarrow y_i \to y$ as $i \to \infty$

Theorem 5.3 (Ax-Sen-Tate)

If F is a complete p-adic field, and if $F \subset K \subset F^{alg}$ (K a possibly infinite ext'n of F)

then $\left(\widehat{F^{alg}}\right)^{Gal(F^{alg}/K)} = \hat{K}$

Proof: Clearly, $\hat{K} \subseteq \widehat{F^{alg}}^{G_K}$

Why do we care?

Answer: Making sure that take completion is okay with Galois theory.

$$\begin{array}{ccccc}
\mathbb{Q}_p^{alg} & & F^{alg} & \rightsquigarrow & \widehat{F^{alg}} \\
| & & |\ )\, G_K & & |\ )\, \text{"}G_K\text{"} \\
\mathbb{Q}_p(\mu_{p^\infty}) & & K & \longrightarrow & \hat{K} \\
| & & | & & |\ )\, \text{"}Gal(\hat{K}/F)\text{"} \\
\text{e.g. } \mathbb{Q}_p & & F & & F
\end{array}$$

Now show the opposite inclusion

If $\alpha \in F^{alg}$, let $\Delta_K(\alpha) := \inf_{g \in G_K} val_p(g(\alpha) - \alpha)$ $\qquad val_p(p) = 1.$

(so if $\alpha \in K$, then $\Delta_K(\alpha) = \infty$.)

Let $\alpha \in \widehat{F^{alg}}^{G_K}$. Write $\alpha = \lim \alpha_n$ with $\alpha_n \in F^{alg}$.

Then $\Delta_K(\alpha_n) \geq val_p(\alpha - \alpha_n) \longrightarrow \infty$ $\Big\}$ $\Rightarrow$ $\exists\, \delta_n \in K$ s.t. $val_p(\alpha_n - \delta_n) \to \infty$ $\Rightarrow \alpha = \lim \delta_n$ ✓.

<u>Lemma 5.4.</u> If $\alpha \in F^{alg}$, then $\exists\, \delta \in K$ s.t.

$$val_p(\alpha - \delta) \geq \Delta_K(\alpha) - \frac{p}{(p-1)^2}$$

Proof: Induction on $n := [K(\alpha):K]$ s.t. we can find such a δ

with $$val_p(\alpha - \delta) \geq \Delta_K(\alpha) - \sum_{k=0}^{m} \frac{1}{p^k(p-1)} \qquad \text{where } p^{m+1} \leq n < p^{m+2}$$

Basically, this says that if $\Delta_K(\alpha)$ is large, can find $\delta \in K$ very close to α.

assume this

Write $Q(x) = \text{minpoly}(\alpha)$ over K

When $K(\alpha)$ is Galois over K,

$$Q(x) = \prod_{g \in Gal(K(\alpha)/K)} (x - g(\alpha)) \approx (x - \alpha)^n \quad \text{has coeffs in } K.$$

So if $p \nmid n$. $\quad \alpha \approx -\dfrac{a_{n-1}}{n}$ ✓

if $n = p^k \cdot r$, then we consider a_{n-p^k}.

Say degree $Q(x) = n$. $\qquad Q(x) = x^n + a_{n-1}x^{n-1} + \cdots \in K[x]$

<u>Case 1</u>: $p \nmid n$, then $val_p(a_{n-1} - n\alpha) \geq \Delta_K(\alpha)$

We are done!

$K(\alpha)$

degree prime-to-p

K_r

p-power

K

Apply Case 1, $\exists\, \alpha' \in K_r$

s.t. $val_p(\alpha - \alpha') \geq \Delta_K(\alpha)$

$\Rightarrow \Delta_K(\alpha') \geq \Delta_K(\alpha)$

Reduces question to α'.

<u>Case 2</u>. $n = p^k$.

Reduce α zero of $Q(x)$

to β zero of $Q^{(p^{k-1})}(x)$

) by our earlier philosophy $\alpha \approx \beta$.

Need to find a β s.t. $\mathrm{val}_p(\alpha - \delta)$ is "large" $\geq \Delta_K(\alpha) - \frac{1}{p^{k-1}(p-1)}$

Let $P(x) = Q(x+\alpha) \in F^{alg}[x]$

Known: All zeros of $P(x)$ have val'n $\geq \Delta_K(\alpha)$

WTS: $P^{(p^{k-1})}(x) = Q^{(p^{k-1})}(x+\alpha)$ has a zero "$\beta-\alpha$" whose val'n $\geq \Delta_K(\alpha) - \frac{1}{p^{k-1}(p-1)}$

Write $P(x) = x^{p^k} + \cdots + a_{p^k}$

$$\Rightarrow \mathrm{val}_p(a_{p^k-p^{k-1}}) \geq (p^k - p^{k-1})\cdot \Delta_K(\alpha)$$

$$\frac{P^{(p^{k-1})}(x)}{p^k \cdot \cdots \cdot (p^k - p^{k-1}+1)} = x^{p^k - p^{k-1}} + \cdots + \frac{a_{p^k-p^{k-1}}}{\binom{p^k}{p^{k-1}}}$$

So at least one zero of $P^{(p^{k-1})}(x)$ has val'n

$$\geq \frac{1}{p^k - p^{k-1}}\left(\mathrm{val}_p(a_{p^k-p^{k-1}}) - \underbrace{\mathrm{val}_p\binom{p^k}{p^{k-1}}}_{=1}\right)$$

$$\geq \Delta_K(\alpha) - \frac{1}{p^k - p^{k-1}}.$$

So question about α $\rightsquigarrow$ question about β $\longrightarrow$ . . .

$[K(\alpha):K] = p^k$ $\qquad$ $[K(\beta):K] < p^k$ $\qquad$ degree p-power part of $[K(\beta):K]$