## Goal :

1. Define $B_{dR}$ and study its properties
2. Define $\tilde{B}_{rig}$ and its subrings
3. Recover $D_{sen}(V)$ from $D^+_{dif}(V)$
   Recover $D_{dR}(V)$ from $D_{dif}(V)$

## 1. The field $B_{dR}$

Let $\tilde{E}^+ = \{ (x_0, x_1, \cdots)$ where $x_i \in \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ $\; x_{i+1}^p = x_i \}$

$$\tilde{A}^+ = W(\tilde{E}^+)$$
$$\tilde{B}^+ = \tilde{A}[\tfrac{1}{p}]$$

And we get a ring homomorphism $\theta : \tilde{A}^+ \to \mathcal{O}_{\mathbb{C}_p}$ given by

$$\theta : \sum_{i \geq 0} p^i [x_i] \to \sum_{i \geq 0} p^i [x_i^{(0)}]$$

Extend it to $\theta : \tilde{B}^+ \to \mathbb{C}_p$.

**Prop. 1.1.** The kernel $\theta: \tilde{A}^+ \to O_{\mathbb{C}_p}$ is a principal ideal generated by any element $y \in \tilde{A}^+$ s.t. $\theta(y)=0$ and $val_E(\bar{y}) = 1$

Pf: Reduce to mod $p$,

$$\theta : \tilde{E}^+ \to O_{\mathbb{C}_p}/pO_{\mathbb{C}_p}$$
$$x \mapsto x^{(0)} \bmod p$$

if $x \in \ker(\theta: \tilde{A}^+ \to O_{\mathbb{C}_p})$

then $\bar{x} \in \ker(\theta: \tilde{E}^+ \to O_{\mathbb{C}_p}/pO_{\mathbb{C}_p})$

$\Rightarrow val_E(\bar{x}) \geq 1$ $\quad$ $\bar{x}/\bar{y} \in \tilde{E}^+$

$\Rightarrow \exists\, a \in \tilde{A}^+$ s.t. $x - ay \in (\ker\theta) \cap p\tilde{A}^+$

$\Rightarrow$ $y \cdot \tilde{A}^+ \to \ker(\theta)$ is surjective modulo $p$

$\Rightarrow$ surj. by NAK

**Rmk 1.2.** We can apply prop 1.1. with

$$y = \frac{[\varepsilon]-1}{[\varepsilon^{\frac{1}{p}}]-1} \quad \text{or} \quad y = [\tilde{p}] - p$$

$\tilde{p} \in \tilde{E}^+$ is any element s.t. $\tilde{p}^{(0)} = p$

- $\theta$ is surjective
- $\theta$ commutes with the action of $G_{\mathbb{Q}_p}$
- $\mathrm{Ker}(\theta)$ not stable under $\varphi$ $\left(\theta(\varphi[\tilde{p}]-p)\right)$
  $= p^p - p \neq 0$

**Definition of $B_{dR}^+$:**

For $h \geq 1$, let $B_h := \tilde{B}^+ / (\ker\theta)^h$

$B_1 = \mathbb{C}_p$ $\qquad B_{h+1} \twoheadrightarrow B_h$

$$B_{dR}^+ := \varprojlim_h B_h$$

Then $B_{dR}^+ / \ker\theta = \mathbb{C}_p$

For $\forall\, x \in B_{dR}^+$, s.t. $\theta(x) \neq 0 \Rightarrow x$ is invertible

$\mathrm{Ker}(\theta : B_{dR}^+ \to \mathbb{C}_p) = ([\varepsilon]-1)$

For $m \geq 1$ $\quad ([\varepsilon]-1)^m \in (\ker\theta)^m$

$$\| $$

$$\left(\frac{[\varepsilon]-1}{[\varepsilon^{\frac{1}{p}}]-1}\right)^m \cdot \underbrace{\left([\varepsilon^{\frac{1}{p}}]-1\right)^m}_{\notin (\ker\theta)^m}$$

$$\theta\left(\left([\varepsilon^{\frac{1}{p}}]-1\right)^m\right) = (\zeta_p - 1)^m$$

$$t_n = ([\varepsilon]-1) - \frac{([\varepsilon]-1)^2}{2} + \cdots \pm \frac{([\varepsilon]-1)^{n-1}}{n-1}$$

$$t_n \in B_n$$

$$t = \sum_{k \geq 1} (-1)^{k-1} \cdot \frac{([\varepsilon]-1)^k}{k} \in B_{dR}^+$$

**Lemma 1.3** If $g \in G_{\mathbb{Q}_p}$, then $g(t) = \chi(g) \cdot t$

Pf: If

$$F_n(x) = x - \frac{x^2}{2} + \cdots \pm \frac{x^n}{n-1},$$ then

$$F_n(x) \equiv \log(1+x) \mod x^n$$

so that if $a \in \mathbb{Z}_p$, then

$$F_n((1+x)^a - 1) \equiv a F_n(x) \mod x^n \quad \text{in } \mathbb{Q}_p[[x]].$$

Since $t_n = F_n([\varepsilon] - 1)$ and

$$g([\varepsilon]) = [\varepsilon^{\chi(g)}] = \left(1 + ([\varepsilon] - 1)\right)^{\chi(g)},$$

we have $g(t_n) = \chi(g) \cdot t_n$ in $B_n$

and therefore $g(t) = \chi(g) \cdot t$.

$\frac{t}{[\varepsilon] - 1}$ is a unit in $B_{dR}^+$      $B_{dR}^+$ is a DVR with the uniformizer $t$

$\Rightarrow \ker(\theta : B_{dR}^+ \to \mathbb{C}_p) = (t)$

$\forall x \in B_{dR}^+ \setminus \{0\}$, $x = t^n \cdot x_0$ where $x_0 \in B_{dR}^+$ and $\theta(x_0) \neq 0$

$B_{dR} := B_{dR}^+[\frac{1}{t}]$ is a field

$B_{dR} \simeq \mathbb{C}_p((t))$

Filtration : $\mathrm{Fil}^h B_{dR} = t^h \cdot B_{dR}^+$ stable under the action of $G_{\mathbb{Q}_p}$

Topology on $B_{dR}^+$ :

1) $(\ker \theta)$ - adic topology :

induces discrete topology on $\mathbb{C}_p = B_{dR}^+ / \ker \theta$

We want p-adic topology on the residue field $\mathbb{C}_p$

2) "The natural topology" on $B_{dR}^+$ :

For $h \geq 1$, $\tilde{A}^+ \to B_h$ the image doesn't contain any $\mathbb{Q}_p$ -line

$V_h(x) = \sup \{ n \in \mathbb{Z} \text{ s.t. } p^{-n} \cdot x \in \mathrm{im}(\tilde{A}^+) \}$

$(\ker \theta)^h$ is a closed subspace of $\tilde{B}^+$ for the p-adic top.

$V_h(\cdot) \simeq$ quotient norm on $\tilde{B}^+ / (\ker \theta)^h$

$B_h$ is cplt. for $V_h(\cdot)$

$\Rightarrow$ $B_h$ is a Banach space

$B_{dR}^{+} = \varprojlim_{n} B_n$ endowed with the structure of a Fréchet space

$\{x_n\}$ $x_n \to x$ in $B_{dR}^{+}$

iff $\forall n$ $\bar{x}_n \to \bar{x}$ in $B_n$

So we have the p-adic topology on $\mathbb{C}_p$ now

**Prop. 1.4.** Every non-constant polynomial $P(T) \in \overline{\mathbb{Q}}_p[T]$ has a root in $B_{dR}^{+}$.

( So for every $x \in \overline{\mathbb{Q}}_p$ we can find a well-defined $\tilde{x} \in B_{dR}^{+}$ s.t. $\theta(\tilde{x}) = x$. i.e.

**Cor. 1.5.** We have a $G_{\mathbb{Q}_p}$-equivalent inclusion $\overline{\mathbb{Q}}_p \subset B_{dR}^{+}$ compatible with $\theta$. )

**Pf of Prop. 1.4.** : Assume $P(T)$ has simple roots.

$\mathbb{C}_p$ is alg. clsd. $\Rightarrow \exists\, \bar{y} \in \mathbb{C}_p \quad P(\bar{y}) = 0$

$$B_{dR}^{+} / t B_{dR}^{+} = \mathbb{C}_p$$

$\exists\, y \in \,^{+}_{dR}$ s.t. $P(y) \in t B_{dR}^{+}$

$y_n \in B_{dR}^{+}$ s.t. $P(y_n) \in t^n B_{dR}^{+}$

$$P(y_n + t^n z) = P(y_n) + t^n z \cdot \underbrace{P'(y_n)}_{\neq 0} + O(t^{n+1})$$

$\exists\, y_{n+1} \equiv y_n \mod t^n$ s.t. $P(y_{n+1}) \in t^{n+1} B_{dR}^{+}$

$\{y_n\}$ s.t. $y_n \to y \in B_{dR}^{+}$ s.t. $P(y) = 0$

**Prop. 1.6.** If $K$ is a finite extension of $\mathbb{Q}_p$, then $B_{dR}^{G_K} = K$.

Pf: If $h \in \mathbb{Z}$, we have an exact sequence

$$0 \to t^{h+1} B_{dR}^{+} \to t^{h} B_{dR}^{+} \to \mathbb{C}_p(h) \to 0 .$$

And by taking $G_K$ - invariants

$$0 \to (t^{h+1} B_{dR}^{+})^{G_K} \to (t^{h} B_{dR}^{+})^{G_K} \to (\mathbb{C}_p(h))^{G_K}$$

$\mathbb{C}_p(h)^{G_K} = \{0\}$ unless $h=0$

$(t^{h+1} B_{dR}^{+})^{G_K} \cong (t^{h} B_{dR}^{+})^{G_K}$ for $h \leq -1$, $h \geq 1$

Let $h \to \pm\infty$ $\Rightarrow (t^{h} B_{dR}^{+})^{G_K} = \{0\}$ for $h \neq 0$

$$\begin{array}{ccc} 0 \to (B_{dR}^{+})^{G_K} & \to & \mathbb{C}_p^{G_K} \\ & & \| \\ & & K \end{array}$$

by Ax-Sen-Tate's thm

$K \subset B_{dR}^{+}$

$\Rightarrow (B_{dR}^{+})^{G_K} \xrightarrow{\sim} K$

$(B_{dR})^{G_K} = K$

## 2. The ring $\tilde{B}_{rig}$

**Def.** If $s \geq r$, define a valuation $v_{[r;s]}$ on $\tilde{B}^{\dagger,r}$ by

$$v_{[r;s]}(f) = \min\left(v_r(f), v_s(f)\right) = \min_{t\in[r,s]} v_t(f)$$

$v_t(f)$ convex for $t$

$\tilde{B}_{[r;s]} :=$ the completion of $\tilde{B}^{\dagger,r}$ for $v_{[r;s]}$

$\tilde{A}_{[r;s]} :=$ the ring of integers of $\tilde{B}_{[r;s]}$ for $v_{[r;s]}$

- The action of $G_{\mathbb{Q}_p}$ extends to the rings $\tilde{A}_{[r;s]}$ & $\tilde{B}_{[r;s]}$

- The Frobenius $\varphi$ gives a bijective map

$$\varphi : \tilde{A}_{[r;s]} \to \tilde{A}_{[pr;ps]}$$

$$\varphi : \tilde{B}_{[r;s]} \to \tilde{B}_{[pr;ps]}$$

$$\varphi : \tilde{A}^{\dagger,r} \to \tilde{A}^{\dagger,pr}$$

$$\tilde{B}^{\dagger,r} \to \tilde{B}^{\dagger,pr}$$

$x_n \to x$ by $v_{[r;s]}$ $\qquad$ $\varphi(x_n) \to \varphi(x)$ by $v_{[pr;ps]}$

$$v_{[pr;ps]}(\varphi(x)) = p\, v_{[r;s]}(x)$$

If $r_0 = \frac{p-1}{p}$, let $\quad A_{max} := \tilde{A}_{[0;r_0]}$

$\qquad\qquad\qquad\qquad B^+_{max} := \tilde{B}_{[0;r_0]}$

Lemma 2.1. Every element of $\tilde{A}_{[r_0;r_0]}$ can be written as $\sum_{j\in\mathbb{Z}} a_j \left(\frac{[\tilde{p}]}{p}\right)^j$ where $a_j \to 0$ $p$-adically as $j \to \pm\infty$

and likewise every element of $\tilde{A}_{[0;r_0]}$ can be written as $\sum_{j\geq 0} a_j \left(\frac{[\tilde{p}]}{p}\right)^j$ where $a_j \to 0$ $p$-adically as $j \to +\infty$

Pf: If $x = \sum_{k\geq 0} p^k [x_k] \in \tilde{A}^{\dagger, r_0}$, then $V_{r_0}(x) = \inf_k (\mathrm{val}_E(x_k) + k)$ so that the rings of integers of $\tilde{B}^{\dagger, r_0} = \{ x = \sum_{k\geq 0} p^k [x_k] \mid \mathrm{val}_E(x_k) + k \geq 0 \}$

$\tilde{A}^+[\frac{p}{[\tilde{p}]}, \frac{[\tilde{p}]}{p}]$ is dense in this ring of integers of $\tilde{B}^{\dagger, r_0}$
( Sec.2.1. Représentations p-adiques et Équations Différentielles by Laurent Berger )

$V_{r_0} \simeq \mathrm{val}_p$ $\qquad\qquad \tilde{A}^+[\frac{[\tilde{p}]}{p}]$ is dense in $\tilde{A}_{[0;r_0]}$

The set of valuations $\{v_{[r;s]}\}_{s\geq r}$ defines a Fréchet topology on $\tilde{B}^{\dagger,r}$
and we denote $\tilde{B}^{\dagger,r}_{rig}$ = the completion of $\tilde{B}^{\dagger r}$ for that topology

$$\tilde{B}^{\dagger}_{rig} := \bigcup_{r\geq 0} \tilde{B}^{\dagger,r}_{rig}$$

$$\tilde{B}^{+}_{rig} := \text{cplt. of } \tilde{B}^{+} \text{ by } \{v_r\}_{r\geq 0}$$
$$= \bigcap_{n\geq 1} \tilde{B}_{[0;r_n]} = \bigcap_{n\geq 1} \varphi^n(B^{+}_{max})$$
$$r_n = p^{n-1}(p-1) \quad \varphi : \tilde{B}_{[0;r_{n-1}]} \to \tilde{B}_{[0;r_n]}$$

Properties of $\tilde{B}^{+}_{rig}$ :

- $\tilde{B}^{+}_{rig}$ is a subring of $\tilde{B}^{\dagger,r}_{rig}$ for $\forall r>0$
  $\tilde{B}^{+}_{rig} \subset \tilde{B}_{[0;r]}$
- $\varphi$ is bijective on $\tilde{B}^{+}_{rig}$
  $\varphi$ is bijective $\tilde{B}_{[0;r_{n-1}]} \to \tilde{B}_{[0;r_n]}$
- $\tilde{B}^{+}_{rig}$ is stable under the action of $G_{\mathbb{Q}_p}$

Recall $B_K^{\dagger,r} := (B^{\dagger,r})^{H_K} = (B \cap \tilde{B}^{\dagger,r})^{H_K}$,

and define $B_{rig,K}^{\dagger,r}$ := the completion of $B_K^{\dagger,r}$ for Fréchet topology.

$B_K^{\dagger,r} = \{$ Laurent series $f(T)$ convergent on $C[r,1)$ bounded $\}$

$C[r,1) = \{ p^{-\frac{1}{r}} \leq |z|_p < 1 \}$

$B_{rig,K}^{\dagger,r} = \{$ Laurent series $f(T)$ convergent on $C[r,1)\}$

Lemma 2.2. If $r>0$, then $\bigcap_{n\geq 1} (\tilde{A}^{+} + p^{n-1}\tilde{A}_{[0;r]}) = \tilde{A}^{+}$

Pf: Suppose that $y = y_n + p^{n-1} \cdot z_n$ with $y_n \in \tilde{A}^{+}$ and $z_n \in \tilde{A}_{[0;r]}$

then $y_{n+1} - y_n = p^{n-1}(z_n - p z_{n+1}) \in p^{n-1} \cdot \tilde{A}_{[0;r]}$

so $v_r(y_{n+1} - y_n) \to \infty$ as $n \to \infty$

Then for fixed $k$, $w_k(y_{n+1} - y_n) \to \infty$

$\{y_n\}$ is Cauchy for the weak top.

$\tilde{A}^{+}$ is cplt. for weak top.

$y = \lim y_n \in \tilde{A}^{+}$

**Lemma 2.3.** We have $(\tilde{B}^+_{rig})^{\varphi=1} = \mathbb{Q}_p$ and

$$(\tilde{B}^+_{rig})^{\varphi=p^{-n}} = \{0\} \quad \text{if } n \geq 1$$

Pf: If $y \in \tilde{A}_{[0;r_0]}$, then $y = \sum_{j\geq 0} a_j \left(\frac{[\tilde{p}]}{p}\right)^j$

and if $y = \varphi^n(y)$, then $y = \sum_{j\geq 0} \varphi^n(a_j) \left(\frac{[\tilde{p}^{p^n}]}{p}\right)^j$

$\Rightarrow y \in \tilde{A}^+ + p^m \tilde{A}_{[0;r_0]}$ for $\forall m \geq 0$

$\Rightarrow y \in \tilde{A}^+$

$$\left(\tilde{A}_{[0;r_0]}\right)^{\varphi=1} = (\tilde{A}^+)^{\varphi=1} = \mathbb{Z}_p$$

$$\Rightarrow (\tilde{B}^+_{rig})^{\varphi=1} = \mathbb{Q}_p$$

If $y \in \left(\tilde{A}_{[0;r_0]}\right)^{\varphi=p^{-n}}$

$$y = p^{kn} \varphi^k(y) \in p^{kn} \tilde{A}_{[0;r_0]} \quad \forall k \geq 0 \Rightarrow y = 0$$

$$\Rightarrow (\tilde{B}^+_{rig})^{\varphi=p^{-n}} = \{0\}$$

$\square$

# 3. Recover Dsen (V)

V is a p-adic rep. of dim $d$

$D_{sen}(V)$ 1) $K_\infty$-mod ⇒ 2) free of dim $d$ ⇒ 3) stable under $\Gamma_K$ 4) $\subseteq (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}$

$\mathbb{C}_p \otimes_{K_\infty} D_{sen}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$

$$\Theta_V = \frac{\log(\gamma)}{\log(\chi(\gamma))} \qquad \gamma \in \Gamma_K, \ \gamma \to 1$$

$D^+_{dif}(V)$ 1) $K_\infty[[t]]$-mod $\subseteq (B^+_{dR} \otimes_{\mathbb{Q}_p} V)^{H_K}$ stable by $\Gamma_K$

2) $d$-dim

3) $B^+_{dR} \otimes_{K_\infty[[t]]} D^+_{dif}(V) = B^+_{dR} \otimes_{\mathbb{Q}_p} V$ Fontaine

( Sec 3.4. P56. Arithmétique Des Représentations Galoisiennes p-adiques)

$$\nabla_V = \frac{\log(\gamma)}{\log(\chi(\gamma))}, \qquad \nabla_V(ax) = a\nabla_V(x) + \nabla(a)\cdot x$$

$$D_{dif}(V) = K_\infty((t)) \otimes_{K_\infty[[t]]} D^+_{dif}(V)$$

$$D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

We say V is de Rham if $\dim_K D_{dR}(V) = d$

$D^{\dagger,r}(V)$ $B^{\dagger,r}_K$-submod in $(\tilde{B}^{\dagger,r} \otimes V)^{H_K}$

Assume V is overconvergent

$$\tilde{B}^{\dagger,r} \otimes V = \tilde{B}^{\dagger,r} \otimes_{B^{\dagger,r}_K} D^{\dagger,r}(V)$$

## Recover $D_{dR}(V)$ from $D_{dif}(V)$

**Prop. 3.1.** If $V$ is a $p$-adic rep. of $G_K$, then the kernel of the connection $\nabla_V$ operating on $D_{dif}(V)$ is $K_\infty \otimes_K D_{dR}(V)$.

In particular, $V$ is de Rham iff $\nabla_V$ is trivial.

Pf. Step 1: $K_\infty \otimes_K D_{dR}(V) \subset \ker \nabla_V$

$$D_{dR}(V) = (B_{dR} \otimes V)^{G_K} \qquad \gamma \in \Gamma_K$$

$\nabla_V$ is trivial on $D_{dR}(V)$

$\nabla_V$ is trivial on $K_\infty$ : $\forall a \in K_\infty$ $n$ large $a \in K_n$

$\gamma \to 1$ $\gamma$ fix $K_n$ $\nabla_V(a) = \frac{\log(\gamma)}{\log(\chi(\gamma))}(a) = 0.$

Step 2: $\ker \nabla_V \subset K_\infty \otimes_K D_{dR}(V)$

$\ker \nabla_V$ is ① finite dimensional $K_\infty$-v.sp. ② stable under $\Gamma_K$

Thm 3 in CZK's lecture $\Rightarrow$ $\ker \nabla_V$ comes from the ext of scalars of a $K_n$-v.sp. $V_n$ for $n$ large

$\nabla_V$ is trivial on $V_n$

$\Rightarrow$ Lie alg. of $\Gamma_K$ acts trivially on $V_n$

$\Rightarrow$ The action of $\Gamma_K$ is discrete

For $m$ large, $\Gamma_{K_m}$ acts trivially on $V_n$

$V_n \subseteq K_m \otimes D_{dR}(V)$ $\Rightarrow$ $\ker \nabla_V \subseteq K_\infty \otimes D_{dR}(V)$

# Recover $D_{sen}(V)$ from $D_{dif}^+(V)$

We already have $\theta : B_{dR}^+ \to \mathbb{C}_p$,
which gives rise to
$$\theta : \underset{\substack{\| \\ D_{dif}^+(V)}}{(B_{dR}^+ \otimes_{\mathbb{Q}_p} V)^{H_K}} \to \underset{\substack{\| \\ D_{sen}(V)}}{(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{H_K}}$$

Goal: $\text{image}(\theta) = D_{sen}(V)$

If $x \neq 0 \in \tilde{E}^+$ then $[x] \in B_{dR}^+$ and $\theta([x]) \neq 0$,
so that $[x]$ is invertible in $B_{dR}^+$ and hence
if $y \neq 0 \in \tilde{E}$, then $[y]$ makes sense in $B_{dR}^+$.
We want to check when the series $x = \sum_{k \gg -\infty} p^k [x_k]$ converges in $B_{dR}^+$

Lemma. 3.2. If $\{x_k\}_{k \gg -\infty}$ is a sequence of elements of $\tilde{E}$, then the series $\sum_{k \gg -\infty} p^k [x_k]$ converges in $B_{dR}^+$
iff $\text{val}_E(x_k) + k \to \infty$ as $k \to \infty$.

Pf: "$\Rightarrow$" If $\sum_{k \gg -\infty} p^k [x_k]$ converges in $B_{dR}^+$

then $\sum_{k \gg -\infty} p^k \theta[x_k]$ converges in $\mathbb{C}_p$

so that $k + \underbrace{\text{val}_p(x_k^{(0)})}_{= \text{val}_E(x_k)} \to \infty$ as $k \to \infty$

"$\Leftarrow$" We have
$val_E(x_k) + k \to \infty$ as $k \to \infty$

$$p^k[x_k] = a_k \left(\frac{p}{[\tilde{p}]}\right)^k = a_k \left(1 + \left(\frac{[\tilde{p}]}{p} - 1\right)\right)^{-k}$$

in $B_h$ $= a_k \left(1 + \binom{-k}{1}\left(\frac{[\tilde{p}]}{p} - 1\right) + \cdots + \binom{-k}{h-1}\left(\frac{[\tilde{p}]}{p} - 1\right)^{h-1}\right)$

since $([\tilde{p}] - p) \in \ker\theta$ $\quad \in p^{-(h-1)}\tilde{A}^+$

$$V_h(x) = \sup\{n \in \mathbb{Z} \text{ s.t. } p^{-n}x \in \mathrm{im}(\tilde{A}^+)\}$$

For fixed $h$, $val_E(x_k) + k \to \infty$
$p^k[x_k] \to 0$ in $B_h$

$\sum p^k[x_k]$ converges in $B_{dR}^+$

Define $\iota_0 : \tilde{B}^{+,r_0} \to B_{dR}^+$

$$\sum_{k \gg -\infty} p^k[x_k] \mapsto \sum_{k \gg -\infty} p^k[x_k]$$

and $\iota_n : \tilde{B}^{+,r_n} \to B_{dR}^+$ $\qquad \iota_n = \iota_0 \circ \varphi^{-n}$

$$\sum_{k \gg -\infty} p^k[x_k] \mapsto \sum_{k \gg -\infty} p^k[x_k^{p^{-n}}]$$

R: over $D_{sen}(V)$ from $D^+_{dif}(V)$ $\quad \theta : D^+_{dif}(V) \twoheadrightarrow D_{sen}(V)$

**Prop. 3.3.** If $n$ is large enough, then $\iota_n : K_\infty[[t]] \otimes_{B_K^{\dagger, r_n}} D^{\dagger, r_n}(V) \to D^+_{dif}(V)$ is an isomorphism between $K_\infty[[t]]$-mod with connection.

Pf: Take $n$ large s.t. $\iota_n(B_K^{\dagger, r_n}) \subseteq K_n[[t]]$

Step 1. Properties of $K_\infty[[t]] \otimes_{\iota_n(B_K^{\dagger, r_n})} \iota_n(D^{\dagger, r_n}(V))$

It's a $K_\infty[[t]]$-mod and it's

- a mod of finite type
- submod of $(B^+_{dR} \otimes_{\mathbb{Q}_p} V)^{H_K}$
- stable under $\Gamma_K$

Step 2: $\theta : D^+_{dif}(V) \to D_{sen}(V)$

$\rightsquigarrow$ $\theta \circ \iota_n : D^{\dagger, r_n}(V) \to D_{sen}(V)$ is surjective

**Lemma**: $\ker(\theta \circ \iota_n) = \varphi^{n-1}(q) \cdot D^{\dagger, r_n}(V)$ $\qquad q = \dfrac{[\varepsilon^p]-1}{[\varepsilon]-1}$

1. $I = [r, s]$ $\quad r_n \in I$

$\tilde{A}^+\left[\frac{p}{[\bar{\pi}]^r}, \frac{[\bar{\pi}]^s}{p}\right]$ is dense in $\tilde{A}_I$

$\bar{\pi} = [\varepsilon] - 1$

$x \in \tilde{A}_I \Rightarrow x = \sum a_k \left(\frac{p}{[\bar{\pi}]^r}\right)^k + \sum b_j \left(\frac{[\bar{\pi}]^s}{p}\right)^j$

direct calculation

$\ker(\theta \circ \iota_n : \tilde{A}_I \to \mathbb{C}_p) = \varphi^{n-1}(q) \tilde{A}_I$

$\ker(\theta \circ \iota_n : \tilde{B}_I \to \mathbb{C}_p) = \varphi^{n-1}(q) \cdot \tilde{B}_I$

2. $\ker(\theta \circ \iota_n : B_K^{\dagger, r_n} \to \mathbb{C}_p) = \varphi^{n-1}(q) \cdot B_K^{\dagger, r_n}$

$I = [r_n, +\infty)$ $\quad \ker(\theta \circ \iota_n : \tilde{B}_{rig}^{\dagger, r_n} \to \mathbb{C}_p) = \varphi^{n-1}(q) \tilde{B}_{rig}^{\dagger, r_n}$

$x \in B_K^{\dagger,r}$ $\quad x = \varphi^{n-1}(q)\cdot y \Rightarrow y \in B_K^{\dagger,r}$

$\Rightarrow \ker\left(\theta\circ \iota_n : B_K^{\dagger,r_n} \to \mathbb{C}_p\right) = \varphi^{n-1}(q)\cdot B_K^{\dagger,r_n}$

$\theta\circ\iota_n$ is an injective map

$$D^{\dagger,r_n}(V)\Big/\varphi^{n-1}(q)D^{\dagger,r_n}(V) \to D_{sen}(V)$$

$B_K^{\dagger,r_n}\Big/\varphi^{n-1}(q)$ -mod of rk $d$

$\iota_n(B_K^{\dagger,r_n}) \subseteq K_n[[t]]$

image = a $K_n$-v.sp. $V_n$ of dim $d$ stable under $\Gamma_K$

$V$ is overconvergent,

$\mathbb{C}_p \otimes_{K_n} V_n \xrightarrow{\sim} \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ for $n \gg 0$

$\Rightarrow$ $K_\infty \otimes_{K_n} V_n \to D_{sen}(V)$ is injective

dim (image) $\geq d$

Since dim $D_{sen}(V) = d$

$\Rightarrow$ image $= D_{sen}(V)$

We can recover $D_{sen}(V)$ from $\theta: D_{dif}^{+}(V) \to D_{sen}(V)$

$\theta\circ\iota_n$ is surjective

$\Rightarrow \theta$ is surjective

$$K_\infty[[t]] \otimes_{\iota_n} \left(B_K^{\dagger, r_n}\right) \iota_n\left(D^{\dagger, r_n}(V)\right) \hookrightarrow D_{dif}^+(V) \xrightarrow{\theta} D_{sen}(V)$$

$\theta(t) = 0$

determinant of this injection isn't divided by t

dim $= d$      dim $= d$

And $(t)$ = max. ideal of $K_\infty[[t]]$

$\Rightarrow$ The injection above is in fact an iso.

$\square$