CRYSTALLINE AND SEMI-STABLE REPRESENTATIONS AND (φ, Γ)-MODULES

by

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1. Introduction

Recall that we defined p-adic rings of periods $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$ and defined the corresponding admissible representations, which are respectively crystalline and semi-stable representations. The goal of this lecture is to explain how one can recover the invariants $\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{D}_{\mathrm{st}}(V)$ attached to a p-adic representation V of $\mathcal{G}_K$ in terms of its (φ, Γ)-module $D(V)$. Indeed, as the data of $D(V)$ is equivalent to that of V, one should be able to compute these invariants directly from $D(V)$. In order to do so, we need a ring which contains both $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{+}[1/t]$ and $\mathbf{B}^{\dagger}$.

We actually already defined such a ring: this is exactly $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}[1/t]$. Of course, this does not contain $\mathbf{B}_{\mathrm{st}}$ and we will introduce an other ring $\widetilde{\mathbf{B}}_{\log}^{\dagger}$, which is to $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger}$ what $\mathbf{B}_{\mathrm{st}}$ is to $\mathbf{B}_{\mathrm{cris}}$, so that $\widetilde{\mathbf{B}}_{\log}^{\dagger}[1/t]$ will contain both $\mathbf{B}_{\mathrm{st}}$ and $\mathbf{B}^{\dagger}$.

Using these rings, we will show that

$$\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)[1/t])^{\Gamma_K},$$

where $\mathbf{D}_{\mathrm{rig}}^{\dagger}(V)$ is the overconvergent (φ, Γ)-module $D^{\dagger}(V)$ tensored over $\mathbf{B}_K^{\dagger}$ by $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$ (we will also show that the corresponding statement relatively to $\mathbf{D}_{\mathrm{st}}(V)$ holds).

Before expanding a bit more on the ideas of the proofs, we give the following interpretation of some of the rings we have constructed (this is developped in [**Ber04**]):

1.1. Rings of periods and limits of algebraic functions. — One should think of most rings of periods as rings of "limits of algebraic functions" on certain subsets of $\mathbf{C}_p$. For example, the formula $\mathbf{B} = \widehat{\mathbf{B}}_F^{\mathrm{unr}}$ tells us that $\mathbf{B}$ is the ring of limits of (separable) algebraic functions on the boundary of the open unit disk. The ring $\widetilde{\mathbf{B}}$ is then the ring of all limits of algebraic functions on the boundary of the open unit disk.

Heuristically, one should view other rings in the same fashion: the ring $\mathbf{B}_{\mathrm{cris}}^{+}$ "is" the ring of limits of algebraic functions on the disk $D(0, |\varepsilon^{(1)} - 1|_p)$, and $\mathbf{B}_{\max}^{+}$ "is" the ring of limits of algebraic functions on a slightly smaller disk $D(0, r)$. One should therefore think of $\varphi^n(\mathbf{B}^+_{\mathrm{cris}})$ as the ring of limits of algebraic functions on the disk $D(0, |\varepsilon^{(n)} - 1|_p)$, and finally $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ "is" the ring of limits of algebraic functions on the open unit disk $D(0,1)$.

Similarly, $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ "is" the ring of limits of algebraic functions on an annulus $C[s,1[$, where s depends on r, and $\varphi^{-n}(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})$ "is" the ring of limits of algebraic functions on an annulus $C[s_n,1[$, where $s_n \to 0$, so that $\cap_{n=0}^{+\infty}\varphi^{-n}(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})$ "is" the ring of limits of algebraic functions on the open unit disk $D(0,1)$ minus the origin; furthermore, if an element of that ring satisfies some simple growth properties near the origin, then it "extends" to the origin (remember that in complex analysis, a holomorphic function on $D(0,1^-) - \{0\}$ which is bounded near 0 extends to a holomorphic function on $D(0,1^-)$).

As for the ring $\mathbf{B}^+_{\mathrm{dR}}$, it behaves like a ring of local functions around a circle (in particular, there is no Frobenius map defined on it). Via the map $\varphi^{-n} : \mathbf{B}^{\dagger,r_n}_{\mathrm{rig}} \to \mathbf{B}^+_{\mathrm{dR}}$, we have for $n \geq 1$ a filtration on $\mathbf{B}^{\dagger,r_n}_{\mathrm{rig}}$, which corresponds to the order of vanishing at $\varepsilon^{(n)} - 1$.

1.2. Heuristic of the proof. — Using the point of view from above, we now explain how to prove that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^\dagger_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$. We have seen that the periods of p-adic representations live inside $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$ and up to twisting by a power of the cyclotomic character, we can actually assume that the crystalline periods of V live inside $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ so that $\mathbf{D}_{\mathrm{cris}}(V) = (\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$.

Now the elements of $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ form a finite dimensional F-vector space, so that there is an r such that $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K} = (\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$, and furthermore this F-vector space is stable by Frobenius, so that the periods of V (in this setting) not only live in $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ but actually in $\cap_{n=0}^{+\infty}\varphi^{-n}(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})$ and they also satisfy some simple growth conditions (depending, say, on the size of $\det(\varphi)$), which ensure that they too can be seen as limits of algebraic functions on the open unit disk $D(0,1)$, that is as elements of $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$. In particular, we have $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K} = (\widetilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$. This is what we get by *regularization* (of the periods).

It's easy to show that $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{H_K} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}(V)$, and the last step is to show that $(\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},K}} \mathbf{D}^\dagger_{\mathrm{rig}}(V))^{\mathcal{G}_K} = \mathbf{D}^\dagger_{\mathrm{rig}}(V)^{\mathcal{G}_K}$. This is a decompletion process, very similar to the one used in order to prove the Colmez-Sen-Tate conditions for $\widetilde{\mathbf{A}}^\dagger_K$, which will take us from $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}$ to $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Similarly to the $\widetilde{\mathbf{A}}^\dagger_K$-case, the main idea is that the ring extension $\widetilde{\mathbf{B}}^\dagger_{\mathrm{rig},K}/\mathbf{B}^\dagger_{\mathrm{rig},K}$ looks very much like $\widehat{K_\infty}/K$, and we will define "decompletion maps" (which are an analogue of Tate's trace maps). This will show that in fact, $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_K} \mathbf{D}^\dagger(V))^{\mathcal{G}_K}$. In particular, V is crystalline if and only if $(\mathbf{B}^\dagger_{\mathrm{rig},K} \otimes_{\mathbf{B}^\dagger_K} \mathbf{D}^\dagger(V))^{\mathcal{G}_K}$ is a d-dimensional F-vector space.

We will now explain how this works exactly, following [**Ber02**].

2. The ring $\widetilde{\mathbf{B}}^{\dagger}_{\log}$ and the maps R_k

We let $\widetilde{\mathbf{B}}^{\dagger,r}_{\log} = \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[\log[\widetilde{p}]]$ endowed with the actions of $\mathcal{G}_F$ and φ given by $g(\log[\widetilde{p}]) = \log[\widetilde{p}] + c(g)t$ and $\varphi(\log[\widetilde{p}]) = p \cdot \log[\widetilde{p}]$. This allows us to extend the maps $\iota_n : \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \to \mathbf{B}^+_{\mathrm{dR}}$ for n big enough to $\widetilde{\mathbf{B}}^{\dagger,r}_{\log}$ by setting $\iota_n(\log[\widetilde{p}]) = p^{-n}\log[\widetilde{p}]$.

Proposition 2.1. — *The map $\iota_n : \widetilde{\mathbf{B}}^{\dagger,r_n}_{\mathrm{rig}}[\log[\widetilde{p}]] \to \mathbf{B}^+_{\mathrm{dR}}$ that extends ι_n by $\iota_n(\log[\widetilde{p}]) = p^{-n}\log[\widetilde{p}]$ is injective, commutes with the Galois action and its restriction to $\widetilde{\mathbf{B}}^+_{\log}$ is φ^{-n}.*

Proof. — Everything follows from the previous lectures. □

Recall that we actually have $\widetilde{\mathbf{B}}^{\dagger,r}_{\log} = \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[\log[x]]$ for any $x \neq 0 \in \widetilde{\mathbf{E}}$ which does not belong to $(\widetilde{\mathbf{E}}^+)^\times$. Since the series defining $\log([\overline{\pi}]/\pi)$ converges in $\widetilde{\mathbf{A}}^{\dagger,r_0}$, we actually have that $\widetilde{\mathbf{B}}^{\dagger,r}_{\log} = \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}[\log[\overline{\pi}]]$. We endow $\widetilde{\mathbf{B}}^{\dagger}_{\log}$ of the monodromy operator N defined by

$$N\left(\sum_{k=0}^{d} a_k \log(\pi)^k\right) = -\sum_{k=0}^{d} k a_k \log(\pi)^{k-1}$$

which means that $N = -d/d\log(\pi)$. One can check that N commutes with the Galois action.

Recall that if $r > 0$ and $p^k r > r_K$ then there exists a map $R_k : \widetilde{\mathbf{A}}^{\dagger,r}_K[1/[\pi]] \to \varphi^{-k}(\mathbf{A}^{\dagger,p^k r}_K[1/[\pi]])$ which is a continuous, $\varphi^{-k}(\mathbf{A}^{\dagger,p^k r}_K[1/[\pi]])$-linear section of the inclusion $\varphi^{-k}(\mathbf{A}^{\dagger,p^k r}_K[1/[\pi]]) \subset \widetilde{\mathbf{A}}^{\dagger,r}_K[1/[\pi]]$ which commutes with the Galois action (i.e. $\gamma \circ R_k = R_k \circ \gamma$ for $\gamma \in \Gamma_K$) and such that $R_k(x) \to x$ for all $x \in \widetilde{\mathbf{A}}^{\dagger,r}_K[1/[\pi]]$.

Proposition 2.2. — *If $r > 0$ and $p^k r > r_K$ then by $\mathbf{Q}_p$-linearity and continuity the above map R_k extends to a map $R_k : \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K} \to \varphi^{-k}(\mathbf{B}^{\dagger,p^k r}_{\mathrm{rig},K})$, such that:*

1. *R_k is a continuous section of the inclusion $\varphi^{-k}(\mathbf{B}^{\dagger,p^k r}_{\mathrm{rig},K}) \subset \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$;*
2. *R_k is $\varphi^{-k}(\mathbf{B}^{\dagger,p^k r}_{\mathrm{rig},K})$-linear;*
3. *if $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$, then $\lim_{k\to+\infty} R_k(x) = x$;*
4. *if $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$, and $\gamma \in \Gamma_K$, then $\gamma \circ R_k(x) = R_k \circ \gamma(x)$.*

Proof. — Once again this follows from statements made in the previous lectures, where it was shown that the maps R_k were continuous for the topology induced by the valuations $V_{[r,s]}$ for all $s \geq r$ and satisfied an inequality of the form $V_{[r,s]}(R_k(x)) \geq V_{[r,s]}(x) - c_K(r)$ where $c_K(r)$ only depends on K and r. Since $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ is the completion of $\widetilde{\mathbf{B}}^{\dagger,r}$ for the topology induced by the valuations $V_{[r,s]}, s \geq r$, this already shows that the maps R_k extend by $\mathbf{Q}_p$-linearity and continuity to maps $R_k : \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K} \to \varphi^{-k}(\mathbf{B}^{\dagger,p^k r}_{\mathrm{rig},K})$. It remains to see that if $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$, then $\lim_{k\to+\infty} R_k(x) = x$.

Let $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},K}$, let $s \geq r$ and let $M \geq 0$. We have to show that there exists $k_0 \in \mathbf{N}$ such that $V_{[r,s]}(x - R_k(x)) \geq M$ for $k \geq k_0$. By density, there exists $y \in \widetilde{\mathbf{B}}^{\dagger,r}$ such that

$V_{[r,s]}(x-y) \geq M + c_K(r)$. Let $n \in \mathbf{N}$ be such that $z = p^n y \in \widetilde{\mathbf{A}}^{\dagger,r}[1/[\pi]]$. Then there exists $k_0 \geq 0$ such that $v_r(z - R_k(z)) \geq M + ns$ if $k \geq k_0$. We then have

$$v_r(y - R_k(y)) = v_r(z - R_k(z)) - nr \geq M,$$

and

$$v_s(y - R_k(y)) = v_s(z - R_k(z)) - ns \geq v_r(z - R_k(z)) - ns \geq M.$$

Thus we have $V_{[r,s]}(y - R_k(y)) \geq M$ if $k \geq k_0$. To conclude it suffices to write $x - R_k(x)$ as $(x-y) - R_k(x-y) + (y - R_k(y))$. □

We also define $\mathbf{B}^{\dagger}_{\log,K} = \mathbf{B}^{\dagger}_{\mathrm{rig},K}[\log(\pi)]$, which is a ring stable by the actions of φ and Γ_K since $\varphi(\log(\pi)) = \log(\varphi(\pi)) = p\log(\pi) + \log(\varphi(\pi)/\pi^p)$ and $\gamma(\log(\pi)) = \log(\pi) + \log(\gamma(\pi)/\pi)$ and that the series defining $\log(\varphi(\pi)/\pi^p)$ and $\log(\gamma(\pi)/\pi)$ converge in $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$.

Definition 2.3. — We extend the maps R_k to a $\varphi^{-k}(\mathbf{B}^{\dagger,p^k r}_{\mathrm{rig},K})$-linear section of the inclusion $\varphi^{-k}(\mathbf{B}^{\dagger,p^k r}_{\log,K}[1/t])$ in $\widetilde{\mathbf{B}}^{\dagger,r}_{\log,K}[1/t]$: those maps still commute with Γ_K, and satisfy $\lim_{k\to+\infty} R_k(x) = x$ si $x \in \widetilde{\mathbf{B}}^{\dagger,r}_{\log,K}[1/t]$.

3. Regularization

We now prove the result of regularization by the Frobenius on which the rest of the proofs will rely on. Recall that the Frobenius φ is a bijection from $\widetilde{\mathbf{B}}_I$ to $\widetilde{\mathbf{B}}_{pI}$ and thus induces a bijection from $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ to $\widetilde{\mathbf{B}}^{\dagger,pr}_{\mathrm{rig}}$ and from $\widetilde{\mathbf{B}}^{\dagger,r}_{\log}$ to $\widetilde{\mathbf{B}}^{\dagger,pr}_{\log}$ since $\varphi(\log[\overline{\pi}]) = p \cdot \log[\overline{\pi}]$.

Lemma 3.1. — *We have* $\widetilde{\mathbf{A}}^{[r;s]}/(p) = \widetilde{\mathbf{E}}^{+}[X, \overline{\pi}^{s-r}X^{-1}]/(\overline{\pi}^s, \overline{\pi}^r X)$. *In particular if* $r = s$, *then* $\widetilde{\mathbf{A}}^{[r;r]}/(p) = \widetilde{\mathbf{E}}^{+}/(\overline{\pi}^r)[X, X^{-1}]$.

Proof. — Let $A = \widetilde{\mathbf{A}}^{+}\{X, Y\}$ and $I = (XY - [\overline{\pi}]^{s-r}, p - X[\overline{\pi}]^r, [\overline{\pi}]^s - pY)$ so that $\widetilde{\mathbf{A}}^{[r;s]}$ can be identified with A/I and so that $\widetilde{\mathbf{A}}^{[r;s]}/(p) = (A/I)/(p)$. We have an exact sequence $0 \to I \to A \to A/I \to 0$ and the multiplication by p induces the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & I/p & \longrightarrow & A/p & \longrightarrow & (A/I)/p & \longrightarrow & 0
\end{array}$$

and since A/I has no p-torsion, the snake's lemma shows that $(A/I)/p$ is identified with the quotient of A/p by the image of I inside it. In our setting we have $A/p = \widetilde{\mathbf{E}}^{+}[X, Y]$ and the image of I is $(XY - \overline{\pi}^{s-r}, -X\overline{\pi}^r, \overline{\pi}^s)$, hence the lemma. □

Lemma 3.2. — *The natural inclusions* $\widetilde{\mathbf{A}}^{[0,r_0]} \subset \widetilde{\mathbf{A}}^{[r_0,r_0]}$ *and* $\widetilde{\mathbf{A}}^{\dagger,r_0} \subset \widetilde{\mathbf{A}}^{[r_0,r_0]}$ *induce the following exact sequence:*

$$0 \to \widetilde{\mathbf{A}}^{+} \to \widetilde{\mathbf{A}}^{[0,r_0]} \oplus \widetilde{\mathbf{A}}^{\dagger,r_0} \to \widetilde{\mathbf{A}}^{[r_0,r_0]} \to 0.$$

Proof. — The fact that the arrow $\widetilde{\mathbf{A}}^{[0,r_0]} \oplus \widetilde{\mathbf{A}}^{\dagger,r_0} \to \widetilde{\mathbf{A}}^{[r_0,r_0]}$ is surjective follows from the decomposition of an element of $\widetilde{\mathbf{A}}^{[r_0,r_0]}$ in two parts (recall that $\widetilde{\mathbf{A}}^{[r_0,r_0]}$ is the p-adic completion of $\widetilde{\mathbf{A}}^+[\frac{p}{[\overline{\pi}]}, \frac{[\overline{\pi}]}{p}]$). We also know that $\widetilde{\mathbf{A}}^+$ is contained in the intersection of $\widetilde{\mathbf{A}}^{[0,r_0]}$ and $\widetilde{\mathbf{A}}^{\dagger,r_0}$ and thus it remains to prove that the map

$$\widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^{\dagger,r_0} \cap \widetilde{\mathbf{A}}^{[0,r_0]}$$

is also surjective. First we prove that it is true modulo $p\widetilde{\mathbf{A}}^{[r_0,r_0]}$ (note that the arrow is no longer injective mod p). Recall that the rings $\widetilde{\mathbf{A}}^{[0,r_0]}$ et $\widetilde{\mathbf{A}}^{\dagger,r_0}$ can be identified with $\widetilde{\mathbf{A}}^+\{X\}/(pX-[\widetilde{p}])$ and $\widetilde{\mathbf{A}}^+\{Y\}/([\widetilde{p}]Y-p)$ and that $\widetilde{\mathbf{A}}^{[r_0,r_0]}/(p) = \widetilde{\mathbf{E}}^+/(\widetilde{p})[X,X^{-1}]$. The image of $\widetilde{\mathbf{A}}^{\dagger,r_0}$ inside this ring is then $\widetilde{\mathbf{E}}^+/(\widetilde{p})[1/X]$ and the one of $\widetilde{\mathbf{A}}^{[0,r_0]}$ is $\widetilde{\mathbf{E}}^+/(\widetilde{p})[X]$ so that the image of their intersection is a subring of $\widetilde{\mathbf{E}}^+/(\widetilde{p})$ and thus the arrow $\widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{A}}^{\dagger,r_0} \cap \widetilde{\mathbf{A}}^{[0,r_0]}$ is surjective modulo $p\widetilde{\mathbf{A}}^{[r_0,r_0]}$. If we let x in $\widetilde{\mathbf{A}}^{\dagger,r_0} \cap \widetilde{\mathbf{A}}^{[0,r_0]}$ it therefore exists $y \in \widetilde{\mathbf{A}}^+$ such that $x-y \in p\widetilde{\mathbf{A}}^{[r_0,r_0]}$. This means that $x-y \in p\widetilde{\mathbf{A}}^{[0,r_0]}$ and $\in p\widetilde{\mathbf{A}}^{\dagger,r_0} + [\widetilde{p}]\widetilde{\mathbf{A}}^+$ (it suffices to apply lemma 3.1 to these rings). Since p divides $[\widetilde{p}]$ in $\widetilde{\mathbf{A}}^{[0,r_0]}$ there exists $z \in [\widetilde{p}]\widetilde{\mathbf{A}}^+$ such that $x-y-z \in p(\widetilde{\mathbf{A}}^{\dagger,r_0} \cap \widetilde{\mathbf{A}}^{[0,r_0]})$. Since $\widetilde{\mathbf{A}}^+$ is complete for the p-adic topology, it suffices to iterate this to prove the lemma. □

Lemma 3.3. — *Let h be a positive integer. Then*

$$\cap_{s=0}^{+\infty} p^{-hs}\widetilde{\mathbf{A}}^{\dagger,p^{-s}r} = \widetilde{\mathbf{A}}^+ \text{ and } \cap_{s=0}^{+\infty} p^{-hs}\widetilde{\mathbf{A}}_{\mathrm{rig}}^{\dagger,p^{-s}r} \subset \widetilde{\mathbf{B}}_{\mathrm{rig}}^+$$

where $\widetilde{\mathbf{A}}_{\mathrm{rig}}^{\dagger,r}$ denotes the ring of integers of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ for the valuation $V_{[r;r]}$.

Proof. — Let us start with the first point: since $x \in \widetilde{\mathbf{A}}^{\dagger,r}$ it can be uniquely written as $\sum_{k\geq 0} p^k[x_k]$ and we also have $p^{hs}x = \sum p^{k+hs}[x_k]$. Since $p^{hs}x \in \widetilde{\mathbf{A}}^{\dagger,p^{-s}r}$ this means that

$$v_{\mathbf{E}}(x_k) + \frac{rp^{1-s}}{p-1}(k+hs) \geq 0$$

so that

$$v_{\mathbf{E}}(x_k) \geq -\frac{(k+hs)r}{p^{s-1}(p-1)}$$

and thus (when $s\to+\infty$) that $v_{\mathbf{E}}(x_k)\geq 0$ so that $x \in \widetilde{\mathbf{A}}^+$.

For the second point, note that for all s one can write $x = a_s + b_s$ with $a_s \in p^{-hs}\widetilde{\mathbf{A}}^{\dagger,p^{-s}r}$ and $b_s \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^+$. By the lemma 3.2 we have $a_s - a_{s+1} \in \widetilde{\mathbf{B}}^+$ and we also know that $a_s - a_{s+1} \in p^{-h(s+1)}\widetilde{\mathbf{A}}^{\dagger,p^{-s}r}$ so that $a_s - a_{s+1} \in p^{-h(s+1)}\widetilde{\mathbf{A}}^+$ and so that up to changing a_{s+1} we can assume that $a_s = a_{s+1} = a$. We then have $a \in \cap_{s=0}^{+\infty} p^{-hs}\widetilde{\mathbf{A}}^{\dagger,p^{-s}r} = \widetilde{\mathbf{A}}^+$ and thus $x \in \widetilde{\mathbf{B}}_{\mathrm{rig}}^+$. □

Proposition 3.4. — *Let r and u be two positive integers and let $A \in \mathrm{M}_{u\times r}(\widetilde{\mathbf{B}}_{\log}^\dagger)$. Assume that there exists $P \in \mathrm{GL}_u(F)$ such that $A = P\varphi^{-1}(A)$. Then $A \in \mathrm{M}_{u\times r}(\widetilde{\mathbf{B}}_{\log}^+)$.*

Proof. — Let $A = (a_{ij})$ and $a_{ij} = \sum_{n=0}^d a_{ij,n}\log[\overline{\pi}]^n$. Let $h_0 \in \mathbf{Z}$ such that $p^{h_0}P \in \mathrm{M}_u(\mathcal{O}_F)$ and $h = h_0 + d$. The assumption on A and P can be written as:

$$p_{i1}\varphi^{-1}(a_{1j}) + \cdots + p_{iu}\varphi^{-1}(a_{uj}) = a_{ij} \qquad \forall i \leq u,\ j \leq r$$

and since $\varphi^{-1}(\log[\overline{\pi}]^n) = p^{-n}\log[\overline{\pi}]^n$, this implies that if $a_{ij,n} \in p^{-c}\widetilde{\mathbf{A}}^{\dagger,r}_{\mathrm{rig}}$, then since $p^{h_0}p_{ik} \in \mathcal{O}_F$ and $\varphi^{-1}(a_{ik,n}) \in p^{-c}\widetilde{\mathbf{A}}^{\dagger,r/p}_{\mathrm{rig}}$, we have $a_{ij,n} \in p^{-h-c}\widetilde{\mathbf{A}}^{\dagger,r/p}_{\mathrm{rig}}$. By iterating this we get that $a_{ij,n} \in \cap_{s=0}^{+\infty} p^{-hs-c}\widetilde{\mathbf{A}}^{\dagger,rp^{-s}}_{\mathrm{rig}}$. This allows us to apply the above lemma to $p^c a_{ij,n}$ and this finishes the proof. □

4. Applications for semi-stable periods

Let $\mathbf{B}^{\dagger,r}_{\log,K} = \mathbf{B}^{\dagger,r}_{\mathrm{rig},K}[\log(\pi)]$ and let

$$\mathbf{D}^{\dagger}_{\mathrm{rig}}(V) = \mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V) \quad \text{et} \quad \mathbf{D}^{\dagger}_{\log}(V) = \mathbf{B}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V)$$

The fact that p-adic representations are overconvergent shows that both $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ and $\mathbf{D}^{\dagger}_{\log}(V)$ are $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$- and $\mathbf{B}^{\dagger}_{\log,K}$- free modules of rank $d = \dim_{\mathbf{Q}_p}(V)$.

If V is a p-adic representation, or more generally, a B-representation where B is a $\mathbf{Q}_p$-algebra endowed with an action of $\mathcal{G}_K$, we let $V(i)$ denote the twist of V by χ^i, so that the action of $g \in \mathcal{G}_K$ on $V(i)$ is the action of g on V multiplied by $\chi(g)^i$.

Proposition 4.1. — *One has*

$$\{x \in \widetilde{\mathbf{B}}^{\dagger}_{\log},\ g(x) = \chi^i(g)x,\ \forall g \in \mathcal{G}_K\} = \begin{cases} Ft^i & \text{if } i \geq 0; \\ 0 & \text{if } i < 0. \end{cases}$$

Proof. — Let $V_i^n = (\widetilde{\mathbf{B}}^{\dagger,r_n}_{\log}(-i))^{\mathcal{G}_K}$. This is a finite dimensional F-vector space stable by Frobenius and the previous proposition implies that $V_i^n = (\widetilde{\mathbf{B}}^{+}_{\log}(-i))^{\mathcal{G}_K}$. But $(\widetilde{\mathbf{B}}^{+}_{\log}[1/t](i))^{\mathcal{G}_K}$ is the (φ, N)-module attached to the crystalline representation χ^i, so that it is of dimension 1 and thus any non zero element of $(\widetilde{\mathbf{B}}^{+}_{\log}[1/t](i))^{\mathcal{G}_K}$ generates this F-vector space. Since t^i works, this means that

$$V_i^n = \begin{cases} Ft^i & \text{if } i \geq 0; \\ 0 & \text{if } i < 0. \end{cases}$$

Since $V_i = \cup_{n=0}^{+\infty} V_i^n$ this proves the result. □

We let $\mathbf{D}^{+}_{\mathrm{st}}(V) = (\mathbf{B}^{+}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$. Recall that $\mathbf{D}^{+}_{\mathrm{st}}(V) = (\widetilde{\mathbf{B}}^{+}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$. A quick computation shows that $\mathbf{D}_{\mathrm{st}}(V) = t^{-d}\mathbf{D}_{\mathrm{st}}(V(-d))$ and thus for $d \gg 0$, we have that $\mathbf{D}_{\mathrm{st}}(V) = t^{-d}\mathbf{D}^{+}_{\mathrm{st}}(V(-d))$.

Proposition 4.2. — *If V is a p-adic representation then $(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ is a finite dimensional F-vector space and the morphism*

$$\mathbf{D}^{+}_{\mathrm{st}}(V) \to (\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$$

induced by the inclusion $\widetilde{\mathbf{B}}^{+}_{\log} \subset \widetilde{\mathbf{B}}^{\dagger}_{\log}$ is an isomorphism of (φ, N)-modules.

Proof. — If $n \in \mathbf{N}$, then $D_n = (\widetilde{\mathbf{B}}^{\dagger,r_n}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ is a F-vector space whose dimension is $\leq [K:F]d$, since ι_n induces an injection of D_n in $\mathbf{D}_{\mathrm{dR}}(V)$, which is a finite dimensional K-vector space of dimension $\leq d$. If we take $[K:F]d+1$ elements of $(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$, they live inside D_n for $n \gg 0$, and thus are linearly dependent. This means that $(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is a F-vector space of dimension $\leq [K:F]d$.

For the second point, let $v_1, \cdots, v_r$ and $d_1, \cdots, d_u$ be respectively $\mathbf{Q}_p$- and F- basis of V and $(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$. There exists a matrix $A \in \mathrm{M}_{r\times u}(\widetilde{\mathbf{B}}^{\dagger}_{\log})$ such that $(d_i) = A(v_i)$. Let $P \in \mathrm{GL}_u(F)$ be the matrix of φ in the basis (d_i) (which is invertible since $\varphi : \widetilde{\mathbf{B}}^{\dagger}_{\log} \to \widetilde{\mathbf{B}}^{\dagger}_{\log}$ is bijective). We then have $\varphi(A) = PA$ and thus $A = \varphi^{-1}(P)\varphi^{-1}(A)$; proposition 3.4 shows that $A \in \mathrm{M}_{r\times u}(\widetilde{\mathbf{B}}^{+}_{\log})$ and thus that $(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K} \subset (\widetilde{\mathbf{B}}^{+}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K} = \mathbf{D}^{+}_{\mathrm{st}}(V)$, which is what we wanted. □

Up to twisting V, this implies that V is semi-stable if and only if it is $\widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]$-admissible, and that it is crystalline if and only if it is $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t]$-admissible.

Proposition 4.3. — *If V is semi-stable we have the following comparison isomorphism:*

$$\widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \otimes_F \mathbf{D}_{\mathrm{st}}(V) = \widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \otimes_{\mathbf{Q}_p} V$$

Proof. — This comes from the fact that in this case we already have:

$$\widetilde{\mathbf{B}}^{+}_{\log}[1/t] \otimes_F \mathbf{D}_{\mathrm{st}}(V) = \widetilde{\mathbf{B}}^{+}_{\log}[1/t] \otimes_{\mathbf{Q}_p} V$$

and thus it suffices to tensor this equality by $\widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t]$ over $\widetilde{\mathbf{B}}^{+}_{\log}[1/t]$. □

Theorem 4.4. — *If V is a p-adic representation of $\mathcal{G}_K$ then*

$$\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{D}^{\dagger}_{\log}(V)[1/t])^{\Gamma_K} \quad \textit{and} \quad \mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$$

In particular V is semi-stable (resp. crystalline) if and only if $(\mathbf{D}^{\dagger}_{\log}(V)[1/t])^{\Gamma_K}$ (resp. $(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$) is a F-vector space of dimension $d = \dim_{\mathbf{Q}_p}(V)$.

Proof. — The second point follows from the first one by taking the invariants under $N = 0$. Since $\mathbf{D}^{\dagger}_{\log}(V)[1/t] \subset (\widetilde{\mathbf{B}}^{\dagger}_{\log}[1/t] \otimes_{\mathbf{Q}_p} V)$ (and since $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t] \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{\mathbf{Q}_p} V)$) the previous results show that $(\mathbf{D}^{\dagger}_{\log}(V)[1/t])^{\Gamma_K}$ (resp. $(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$) is contained in $\mathbf{D}_{\mathrm{st}}(V)$ (resp. in $\mathbf{D}_{\mathrm{cris}}(V)$).

Let us now show that $\mathbf{D}_{\mathrm{st}}(V) \subset (\mathbf{D}^{\dagger}_{\log}(V)[1/t])^{\Gamma_K}$, and that $\mathbf{D}_{\mathrm{cris}}(V) \subset (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$. It suffices to show the semi-stable case because the crystalline one will follow by taking the invariants under $N = 0$. Let $r = \dim_F(\mathbf{D}_{\mathrm{st}}(V))$. Up to twisting V, we can assume that $\mathbf{D}_{\mathrm{st}}(V) = (\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$ and we know that we have $(\widetilde{\mathbf{B}}^{\dagger}_{\log} \otimes_{\mathbf{Q}_p} V)^{H_K} = \widetilde{\mathbf{B}}^{\dagger}_{\log,K} \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V)$ since $\mathbf{D}^{\dagger}(V)$ has the right dimension. This implies that if we chose a basis $\{e_i\}$ of $\mathbf{D}^{\dagger}(V)$ and $\{d_i\}$ a basis of $\mathbf{D}_{\mathrm{st}}(V)$, then the matrix $M \in \mathrm{M}_{r\times d}(\widetilde{\mathbf{B}}^{\dagger}_{\log,K})$ defined by $(d_i) = M(e_i)$ is of rank r et and satisfies $\gamma_K(M)G - M = 0$ where $G \in \mathrm{GL}_d(\mathbf{B}^{\dagger}_K)$ is the matrix of γ_K in the basis $\{e_i\}$.

The trace maps R_m defined previously are $\mathbf{B}^{\dagger}_{\log,K}$-linear and commute with Γ_K so that $\gamma_K(R_m(M))G - R_m(M) = 0$. Moreover $R_m(M) \to M$ and if $M \in \mathrm{M}_{r\times d}(\widetilde{\mathbf{B}}^{\dagger,r_n}_{\log,K})$, then

$R_m(M) \in \mathrm{M}_{r\times d}(\widetilde{\mathbf{B}}^{\dagger,r_n}_{\log,K})$. Let $N = \varphi^m(R_m(M))$. We then have $\gamma_K(N)\varphi^m(G) = N$ and since the actions of φ and Γ_K commute on $\mathbf{D}^\dagger_{\mathrm{rig}}(V)$ we have $\varphi(G) = \gamma_K(P)GP^{-1}$ (P is the matrix of φ and is invertible since φ is overconvergent and $\mathbf{B}^\dagger_K$ is a field) so that if $Q = \varphi^{m-1}(P)\cdots\varphi(P)P$, then $\varphi^m(G) = \gamma_K(Q)GQ^{-1}$ and thus $\gamma_K(NQ)G = (NQ)$. The matrix NQ determines r elements of $\mathbf{D}^\dagger_{\log}(V)$ which are fixed by γ_K. It remains to show that these elements are linearly independent over F for m big enough. But since $R_m(M) \to M$, the matrix NQ is of rank $r = \mathrm{rank}(M)$ for $m \gg 0$ and thus will determine a free sub-module of rank r of $\mathbf{D}^\dagger_{\log}(V)$. The F-vector space generated by the elements determined by NQ is thus of dimension r and thus equal to $\mathbf{D}_{\mathrm{st}}(V)$. □

Proposition 4.5. — *We have the following comparison isomorphisms:*

1. *if V is a semi-stable representation then*
$$\mathbf{D}^\dagger(V) \otimes_{\mathbf{B}^\dagger_K} \mathbf{B}^\dagger_{\log,K}[1/t] = \mathbf{D}_{\mathrm{st}}(V) \otimes_F \mathbf{B}^\dagger_{\log,K}[1/t]$$
2. *if V is a crystalline representation then*
$$\mathbf{D}^\dagger(V) \otimes_{\mathbf{B}^\dagger_K} \mathbf{B}^\dagger_{\mathrm{rig},K}[1/t] = \mathbf{D}_{\mathrm{cris}}(V) \otimes_F \mathbf{B}^\dagger_{\mathrm{rig},K}[1/t]$$

Proof. — Once again it suffices to prove the semi-stable case as the crystalline case will follow by taking the invariants under $N = 0$. Up to twisting V, we are reduced to the case where V is $\widetilde{\mathbf{B}}^+_{\log}$-admissible and we then know that $\mathbf{D}_{\mathrm{st}}(V) \subset \widetilde{\mathbf{B}}^\dagger_{\log,K} \otimes_{\mathbf{B}^\dagger_K} \mathbf{D}^\dagger(V)$ and that
$$\widetilde{\mathbf{B}}^\dagger_{\log,K}[1/t] \otimes_{\mathbf{B}^\dagger_K} \mathbf{D}^\dagger(V) = \widetilde{\mathbf{B}}^\dagger_{\log,K}[1/t] \otimes_F \mathbf{D}_{\mathrm{st}}(V)$$
so that if we chose basis $\{d_i\}$ of $\mathbf{D}_{\mathrm{st}}(V)$ and $\{e_i\}$ of $\mathbf{D}^\dagger(V)$, then $(e_i) = B(d_i)$ with $B \in \mathrm{M}_d(\widetilde{\mathbf{B}}^\dagger_{\log,K}[1/t])$; proposition 4.4 implies that $(d_i) = A(e_i)$ with $A \in \mathrm{M}_d(\mathbf{B}^\dagger_{\log,K}[1/t])$; and moreover we have that $AB = \mathrm{Id}$. We can apply the operator R_0 which is $\mathbf{B}^\dagger_{\log,K}[1/t]$-linear and we obtain that $AR_0(B) = \mathrm{Id}$ so that $B = R_0(B)$ and thus B has its coefficients in $\mathbf{B}^\dagger_{\log,K}[1/t]$ and $A \in \mathrm{GL}_d(\mathbf{B}^\dagger_{\log,K}[1/t])$. This finishes the proof. □

Proposition 4.6. — *Let V be a semi-stable representation of $\mathcal{G}_K$ and let M be the transfer matrix from a basis of $\mathbf{D}_{\mathrm{st}}(V)$ to a basis of $\mathbf{D}^\dagger(V)$. It then exists $r \in \mathbf{Z}$ and $\lambda \in \mathbf{B}^\dagger_K$ such that* $\det(M) = \lambda t^r$.

Proof. — The determinant of the transfer matrix is equal to the coefficient of the transfer matrix for $\det(V)$ and thus it suffices to prove the result in dimension 1. But semi-stable characters are of the form $\omega\chi^r$ where ω is an unramified character and its period is an element $\beta \in W(\overline{k})$, so that $\mathbf{D}_{\mathrm{st}}(V) = F \cdot \beta t^{-r}$ and $\mathbf{D}^\dagger(V) = \mathbf{B}^\dagger_K \cdot \beta$. □

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