CRYSTALLINE AND SEMI-STABLE REPRESENTATIONS

by

The goal of this lecture is to define crystalline and semi-stable representations, to explain the attached notions and to explain the links with (φ, Γ)-modules. In what follows, we keep the notations defined and used in the previous lectures. In particular, recall that K is a finite extension of $\mathbf{Q}_p$ and that we note $F = \mathbf{Q}_p{}^{\mathrm{unr}} \cap K$.

Fontaine's strategy to study p-adic representations of $\mathcal{G}_K$ is to construct some "p-adic rings of periods", which are topological $\mathbf{Q}_p$-algebras B endowed with an action of $\mathcal{G}_K$ and with additional structures (like a Frobenius, a filtration, ...) and to attach to V the $B^{\mathcal{G}_K}$-module $D_B(V) = (B \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$, so that D_B inherits the structures coming from those on B. When V is such that $B \otimes_{\mathbf{Q}_p} V \simeq B^{\dim_{\mathbf{Q}_p} V}$ as $B[\mathcal{G}_K]$-modules, we say that V is B-admissible. If the structures on B are "nice enough", one can recover V from $D_B(V)$ when V is B-admissible. Of course, we would also like to define rings of periods that will be large enough so that representations arising from geometric situations, like the Tate modules of elliptic curves are admissible for these rings.

In the last lecture we defined $\mathbf{B}_{\mathrm{dR}}^+$ as the completion of $\widetilde{\mathbf{B}}^+$ for the $\ker\theta$-adic tolopogy, and we have denoted by t the element of $\mathbf{B}_{\mathrm{dR}}^+$ defined by the power series $\log[\varepsilon]$. We have seen that $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$ is a field, and we defined the notion of de Rham representations, which are p-adic representations of $\mathcal{G}_K$ that are $\mathbf{B}_{\mathrm{dR}}$-admissible.

However, note that we can't recover V from $\mathbf{D}_{\mathrm{dR}}(V)$, even when V is de Rham, because the functor $V \mapsto \mathbf{D}_{\mathrm{dR}}(V)$, from the category of de Rham representations to the category of filtered K-vector spaces, is not fully faithful. The rings $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$ we will introduce are meant to answer this problem while still taking into account (some) representations arising from geometric situations.

In this lecture, we will mainly follow [**Ber04**] and [**Ber10**].

1. The rings $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$

1.1. The ring $\mathbf{B}_{\mathrm{cris}}$. — One problem with $\mathbf{B}_{\mathrm{dR}}^+$ is that completing $\widetilde{\mathbf{B}}^+$ for the $\ker\theta$-adic topology is too coarse: there is no natural extension of the Frobenius $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$ to $\mathbf{B}_{\mathrm{dR}}^+$. One way to see this is that since ${\mathbf{B}_{\mathrm{dR}}}^{\mathcal{G}_L} = L$ for every finite extension L/K, the existence of a canonical Frobenius map $\varphi : \mathbf{B}_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}}$ would imply the existence of a Frobenius map $\varphi : \overline{K} \to \overline{K}$, and such a map does not exist.

An other way to see it and which will explain a bit more the construction of $\mathbf{B}_{\mathrm{cris}}$ is that there are elements $x \in \widetilde{\mathbf{B}}^+$ that are killed by θ but such that $\theta(\varphi(x)) \neq 0$. For example, $\theta([\widetilde{p}^{1/p}]-p) \neq 0$, so that $[\widetilde{p}^{1/p}]-p$ is invertible in $\mathbf{B}^+_{\mathrm{dR}}$, and so $1/([\widetilde{p}^{1/p}]-p) \in \mathbf{B}^+_{\mathrm{dR}}$. But if φ is a natural extension of $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$, then one should have $\varphi(1/([\widetilde{p}^{1/p}]-p)) = 1/([\widetilde{p}]-p)$, and since $\theta([\widetilde{p}]-p) = 0$, $1/([\widetilde{p}]-p) \notin \mathbf{B}^+_{\mathrm{dR}}$.

In particular, we want to complete $\widetilde{\mathbf{B}}^+$ in a more subtle way, so that $t = \log[\varepsilon]$ still lies in this completion, but such that we avoid to invert elements like $[\widetilde{p}^{1/p}] - p$. The way to do this is to impose some growth condition. Recall that $\pi = [\varepsilon] - 1$ and that

$$\omega = \frac{\pi}{\varphi^{-1}(\pi)} = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{1/p}]^{p-1}.$$

Definition 1.1. — We define

$$\mathbf{A}_{\mathrm{cris}} := \left\{ \begin{array}{c} x \in \mathbf{B}_{\mathrm{dR}} \text{ such that } x \text{ can be written as } x = \sum_{n=0}^{+\infty} x_n \dfrac{\omega^n}{n!} \\ \text{where } x_n \in \widetilde{\mathbf{A}}^+ \text{ and the } x_n \to 0 \text{ in } \widetilde{\mathbf{A}}^+ \text{ for the p-adic topology} \end{array} \right\},$$

and

$$\mathbf{B}^+_{\mathrm{cris}} = \mathbf{A}^+_{\mathrm{cris}}[1/p] = \left\{ \begin{array}{c} x \in \mathbf{B}_{\mathrm{dR}} \text{ such that } x \text{ can be written as } x = \sum_{n=0}^{+\infty} x_n \dfrac{\omega^n}{n!} \\ \text{where } x_n \in \widetilde{\mathbf{A}}^+ \text{ and the } x_n \to 0 \text{ in } \widetilde{\mathbf{B}}^+ \text{ for the p-adic topology} \end{array} \right\}.$$

Remark 1.2. — Note that, since $\ker(\theta_{|\widetilde{\mathbf{A}}^+})$ is a principal ideal of $\widetilde{\mathbf{A}}^+$, one can replace ω in the definition by any generator of $\ker(\theta)$. Also note that if one can write $x = \sum_{n=0}^{+\infty} x_n \frac{\omega^n}{n!} \in \mathbf{A}_{\mathrm{cris}}$ with $x_n \to 0$, such a sequence is not unique.

Proposition 1.3. — *The Frobenius and the action of $\mathcal{G}_K$ on $\widetilde{\mathbf{A}}^+$ and $\widetilde{\mathbf{B}}^+$ extend naturally to actions on $\mathbf{A}_{\mathrm{cris}}$ and $\mathbf{B}^+_{\mathrm{cris}}$.*

Proof. — The fact that $\mathbf{A}_{\mathrm{cris}}$ is stable under the action of $\mathcal{G}_K$ comes from the fact that $g(\omega) = x\omega$ with $x \in \widetilde{\mathbf{A}}^+$ and from the definition of $\mathbf{A}_{\mathrm{cris}}$.

For the Frobenius, we will use the fact that if $x \in \widetilde{\mathbf{A}}^+$, then $\varphi(x) \equiv x^p \mod p$. In particular, we have

$$\varphi(\omega) = \omega^p + px = p\left(x + (p-1)!\frac{\omega^p}{p!}\right)$$

with $x \in \widetilde{\mathbf{A}}^+$, and thus

$$\varphi(\omega^m) = p^m\left(x + (p-1)!\frac{\omega^p}{p!}\right)^m$$

which gives

$$\varphi\left(\frac{\omega^m}{m!}\right) = \frac{p^m}{m!}\left(x + (p-1)!\frac{\omega^p}{p!}\right)^m \in \widetilde{\mathbf{A}}^+\left[\frac{\omega^p}{p!}\right] \subset \mathbf{A}_{\mathrm{cris}}.$$

This allows us to define φ on $\mathbf{A}_{\text{cris}}$ et $\mathbf{B}^+_{\text{cris}}$. □

Proposition 1.4. — *We have $t \in \mathbf{A}_{\text{cris}}$.*

Proof. — Write $[\varepsilon] - 1 = x\omega$ with $x \in \widetilde{\mathbf{A}}^+$. Then

$$\frac{([\varepsilon]-1)^n}{n} = (n-1)!x^n\frac{\omega^n}{n!}$$

and since $(n-1)! \to 0$ p-adically, we deduce that $t \in \mathbf{A}_{\text{cris}}$. □

We now define $\mathbf{B}_{\text{cris}}$ by $\mathbf{B}_{\text{cris}} = \mathbf{B}^+_{\text{cris}}[1/t]$ and we extend the actions of $\mathcal{G}_K$ and of the Frobenius to $\mathbf{B}_{\text{cris}}$ by setting $\varphi(1/t) = \frac{1}{pt}$ and $g(1/t) = \chi(\gamma)^{-1}1/t$. Note that $\mathbf{B}_{\text{cris}}$ is a domain but not a field because for example $\omega - p$ belongs to $\mathbf{B}_{\text{cris}}$ but not its inverse.

We now recall a useful lemma. Let N be a torsion free $\mathbf{Z}_p$-module which is separated and complete for the p-adictopology and let $M \subset N$. If $\widehat{M}$ denotes the completion of M for the p-adic topology, then the inclusion map $M \to N$ extends to a map $\widehat{M} \to N$ whose image is the closure of M in N.

Lemma 1.5. — *If there exists $i > 1$ such that $p^iN \cap M \subset pM$, then the map above is injective.*

Proof. — Every element of $\widehat{M}$ can be written as $m = \sum_{n\geq 0} p^{in}m_n$ with $m_n \in M$ and if $m \in \ker(\widehat{M} \to N)$ then $m_0 \in p^iN \cap M \subset pM$ so that $\ker(\widehat{M} \to N) \subset p\ker(\widehat{M} \to N)$. By iterating this, we get that $\ker(\widehat{M} \to N) \subset \cap_{k\geq 0}p^k\widehat{M} = 0$. □

Proposition 1.6. — *The natural map $K \otimes_F \mathbf{B}^+_{\text{cris}} \to \mathbf{B}^+_{\text{dR}}$ is injective.*

Proof. — We will prove that the natural map $K \otimes_F \widetilde{\mathbf{B}}^+ \to \mathbf{B}^+_{\text{dR}}$ is injective. Actually it suffices to prove that the natural map $\mathcal{O}_K \otimes_{\mathcal{O}_F} \widetilde{\mathbf{A}}^+ \to \mathbf{B}^+_{\text{dR}}$ is injective. This map is obtained by gluing together the maps $\mathcal{O}_K \otimes_{\mathcal{O}_F} \widetilde{\mathbf{A}}^+ \to B_h = \widetilde{\mathbf{B}}^+/(\ker\theta)^h$, so that its kernel is $\cap_{h\geq 1}([\widetilde{\pi}_K] - \pi_K)^h \cdot \mathcal{O}_K \otimes_{\mathcal{O}_F} \widetilde{\mathbf{A}}^+ = \{0\}$.

By definition of $\mathbf{B}^+_{\text{cris}}$ and with the previous lemma, this finishes the proof. □

Proposition 1.7. — *We have $\operatorname{Frac}(\mathbf{B}^+_{\text{cris}})^{\mathcal{G}_K} = F$.*

Proof. — We already know that $F \subset \operatorname{Frac}(\mathbf{B}^+_{\text{cris}})^{\mathcal{G}_K} \subset {\mathbf{B}_{\text{dR}}}^{\mathcal{G}_K} = K$. Let $a/b \in \mathbf{B}^+_{\text{cris}}$ such that $a/b = x \in K$. Then $1 \otimes a - x \otimes b$ is in the kernel of the map $K \otimes_F \mathbf{B}^+_{\text{cris}} \to \mathbf{B}^+_{\text{dR}}$ and thus $1 \otimes a = x \otimes b$ and therefore $x \in F$. □

Definition 1.8. — Let V be a p-adic representation of $\mathcal{G}_K$. We let $\mathbf{D}_{\text{cris}}(V) = (\mathbf{B}_{\text{cris}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$.

We say that V is crystalline if it is $\mathbf{B}_{\text{cris}}$-admissible, that is if $\mathbf{D}_{\text{cris}}(V)$ is an F-vector space whose dimension is the same as $\dim_{\mathbf{Q}_p} V$.

Since $K \otimes_F \mathbf{B}^+_{\text{cris}} \subset \mathbf{B}^+_{\text{dR}}$, we get that $K \otimes_F \mathbf{D}_{\text{cris}}(V) \subset \mathbf{D}_{\text{dR}}(V)$ and thus is endowed with a filtration induced by the one on $\mathbf{D}_{\text{dR}}(V)$. Moreover, since $\mathbf{B}_{\text{cris}}$ is endowed with a Frobenius φ, so is $\mathbf{D}_{\text{cris}}(V)$.

Definition 1.9. — We define the category of filtered φ-modules as follow: A filtered φ-module over K is a F-vector space D equipped with a bijective map $\varphi : D \to D$ which is semilinear relatively to σ_F, the absolute Frobenius on F, and with a decreasing, separated and exhaustive filtration indexed by $\mathbf{Z}$ on $K \otimes_F D$.

One can associate to a filtered φ-module D two polygons: its Hodge polygon $P_H(D)$, coming from the filtration, and its Newton polygon $P_N(D)$, coming from the slopes of φ.

Remark 1.10. — The topology on $\mathbf{B}_{\text{cris}}$ is actually quite bad. Let us see why by following an example given by Colmez [**Col98**, III.2]. The sequence $x_n = \frac{\omega^{p^n-1}}{(p^n-1)!}$ does not tend to 0 in $\mathbf{B}^+_{\text{cris}}$ by construction, but the sequence $\omega x_n = p^n \frac{\omega^{p^n}}{(p^n)!}$ tends to 0, so that tx_n tends to 0 in $\mathbf{B}^+_{\text{cris}}$ and so x_n tends to 0 in $\mathbf{B}_{\text{cris}}$. This is one of the reasons why we usually prefer to work with the ring $\mathbf{B}^+_{\max}$ instead of $\mathbf{B}^+_{\text{cris}}$, which has been defined in the last talk. Just recall that we can define $\mathbf{B}^+_{\max}$ by $\mathbf{B}^+_{\max} := \mathbf{A}_{\max}[1/p]$ and where $\mathbf{A}_{\max}$ is defined as follows:

$$\mathbf{A}_{\max} = \left\{ \begin{array}{c} x \in \mathbf{B}^+_{\text{dR}} \text{ such that } x \text{ can be written as } x = \sum_{n\geq 0} a_n \omega^n / p^n \\ \text{where } a_n \in \widetilde{\mathbf{A}}^+ \text{ tends to 0 for the p-adic topology} \end{array} \right\}.$$

Remark 1.11. — One can check that we have

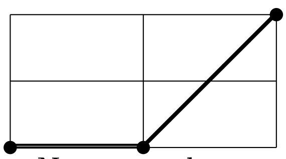
$$\varphi(\mathbf{B}_{\max}) \subset \mathbf{B}_{\text{cris}} \subset \mathbf{B}_{\max}.$$

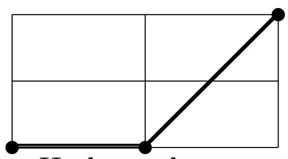
Thus, since the periods of crystalline representations live inside finite dimensional F-vector spaces that are stable by φ, they also live inside $\mathbf{B}_{\max}$ and thus crystalline representations are the same as $\mathbf{B}_{\max}$-admissible representations, and the same as $\widetilde{\mathbf{B}}^+_{\text{rig}}[1/t]$-admissible representations (recall that $\widetilde{\mathbf{B}}^+_{\text{rig}} = \cap_{n=0}^{\infty} \mathbf{B}^+_{\max} = \cap_{n=0}^{\infty} \mathbf{B}^+_{\text{cris}}$).

1.2. Example: elliptic curves. — *The following examples come from* [**Ber04**]. If $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E$, where E is an elliptic curve over F with good ordinary reduction, then $\mathbf{D}_{\text{cris}}(V)$ is a 2-dimensional F-vector space with a basis x, y, and there exists $\lambda \in F$ and $\alpha_0, \beta_0 \in \mathcal{O}_F^*$ depending on E such that:

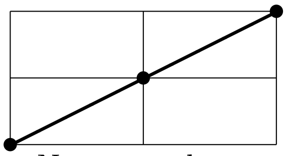
$$\begin{cases} \varphi(x) &= \alpha_0 p^{-1} x \\ \varphi(y) &= \beta_0 y \end{cases} \quad \text{and} \quad \text{Fil}^i \mathbf{D}_{\text{cris}}(V) = \begin{cases} \mathbf{D}_{\text{cris}}(V) & \text{if } i \leq -1 \\ (y + \lambda x) F & \text{if } i = 0 \\ \{0\} & \text{if } i \geq 1 \end{cases}$$

The Newton and Hodge polygons of $\mathbf{D}_{\text{cris}}(V)$ are then as follows:

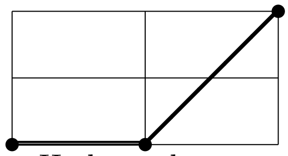

Newton polygon


Hodge polygon

If on the other hand, an elliptic curve E has good supersingular reduction, then the operator $\varphi : \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{cris}}(V)$ is irreducible and the Newton and Hodge polygons are as follows:



Newton polygon



Hodge polygon

1.3. The case of bad semi-stable reduction. — *In what follows we consider the case where $K = F$ for simplicity.* Let q be a formal parameter and define

$$s_k(q) = \sum_{n=1}^{+\infty} \frac{n^k q^n}{1-q^n} \qquad a_4(q) = -s_3(q) \qquad a_6(q) = -\frac{5s_3(q)+7s_5(q)}{12}$$

$$x(q,v) = \sum_{n=-\infty}^{+\infty} \frac{q^n v}{(1-q^n v)^2} - 2s_1(q) \qquad y(q,v) = \sum_{n=-\infty}^{+\infty} \frac{(q^n v)^2}{(1-q^n v)^3} + s_1(q).$$

All those series are convergent if $q \in p\mathcal{O}_F$ and $v \notin q^{\mathbf{Z}} = \langle q \rangle$ (the multiplicative subgroup of $F^\times$ generated by q). For such $q \neq 0$, let E_q be the elliptic curve defined by the equation $y^2 + xy = x^3 + a_4(q)x + a_6(q)$. The theorem of Tate is then: the elliptic curve E_q is defined over F, it has bad semi-stable reduction, and E_q is uniformized by $\overline{F}^\times$, that is, there exists a map $\alpha : \overline{F}^\times \to E_q(\overline{F})$, given by

$$v \mapsto \begin{cases} (x(q,v), y(q,v)) & \text{if } v \notin q^{\mathbf{Z}} \\ 0 & \text{if } v \in q^{\mathbf{Z}} \end{cases}$$

which induces an isomorphism of groups with $\mathcal{G}_F$-action $\overline{F}^\times/\langle q \rangle \to E_q(\overline{F})$.

Furthermore, if E is an elliptic curve over F with bad semi-stable reduction, then there exists q such that E is isomorphic to E_q over F.

1.3.1. *The p-adic representation attached to E_q.* — Using Tate's theorem, we can give an explicit description of $T_p(E_q)$. Let $\widetilde{q} \in \widetilde{\mathbf{E}}^+$ be such that $q^{(0)} = q$. Then α induces isomorphisms

$$\begin{array}{ccc} \overline{F}^\times/\langle q \rangle & \longrightarrow & E_q(\overline{F}) \\ \{x \in \overline{F}^\times/\langle q \rangle, x^{p^n} \in \langle q \rangle\} & \longrightarrow & E_q(\overline{F})[p^n] \end{array}$$

and one sees that $\{x \in \overline{F}^\times/\langle q \rangle, x^{p^n} \in \langle q \rangle\} = \{(\varepsilon^{(n)})^i (q^{(n)})^j, 0 \leq i,j < p^n - 1\}$. The elements $\varepsilon^{(n)}$ and $q^{(n)}$ therefore form a basis of $E_q(\overline{F})[p^n]$, so that a basis of $T_p(E_q)$ is given by $e = \varprojlim_n \varepsilon^{(n)}$ and $f = \varprojlim_n q^{(n)}$. This makes it possible to compute explicitly the Galois action on $T_p(E_q)$. We have $g(e) = \varprojlim_n g(\varepsilon^{(n)}) = \chi(g)e$ and $g(f) = \varprojlim_n g(q^{(n)}) = \varprojlim_n q^{(n)}(\varepsilon^{(n)})^{c(g)} = f + c(g)e$ where $c(g)$ is some p-adic integer, determined by the fact that $g(q^{(n)}) = q^{(n)}(\varepsilon^{(n)})^{c(g)}$. Note that $[g \mapsto c(g)] \in H^1(F, \mathbf{Z}_p(1))$. The matrix of g in the basis (e, f) is therefore given by

$$\begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}$$

1.3.2. *p-adic periods of E_q.* — We are looking for p-adic periods of $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(E_q)$ which live in $\mathbf{B}_{\mathrm{dR}}$, that is for elements of $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_F}$. An obvious candidate is $t^{-1} \otimes e$ since $g(t) = \chi(g)t$ and $g(e) = \chi(g)e$. Let us look for a second element of $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_F}$, of the form $a \otimes e + 1 \otimes f$. We see that this element will be fixed by $\mathcal{G}_F$ if and only if $g(a)\chi(g) + c(g) = a$.

Let $\widetilde{q}$ be the element of $\widetilde{\mathbf{E}}^+$ defined by $\widetilde{q} = (q^{(0)}, q^{(1)}, \cdots)$. Observe that we have $g(\widetilde{q}) = (g(q^{(0)}), g(q^{(1)}), \cdots) = \widetilde{q}\varepsilon^{c(g)}$, and that $\theta([\widetilde{q}]/q^{(0)} - 1) = 0$, so that $[\widetilde{q}]/q^{(0)} - 1$ is small in the $\ker(\theta)$-adic topology. The series

$$\log_p(q^{(0)}) - \sum_{n=1}^{+\infty} \frac{(1 - [\widetilde{q}]/q^{(0)})^n}{n}$$

therefore converges in $\mathbf{B}_{\mathrm{dR}}^+$ to an element which we call u. One should think of u as being $u = \log([\widetilde{q}])$. In particular, $g(u) = g(\log([\widetilde{q}])) = \log([g(\widetilde{q})]) = \log([\widetilde{q}]) + c(g)\log([\varepsilon]) = u + c(g)t$, and we see that $a = -u/t$ satisfies the equation $g(a)\chi(g) + c(g) = a$. A basis of $\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_F}$ is therefore given by

$$\begin{cases} x = t^{-1} \otimes e \\ y = -ut^{-1} \otimes e + 1 \otimes f \end{cases}$$

and this shows that $T_p(E_q)$ is $\mathbf{B}_{\mathrm{dR}}$-admissible. Furthermore, one sees that $\theta(u - \log_p(q^{(0)})) = 0$, so that $u - \log_p(q^{(0)})$ is divisible by t and

$$\mathrm{Fil}^i \mathbf{D}_{\mathrm{dR}}(V) = \begin{cases} \mathbf{D}_{\mathrm{dR}}(V) & \text{if } i \leq -1 \\ (y + \log_p(q^{(0)})x)F & \text{if } i = 0 \\ \{0\} & \text{if } i \geq 1 \end{cases}$$

We will now see why $\mathbf{B}_{\mathrm{cris}}$ is not sufficient in order to take into account the representations arising from elliptic curves with bad semi-stable reduction. First, we will define a log map : $x \in \widetilde{\mathbf{E}}^\times \mapsto \log[x] \in \mathbf{B}_{\mathrm{dR}}^+$.

Proposition 1.12. — *There exists a unique map* $\log : x \mapsto \log[x]$ *from* $\widetilde{\mathbf{E}}^\times$ *to* $\mathbf{B}_{\mathrm{dR}}^+$ *satisfying* $\log[xy] = \log[x] + \log[y]$, $\log[x] = 0$ *if* $x \in \overline{\mathbf{F}}_p$,

$$\log[x] = \sum_{n>0} (-1)^{n+1} \frac{([x] - 1)^n}{n}$$

if $v_p(x^{(0)} - 1) \geq 1$ *and such that*

$$\log[\widetilde{p}] = \sum_{i=1}^{+\infty} (-1)^{n+1} \frac{(\frac{[\widetilde{p}]}{p} - 1)^n}{n}.$$

Proof. — If $x \in \widetilde{\mathbf{E}}$ with $v_{\mathbf{E}}(x-1) \geq 1$, then we set
$$\log[x] = \sum_{n>0} (-1)^{n+1} \frac{([x]-1)^n}{n}.$$
The series converges in $\mathbf{A}_{\mathrm{cris}}$ since then $\theta([x]-1) = x^{(0)} - 1$, and thus $\frac{([x]-1)^n}{n!} \in \mathbf{A}_{\mathrm{cris}}$ and
$$\log[x] = \sum_{n>0} (-1)^{n+1} (n-1)! \frac{([x]-1)^n}{n!}$$
which converges in $\mathbf{A}_{\mathrm{cris}}$ since $(n-1)! \to 0$ when $n \to +\infty$.

This map extends uniquely to a map from $1 + \mathfrak{m}_{\widetilde{\mathbf{E}}}$ to $\mathbf{B}^+_{\mathrm{cris}}$ by setting
$$\log[x] := \frac{1}{p^m} \log[x^{p^m}]$$
for m such that $v_{\mathbf{E}}(x^{p^m} - 1) \geq 1$, this value being independant of the choice of such m, and we check that if $x \in \overline{\mathbf{F}}_p^\times$, then $\log[x] = 0$.

If $x \in (\widetilde{\mathbf{E}}^+)^\times$, one can write $x = x_0 y$ with $x_0 \in \overline{\mathbf{F}}_p^\times$ and $y \in 1 + \mathfrak{m}_{\widetilde{\mathbf{E}}}$, and we put
$$\log[x] := \log[y].$$
If $x \in \widetilde{\mathbf{E}}^\times$ with $v_{\mathbf{E}}(x) = \frac{r}{s}$, $r, s \in \mathbf{Z}$ and $s \geq 1$, then $\frac{x^s}{\widetilde{p}^r} = y \in (\widetilde{\mathbf{E}}^+)^\times$, so that the relation
$$\log\left(\frac{[x^s]}{[\widetilde{p}]^r}\right) = \log[y] = s\log[x] - r\log[\widetilde{p}]$$
gives us
$$\log[x] = \frac{1}{s}(r\log[\widetilde{p}] + \log[y])$$
and thus it suffices in order to define $\log[x]$ to define $\log[\widetilde{p}]$. It thus suffices to check that
$$\log[\widetilde{p}] = \sum_{i=0}^{+\infty} (-1)^{n+1} \frac{(\frac{[\widetilde{p}]}{p} - 1)^n}{n}$$
converges in $\mathbf{B}^+_{\mathrm{dR}}$, which is indeed the case since $\theta(\frac{[\widetilde{p}]}{p} - 1) = 0$. $\square$

Remark 1.13. — In the above proof, we actually showed that if $x \in (\widetilde{\mathbf{E}}^+)^\times$, then $\log[x] \in \mathbf{B}^+_{\mathrm{cris}}$.

Proposition 1.14. — *The element* $\log[\widetilde{p}]$ *is transcendental over* $\mathrm{Frac}(\mathbf{B}_{\mathrm{cris}})$.

Proof. — The hardest part is to show that $\log[\widetilde{p}] \notin \mathrm{Frac}(\mathbf{B}_{\mathrm{cris}})$. We assume for now that $\log[\widetilde{p}] \notin \mathrm{Frac}(\mathbf{B}_{\mathrm{cris}})$.

Let $X^d + u_{d-1}X^{d-1} + \ldots + u_0$ be the minimal polynomial of $\log[\widetilde{p}]$ over $\mathrm{Frac}(\mathbf{B}_{\mathrm{cris}})$. By applying g and comparing the coefficients we obtain $g(u_{d-1}) = u_{d-1} + dc(g)t$ so that $u_{d-1} - d\log[\widetilde{p}]$ can be seen as an element of $\mathbf{B}_{\mathrm{dR}}$ stable by $\mathcal{G}_F$ and thus $\log[\widetilde{p}] = d^{-1}(u_{d-1} - c)$ for some $c \in F$, which is impossible.

Let us now see why $\log[\widetilde{p}] \notin \mathrm{Frac}(\mathbf{B}_{\mathrm{cris}})$. Here we follow Fontaine's original proof [**Fon94**, 4.3.2]. Let $\xi = p - [\widetilde{p}]$ and let $\beta = \xi/p$. Then both β and ξ are in $\mathrm{Fil}^1\mathbf{B}_{\mathrm{dR}}$ but not in $\mathrm{Fil}^2\mathbf{B}_{\mathrm{dR}}$. Let $S = \widetilde{\mathbf{A}}^+[\![\beta]\!] \subset \mathbf{B}^+_{\mathrm{dR}}$. For every $i \in \mathbf{N}$, let $\mathrm{Fil}^i S = S \cap \mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}}$. Then $\mathrm{Fil}^i S$ is the principal ideal of S generated by β^i. Let

$$\theta^i : \mathrm{Fil}^i S \to \mathcal{O}_{\mathbf{C}_p}$$

be the map sending $\beta^i\alpha$ to $\theta(\alpha)$. We then have $\theta^i\mathrm{Fil}^i S = \mathcal{O}_{\mathbf{C}_p}$. By construction, we have $\mathbf{A}_{\mathrm{cris}} \subset S$ and thus $\mathrm{Frac}(\mathbf{B}_{\mathrm{cris}}) = \mathrm{Frac}(\mathbf{A}_{\mathrm{cris}}) \subset \mathrm{Frac}(S)$. We will now show that if $\alpha \in S$ is not zero, then $\alpha \log[\widetilde{p}] \notin S$ which will conclude the proof.

Since S is separated for the p-adic topology, it suffices to show that if $r \in \mathbf{N}$ and $\alpha \in S \setminus pS$ then $p^r\alpha\log[\widetilde{p}] \notin S$. If $a \in \widetilde{\mathbf{A}}^+$ satisfies $\theta(a) \in p\mathcal{O}_{\mathbf{C}_p}$ then $a \in (p,\xi)\widetilde{\mathbf{A}}^+$ and hence $a \in pS$. Therefore one can find $i \geq 0$ and a sequence (b_n) of elements of $\widetilde{\mathbf{A}}^+$ such that the $\theta(b_i) \notin p\mathcal{O}_{\mathbf{C}_p}$ and

$$\alpha = p(\sum_{0 \leq n < i} b_n\beta^n) + \sum_{n \geq i} b_n\beta^n.$$

Recall that we have $\log[\widetilde{p}] = -\sum_{n\geq 1}\beta^n/n$. Suppose $j > r$ is an integer such that $p^j > i$. If $p^r\alpha\log[\widetilde{p}] \in S$, then we have $\alpha\cdot\sum_{n>0} p^{j-1}\beta^n/n \in S$. Note also that $\alpha\cdot\sum_{0<n<p^j} p^{j-1}\beta^n/n \in S$ so that $\alpha\cdot\beta^{p^j}/p \in S + \mathrm{Fil}^{2p^j}\mathbf{B}_{\mathrm{dR}}$.

Thus we get that $b_i\beta^{i+p^j}/p \in S + \mathrm{Fil}^{i+p^j+1}\mathbf{B}_{\mathrm{dR}} \cap \mathrm{Fil}^{i+p^j}\mathbf{B}_{\mathrm{dR}} = \mathrm{Fil}^{i+p^j}S + \mathrm{Fil}^{i+p^j+1}\mathbf{B}_{\mathrm{dR}}$.

But we have $\theta^{i+p^j}(\mathrm{Fil}^{i+p^j}S + \mathrm{Fil}^{i+p^j+1}\mathbf{B}_{\mathrm{dR}}) = \mathcal{O}_{\mathbf{C}_p}$ and $\theta^{i+p^j}(b_i\beta^{i+p^j}/p) = \theta(b_i)/p \notin \mathcal{O}_{\mathbf{C}_p}$ and thus we get a contradiction. □

In particular, when E is an elliptic curve with bad semi-stable reduction, then the representation coming from its Tate module is not crystalline. This makes us define a new ring of periods:

Definition 1.15. — We define $\mathbf{B}_{\mathrm{st}} := \mathbf{B}_{\mathrm{cris}}[\log[\widetilde{p}]]$ as the sub-$\mathbf{B}_{\mathrm{cris}}$-algebra of $\mathbf{B}_{\mathrm{dR}}$ generated by $\log[\widetilde{p}]$, and $\mathbf{B}^+_{\mathrm{st}} := \mathbf{B}^+_{\mathrm{cris}}[\log[\widetilde{p}]]$ as the sub-$\mathbf{B}^+_{\mathrm{cris}}$-algebra of $\mathbf{B}^+_{\mathrm{dR}}$ generated by $\log[\widetilde{p}]$.

The Frobenius naturally extends to $\mathbf{B}_{\mathrm{st}}$ by $\varphi(\log[\widetilde{p}]) = p\cdot\log[\widetilde{p}]$.

Note that there exists an element $\log(\pi) \in \mathbf{B}_{\mathrm{st}}$ and that $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{cris}}[\log(\pi)]$. We endow $\mathbf{B}_{\mathrm{st}}$ with a monodromy operator N defined by

$$N\left(\sum_{k=0}^{d} a_k\log(\pi)^k\right) = -\sum_{k=1}^{d} ka_k\log(\pi)^{k-1}.$$

Proposition 1.16. — *One has:*

1. *the natural map* $K \otimes_F \mathbf{B}^+_{st} \to \mathbf{B}^+_{\mathrm{dR}}$ *is injective;*
2. $\mathbf{B}_{\mathrm{st}}^{\mathcal{G}_K} = F$*;*
3. *the operator* N *commutes with* $\mathcal{G}_F$ *and satisfies* $N\varphi = p\varphi N$*;*
4. $\mathbf{B}_{\mathrm{st}}^{N=0} = \mathbf{B}_{\mathrm{cris}}$.

Proof. — The last two items are just straightforward computations. For the first point, note that $\mathrm{Frac}(K \otimes_F \mathbf{B}_{\mathrm{cris}})$ is a finite extension of $\mathrm{Frac}(\mathbf{B}_{\mathrm{cris}})$ and thus $\log[\tilde{p}]$ is transcendental over $\mathrm{Frac}(K \otimes_F \mathbf{B}_{\mathrm{cris}})$. Therefore we have

$$K \otimes_F \mathbf{B}_{\mathrm{st}} = (K \otimes_F \mathbf{B}_{\mathrm{cris}})[\log[\tilde{p}]]$$

and this proves the first point.

For the second point, we already know that $F \subset (\mathbf{B}_{\mathrm{st}})^{G_K} \subset \mathrm{Frac}(\mathbf{B}_{\mathrm{st}})^{G_K} \subset \mathbf{B}_{\mathrm{dR}}{}^{\mathcal{G}_K} = K$ and the result now follows from the first point. □

Definition 1.17. — Let V be a p-adic representation of $\mathcal{G}_K$. We let $\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{Q}_p} V)^{\mathcal{G}_K}$.

We say that V is semi-stable if it is $\mathbf{B}_{\mathrm{st}}$-admissible, that is if $\mathbf{D}_{\mathrm{st}}(V)$ is an F-vector space whose dimension is the same as $\dim_{\mathbf{Q}_p} V$.

Remark 1.18. — Since we have the $\mathcal{G}_K$-equivariant inclusions $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{st}} \subset \mathbf{B}_{\mathrm{dR}}$, it follows that crystalline representations are semi-stable, and that semi-stable representations are de Rham.

As in the crystalline case, since $K \otimes_F \mathbf{B}_{\mathrm{st}}^+ \subset \mathbf{B}_{\mathrm{dR}}^+$, we get that $K \otimes_F \mathbf{D}_{\mathrm{st}}(V) \subset \mathbf{D}_{\mathrm{dR}}(V)$ and thus is endowed with a filtration induced by the one on $\mathbf{D}_{\mathrm{dR}}(V)$. Moreover, since $\mathbf{B}_{\mathrm{st}}$ is endowed with a Frobenius φ and a monodromy map N, so is $\mathbf{D}_{\mathrm{st}}(V)$.

Definition 1.19. — We define the category of filtered (φ, N)-modules over K as follow: A filtered (φ, N)-module over K is a finite dimensional F-vector space D equipped with two maps $\varphi, N : D \to D$ satsfying the following properties:

1. φ is bijective and semilinear with respect to σ_F;
2. N is a F-linear map;
3. $N\varphi = p\varphi N$;

and also equipped with a decreasing, separated and exhaustive filtration indexed by $\mathbf{Z}$ on $K \otimes_F D$.

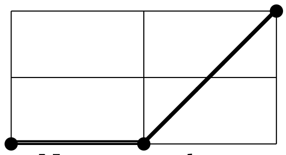
Remark 1.20. — The relation between N and φ in the definition implies that N is nilpotent.

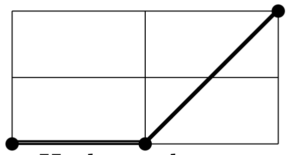
Similarly to the crystalline case, one can also associate to a filtered (φ, N)-module D two polygons: its Hodge polygon $P_H(D)$, coming from the filtration, and its Newton polygon $P_N(D)$, coming from the slopes of φ.

It follows from the definition of $\mathbf{D}_{\mathrm{st}}(V)$ that $\mathbf{D}_{\mathrm{st}}(V)$ is a filtered (φ, N)-module over K. We now go back to the example of Tate's elliptic curve. The computatons we made showed that a basis of $\mathbf{D}_{\mathrm{dR}}(V)$ was given by

$$\begin{cases} x = t^{-1} \otimes e \\ y = -\log[\tilde{p}]t^{-1} \otimes e + 1 \otimes f \end{cases}$$

and since both t^{-1} and $\log[\tilde{p}]$ belong to $\mathbf{B}_{\mathrm{st}}$, this shows that (x,y) is also a basis of $\mathbf{D}_{\mathrm{st}}(V)$. We already computed the filtration on $\mathbf{D}_{\mathrm{dR}}(V)$ above, so it remains to compute φ and N on $\mathbf{D}_{\mathrm{st}}(V)$. A direct computation gives $N(x)=0$ and $N(y)=x$. Note that since $\varphi(\log[\tilde{p}]) = p\cdot\log[\tilde{p}]$, we have $\varphi(x)=p^{-1}x$ and $\varphi(y)=y$. Thus the Newton and Hodge polygons of $\mathbf{D}_{\mathrm{st}}(V)$ are as follow:


Newton polygon


Hodge polygon

Proposition 1.21. — *If $y \in \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ is such that $\varphi^{n}(y) \in t\mathbf{B}^{+}_{\mathrm{dR}}$ for all $n \in \mathbf{Z}$ then $y \in t\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$.*

Proof. — If $y \in \widetilde{\mathbf{A}}^{[0,r_0]} = \mathbf{A}_{\max}$, recall that one can write $y=\sum_{j\geq 0} a_j([\tilde{p}]/p)^j$ with $a_j \in \widetilde{\mathbf{A}}^{+}$ and $a_j \to 0$. Recall that $\omega = \pi/\varphi^{-1}(\pi)$ is a generator of $\ker(\theta : \widetilde{\mathbf{A}}^{+} \to \mathcal{O}_{\mathbf{C}_p})$. Thus $\omega/([\tilde{p}]-p)$ is a unit of $\widetilde{\mathbf{A}}^{+}$ and we can write y as $y=\sum_{j\geq 0} y_j(\omega/p-1)^j$ (to see this, note that $\widetilde{\mathbf{A}}^{[0,r_0]}$ is the p-adic completion of $\widetilde{\mathbf{A}}^{+}[[\tilde{p}]/p] = \widetilde{\mathbf{A}}^{+}[[\tilde{p}]/p-1] = \widetilde{\mathbf{A}}^{+}[\omega/p] = \widetilde{\mathbf{A}}^{+}[\omega/p-1]$). If $Q(\pi) = ((1+\pi)^p-1)/\pi = \varphi(\omega)$, then $\varphi(y) = \sum_{j\geq 0}\varphi(y_j)(Q(\pi)/p-1)^j$ in $\widetilde{\mathbf{A}}^{[0,r_1]}$. Since $Q(\pi)/p-1$ is a multiple of π, this means that we have $\varphi(y)=\varphi(y_0)+\pi z$ in $\widetilde{\mathbf{B}}^{[0,r_1]}$ with $y_0 \in \widetilde{\mathbf{A}}^{+}$.

By assumption, we have $\theta\circ\varphi^{n}(y)=0$ for all $n\geq 1$. Since $\theta\circ\varphi^{-k}(\omega)\neq 0$ for all $k\neq 0$, we get that $y \in \pi/\varphi^{-k}(\pi)\widetilde{\mathbf{A}}^{+}$ for all $k \geq 1$ (because it is then divisible by $\omega, \varphi^{-1}(\omega), \ldots, \varphi^{-k}(\omega)$ and thus by their product). Let I be the ideal of elements of $\widetilde{\mathbf{A}}^{+}$ such that $\theta\circ\varphi^{n}(x)=0$ for all $n\geq 0$ so that $I=\cap_{k\geq 0}\pi/\varphi^{-k}(\pi)\widetilde{\mathbf{A}}^{+}$. In $\widetilde{\mathbf{E}}^{+}$ we have $\overline{I}=\overline{\pi}\widetilde{\mathbf{E}}^{+}$ so that $I=\pi\widetilde{\mathbf{A}}^{+}+p(\widetilde{\mathbf{A}}^{+}\cap I)$ and thus $I=\pi\widetilde{\mathbf{A}}^{+}$. This implies that $\varphi(y_0)\in\pi\widetilde{\mathbf{A}}^{+}$ and hence that $\varphi(y)\in\pi\widetilde{\mathbf{B}}^{[0,r_1]}$ so that $y\in\varphi^{-1}(\pi)\widetilde{\mathbf{B}}^{[0,r_0]}$. Since in addition $\theta(y)=0$, we have $y \in \pi\widetilde{\mathbf{B}}^{[0,r_0]}$. Since t/π is a unit of $\widetilde{\mathbf{B}}^{[0,r_0]}$, we have $y\in t\widetilde{\mathbf{B}}^{[0,r_0]}$.

Since $\varphi^{k} : \widetilde{\mathbf{B}}^{[0,r_0]} \to \widetilde{\mathbf{B}}^{[0,r_k]}$ is a bijection for $k\geq 0$, we see that if $y\in\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ and $\theta\circ\varphi^{n}(y)=0$ for all $n\geq -k$, then $y=t\cdot z_k$ with $z_k\in\widetilde{\mathbf{B}}^{[0,r_k]}$. By construction the sequence $\{z_k\}$ defines an element z of $\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ that satisfies $y=tz$ and thus this proves the proposition. □

Proposition 1.22. — *We have $\mathbf{B}^{\varphi=1}_{\mathrm{cris}} \cap \mathrm{Fil}^{0}\mathbf{B}_{\mathrm{dR}} = \mathbf{Q}_p$.*

Proof. — If $y \in \mathbf{B}^{\varphi=1}_{\mathrm{cris}}$ then a direct computation shows that $y \in (\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}[1/t])^{\varphi=1}$. Let $y\in\mathbf{B}^{\varphi=1}_{\mathrm{cris}}\cap\mathrm{Fil}^{0}\mathbf{B}_{\mathrm{dR}}$ and let $k\geq 1$ be such that $y\in t^{-k}\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$. We have $\varphi^{n}(t^{k}y)=p^{-kn}t^{k}y\in t\mathbf{B}^{+}_{\mathrm{dR}}$ so that by the proposition above we have $t^{k}y\in t\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$ so that by induction we get $y\in\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$. The result then follows from the fact that $(\widetilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{\varphi=1}=\mathbf{Q}_p$. □

We have $\widehat{\mathbf{Q}_p^{\mathrm{unr}}} = W(\overline{\mathbf{F}}_p)[1/p] \subset \widetilde{\mathbf{B}}^{+} \subset \widetilde{\mathbf{B}}^{+}_{\mathrm{rig}}$. If $\lambda_0 \in W(\overline{\mathbf{F}}_p)^{\times}$ then there exists $\mu \in W(\overline{\mathbf{F}}_p)^{\times}$ such that $\lambda_0=\mu/\varphi(\mu)$ (to see this recall that if V is a φ-module then $1-\varphi : V \to V$ is surjective). Every $\lambda \in \widehat{\mathbf{Q}_p^{\mathrm{unr}}}$ can then be written as $\lambda = p^{n}\mu/\varphi(\mu)$.

The map $\mathbf{B}_{\mathrm{cris}}^{\varphi=\lambda} \to \mathbf{B}_{\mathrm{cris}}^{\varphi=p^n}$ given by $y \mapsto \mu y$ is then a bijection which respects the filtration induced by $\mathbf{B}_{\mathrm{dR}}$.

Proposition 1.23. — *If $\lambda \in \widehat{\mathbf{Q}_p^{\mathrm{unr}}}$ and $n = v_p(\lambda)$ then $\mathbf{B}_{\mathrm{cris}}^{\varphi=\lambda} \cap \mathrm{Fil}^{n+1}\mathbf{B}_{\mathrm{dR}} = \{0\}$.*

Proof. — If $y \in \mathbf{B}_{\mathrm{cris}}^{\varphi=\lambda} \cap \mathrm{Fil}^{n+1}\mathbf{B}_{\mathrm{dR}}$ and if μ is as above, then $t^{-n}\mu y \in \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \cap \mathrm{Fil}^{1}\mathbf{B}_{\mathrm{dR}}$ so that it is equal to 0. □

Proposition 1.24. — *The functors $V \mapsto \mathbf{D}_{\mathrm{cris}}(V)$ and $V \mapsto \mathbf{D}_{\mathrm{st}}(V)$ are fully faithful from the categories of crystalline and semistable representations of $\mathcal{G}_K$ to the categories of filtered φ-modules over K and filtered (φ, N)-modules over K.*

Proof. — The proposition above shows that $\mathbf{B}_{\mathrm{cris}}^{\varphi=1} \cap \mathrm{Fil}^{0}\mathbf{B}_{\mathrm{dR}} = \mathbf{Q}_p$, so that if V is crystalline then $V = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)) \cap (\mathbf{B}_{\mathrm{cris}} \otimes_F \mathbf{D}_{\mathrm{cris}}(V))^{\varphi=1}$. When V is semi-stable, we have $V = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_K \mathbf{D}_{\mathrm{dR}}(V)) \cap (\mathbf{B}_{\mathrm{st}} \otimes_F \mathbf{D}_{\mathrm{st}}(V))^{\varphi=1,N=0}$. In both cases, we can recover V from either $\mathbf{D}_{\mathrm{cris}}(V)$ or $\mathbf{D}_{\mathrm{st}}(V)$. □

If $\eta : \mathcal{G}_K \to \mathbf{Z}_p^\times$ is a character, then we say that it is crystalline (resp. semi-stable, resp. de Rham) if the associated representation is. Note that this is the case if and only if there exists $y \in \mathbf{B}_{\mathrm{cris}}$ (resp. $\in \mathbf{B}_{\mathrm{st}}$, resp. $\in \mathbf{B}_{\mathrm{dR}}$) such that $\eta(g) = y/g(y)$.

Theorem 1.25. — *If $\eta : \mathcal{G}_K \to \mathbf{Z}_p^\times$ is a character, then it is de Rham if and only if $\eta = \mu \cdot \chi^h$ where μ is potentially unramified and $h \in \mathbf{Z}$.*

Proof. — The character η is de Rham if and only if there exists $y \in \mathbf{B}_{\mathrm{dR}}^\times$ such that $\eta(g) = y/g(y)$. Let h be such that $y = y_0 t^{-h}$ with $y_0 \in (\mathbf{B}_{\mathrm{dR}}^+)^\times$. The character $\mu = \chi^{-h}\eta$ then satisfies $\mu(g) = \theta(y_0)/g(\theta(y_0))$ and is then potentially unramified by Sen theory (it is $\mathbf{C}_p$-admissible). □

Theorem 1.26. — *If $\eta : \mathcal{G}_K \to \mathbf{Z}_p^\times$ is a character, then it is crystalline if and only if it is semi-stable, and this occurs if and only if $\eta = \mu \cdot \chi^h$ where μ is unramified and $h \in \mathbf{Z}$.*

Proof. — The first part follows from the fact that N is nilpotent on a vector space of dimension 1 and thus $N = 0$ and $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}_{\mathrm{st}}(V)$.

If η is of the form $\eta = \mu \cdot \chi^h$ where μ is unramified and $h \in \mathbf{Z}$ then there exists $y_0 \in W(\overline{\mathbf{F}}_p)^\times$ such that $y_0/\varphi(y_0) = \mu(\mathrm{Frob}_p)$ and thus we have $\eta(g) = y/g(y)$ with $y = y_0 t^{-h} \in \mathbf{B}_{\mathrm{cris}}$.

Assume now that $\eta : \mathcal{G}_K \to \mathbf{Z}_p^\times$ is crystalline. Then it is de Rham and thus of the form $\eta = \mu \cdot \chi^h$ where μ is potentially unramified and $h \in \mathbf{Z}$. Let L be a finite extension of K such that $\mu_{|L}$ is unramified, so that there exists $\lambda_0 \in W(\overline{\mathbf{F}}_p)^\times$ such that $\lambda_0/\varphi(\lambda_0) = \mu(\mathrm{Frob}_p)$ for $g \in \mathcal{G}_L$. Since μ is crystalline, there exists $y_0 \in \mathbf{B}_{\mathrm{cris}}$ such that $\mu(g) = y_0/g(y_0)$ if $g \in \mathcal{G}_K$ and thus $y_0/\lambda_0 \in \mathbf{B}_{\mathrm{cris}}^{\mathcal{G}_L} = L \cap \mathbf{Q}_p^{\mathrm{unr}}$ and thus μ is unramified. □

We say that a filtered (φ, N)-module D is weakly admissible if for every subobject D' of D, the Hodge polygon of D' lies below the Newton polygon of D', and the endpoints of the Hodge and Newton polygons of D are the same. Note that, in the cases of representations arising from elliptic curves, this was always the case. If D is of dimension 1, then there exists a well defined $h \in \mathbf{Z}$ such that $\mathrm{Fil}^h D = D$ and $\mathrm{Fil}^{h+1} D = \{0\}$ and we set $t_H(D) = h$, and if $\varphi(d) = \lambda \cdot d$ with $d \in D$ then $v_p(\lambda)$ does not depend of the choice of $d \neq 0$ and we let $t_N(D) = v_p(\lambda)$. If $\dim(D) > 1$, we set $t_H(D) = t_H(\det D)$ and $t_N(D) = t_N(\det D)$. Note that the terminal point of the Hodge polygon of D is $(\dim D, t_H(D))$ and the terminal point of the Newton polygon of D is $(\dim D, t_N(D))$. One can check that weakly admissibility is the same as asking that $t_H(D) = t_N(D)$ and for every subobject D' of D one has $t_N(D') \geq t_H(D')$.

Proposition 1.27. — *If V is a semi-stable representation, then $\mathbf{D}_{\mathrm{st}}(V)$ is weakly admissible.*

Proof. — Let $D = \mathbf{D}_{\mathrm{st}}(V)$. The fact that $t_H(D) = t_N(D)$ comes from the theorem above on semi-stable characters applied to $\det(V)$.

If D' is a sub-object of rank r of D then we can replace V by $\Lambda^r V$ so that D' is of dimension 1. In this case, $N = 0$ on D' and $D' \subset \mathbf{D}_{\mathrm{cris}}(V)$ is a line so that $D' \subset (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{\varphi=\lambda}$ where $t_N(D) = v_p(\lambda)$ and the proposition follows from the fact that $\mathbf{B}_{\mathrm{cris}}^{\varphi=\lambda} \cap \mathrm{Fil}^{n+1} \mathbf{B}_{\mathrm{dR}} = \{0\}$. □

We say that a filtered (φ, N)-module D is admissible if there exists a p-adic representation V such that $D = D_{\mathrm{st}}(V)$. We should see in one of the following lectures that weakly admissible filtered (φ, N)-modules are actually admissible.

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