Perfectoid Algebras - Tilting equivalence and almost purity

Definition 1. *A complete topological field K with topology induced by a nondiscrete valuation of rank 1 is called* **perfectoid** *if the Frobenius morphism $\Phi : K^\circ/p \to K^\circ/p$ is surjective.*

Example 2. *The completions of $\mathbb{Q}_p(p^{1/p^\infty})$, $\mathbb{Q}_p(\mu_{p^\infty})$, $\overline{\mathbb{Q}_p}$, $\mathbb{F}_p((t))(t^{1/p^\infty})$ and $\overline{\mathbb{F}_p((t))}$ are perfectoid fields. Perfectoid fields of characteristic p are just perfect complete nonarchimedean fields.*

Let K be a perfectoid field of characteristic 0. We can associate to it a characteristic p field $K^\flat$, called the **tilt** of K, by setting

$$K^\flat := (\varprojlim_\Phi K^\circ/\varpi)[1/\varpi],$$

for any $\varpi \in K$, $|p| \leq |\varpi| < 1$.

Example 3. Let $K = \mathbb{Q}_p(p^{1/p^\infty})^\wedge$ and $K^\flat = \mathbb{F}_p((t^{1/p^\infty}))^\wedge$ then we get

$$K^\circ/p = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge/p = \mathbb{F}_p[t^{1/p^\infty}]^\wedge/t = K^{\flat\circ}/t.$$

Under this map t is identified with $(p, p^{1/p}, \ldots)$.
This works similarly for the cyclotomic extension $K = \mathbb{Q}_p(\mu_{p^\infty})^\wedge$. We choose a compatible sequence of p-power roots of unity ε then $K^\flat = \mathbb{F}_p((t))(t^{1/p^\infty})^\wedge$ where $t \mapsto \varepsilon - 1$.

Proposition 4. *As a multiplicative monoid we can identify $K^\flat$ with $\varprojlim_{x\mapsto x^p} K$.*

Proof. We construct a multiplicative map $\varprojlim_\Phi K^\circ/\varpi \to K^\circ$, $x \mapsto x^\sharp$ as follows: let $x = (\overline{x}_0, \overline{x}_1, \ldots) \in \varprojlim_\Phi K^\circ/\varpi$ then, taking any lifts $x_n \in K^\circ$, we set $x^\sharp = \varinjlim x_n^{p^n}$ which is independent of the choice of lifts. This follows from the fact, that for two lifts x_n, x'_n we have $\varpi \mid x_n - x'_n$ and thus $\varpi^{n+1} \mid x_n^{p^n} - x_n'^{p^n}$. The multiplicative map $\varprojlim_\Phi K^\circ/\varpi \to \varprojlim_{x\mapsto x^p} K^\circ$ given by $x \mapsto (x^\sharp, (x^{1/p})^\sharp, \ldots)$ is bijective with the inverse being the projection map. □

We can also use this map to provide $K^\flat$ with a norm $|x|_{K^\flat} := |x^\sharp|_K$ for $x \in K^\flat$. The following theorem is Scholze's generalization of the Fontaine-Wintenberger theorem:

Theorem 5. *Let K be a perfectoid field and let L be a finite field extension of K then L is a perfectoid field. Moreover, the tiliting functor $L \mapsto L^\flat$ induces an equivalence of categories between finite field extensions of K and $K^\flat$.*

Remark. Note that the fact that $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$ and $\mathbb{Q}_p(\mu_{p^\infty})^\wedge$ have isomorphic tilts does not contradict the theorem since the base field $\mathbb{Q}_p$ is not perfectoid.

In the following we give a brief overview of almost mathematics. The main reference for this part is the book of Gabber and Ramero on "Almost ring theory".
Let M be a K°-module and $\mathfrak{m} = \{x \in K \mid |x| < 1\}$. We say that M is *almost zero* if $\mathfrak{m}M = 0$. Taking the quotient of the category of K°-modules K°-**Mod** by the subcategory of almost zero K°-modules we obtain a localization functor

$$K^\circ\text{-}\mathbf{Mod} \to K^{\circ a}\text{-}\mathbf{Mod} : \ M \mapsto M^a.$$

We will state (without proof) some facts of almost ring theory:
For two K°-modules M, N we have $\mathrm{Hom}_{K^{\circ a}\text{-}\mathbf{Mod}}(M^a, N^a) = \mathrm{Hom}_{K^\circ}(\mathfrak{m} \otimes_{K^\circ} M, N)$. The localization functor has a right-adjoint given by $M \mapsto M_* = \mathrm{Hom}_{K^{\circ a}\text{-}\mathbf{Mod}}(K^{\circ a}, M)$ and the adjuntion morphism $(M_*^a \to M$ is an isomorphism. One can define an internal Hom-functor by setting $\mathrm{alHom}_{K^\circ}(M, N) := \mathrm{Hom}_{K^{\circ a}\text{-}\mathbf{Mod}}(M^a, N^a)^a$ and a tensor product by $M \otimes_{K^{\circ a}} N := (M_* \otimes_{K^\circ} N_*)^a$.

Definition 6. *(1.) An $K^{\circ a}$-module M is* **flat** *(resp.* **faithfully flat***) if the functor $N \mapsto M \otimes_{K^{\circ a}} N$ is exact (resp. exact and faithful).*
(2.) An $K^{\circ a}$-module M is **almost projective** *if the functor $N \mapsto \mathrm{alHom}_{K^{\circ a}}(M, N)$ is exact.*

Definition 7. *Let M be an K°-module then M^a is an* **almost finitely generated** *(resp.* **almost finitely presented***) $K^{\circ a}$-module if for every finitely generated subideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exists a finitely*

generated (resp. finitely presented) K°-module N_0 and a homomorphism $f : N_0 \to M$ such that $\ker(f)$ *and* $\mathrm{coker}(f)$ *are annihilated by* $\mathfrak{m}_0$.
We say that M^a is ***uniformly almost finitely generated*** *if there exists some $n \in \mathbb{Z}$ such that for every $\mathfrak{m}_0$ we can choose an N_0 which is generated by n elements.*

Definition 8. *The morphism pf $K^{\circ a}$-algebras $f : A \to B$ is said to be* ***unramified*** *if B is an almost projective $B \otimes_A B$- module via the multiplication map $\mu_{B/A}$, and f is called* ***étale*** *if B is a flat A-module and f is unramified. We will wirte A-$\mathbf{\acute{E}t}_{\mathrm{afp}}$ to denote the category of étale A-algebras B which are almost finitely presented as an A-module.*

Theorem 9. *Let A be a $K^{\circ a}$-algebra and let A be ϖ-adically complete. Then the functor*

$$A\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} \to A/\varpi\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} : \ B \mapsto B \otimes_A A/\varpi,$$

is an equivalence of categories.

Definition 10. *Let K be a perfectoid field and $\varpi \in K$ with $|p| \le |\varpi| < 1$.*
(1.) A Banach K-algebra R with the subset of powerbounded elements $R^\circ \subset R$ a bounded subring is called a ***perfectoid K-algebra*** *if the Frobenius morphism*

$$\Phi : R^\circ/\varpi \to R^\circ/\varpi,$$

is surjective.
(2.) A ϖ-adically complete flat $K^{\circ a}$-algebra A is called a ***perfectoid $K^{\circ a}$-algebra*** *if the Frobenius induces an isomorphism*

$$\Phi : A/\varpi^{1/p} \cong A/\varpi.$$

(3.) A flat $(K^{\circ a}/\varpi)$-algebra $\overline{A}$ is called a ***perfectoid $(K^{\circ a}/\varpi)$-algebra*** *if the Frobenius induces an isomorphism*

$$\Phi : \overline{A}/\varpi^{1/p} \cong \overline{A}.$$

We denote the category of perfectoid K-algebras (resp. $K^{\circ a}$-algebras; resp. $(K^\circ/\varpi)^a$-algebras) with the obvious morphisms by K-**Perf** (resp. $K^{\circ a}$-**Perf**; resp. $(K^{\circ a}/\varpi)$-**Perf**).

Theorem 11. *Let K be a perfectoid field, then we have the following chain of equivalences of categories*

$$K\text{-}\mathbf{Perf} \overset{(1)}{\cong} K^{\circ a}\text{-}\mathbf{Perf} \overset{(2)}{\cong} (K^{\circ a}/\varpi)\text{-}\mathbf{Perf} = (K^{\flat\circ a}/\varpi^\flat)\text{-}\mathbf{Perf} \overset{(3)}{\cong} K^{\flat\circ a}\text{-}\mathbf{Perf} \overset{(4)}{\cong} K^\flat\text{-}\mathbf{Perf}.$$

(1) Let R be a perfectoid K-algebra. If $x^p \in \varpi R^\circ$ then $x^p = \varpi y$ for some $y \in R^\circ$ and because ϖ is invertible in R we get $(x/\varpi^{1/p})^p = y \in R^\circ$ which implies that $x \in \varpi^{1/p}R^\circ$. Therefore Φ induces an isomorphism $R^\circ/\varpi^{1/p} \cong R^\circ/\varpi$. Since R° is flat over K° and $- \otimes_{K^\circ} R^\circ$ induces an exact functor on the category of $K^{\circ a}$-modules we get that $R^{\circ a}$ is flat over $K^{\circ a}$. The localization functor $M \mapsto M^a$ has a left-adjoint and therefore commutes with inverse limits:

$$R^{\circ a} = (\varprojlim R^\circ/\varpi^n)^a = \varprojlim R^{\circ a}/\varpi.$$

Thus $R^{\circ a}$ is an object in $K^{\circ a}$-**Perf**.
Now, let A be a perfectoid $K^{\circ a}$-algebra, we show that $R = A_*[1/\varpi]$ is a perfectoid K-algebra when it is equipped with the Banach K-algebra structure such that A_* is open and bounded .
We show that $\Phi : A_*/\varpi^{1/p} \to A_*/\varpi$ is an *actual* isomorphism. For this, let $x \in A_*$ with $x^p \in \varpi A_*$ then almost injectivity implies that for all $\varepsilon \in \mathfrak{m}$ we have $\varepsilon x \in \varpi^{1/p}A_*$. By definition $A_* = \mathrm{Hom}_{K^{\circ a}}(\mathfrak{m}, A) = \mathrm{Hom}_{K^\circ}(K^\circ, B)$ for B a K°-module with $B^a = A$. Since A is flat we can write this as the set of $x \in \mathrm{Hom}_K(K, B[\varpi^{-1}])$ such that $\varepsilon x \in B$ for all $\varepsilon \in \mathfrak{m}$. Thus we have shown $x \in (\varpi^{1/p}A_*)_* = \varpi^{1/p}A_*$.
By the almost surjectivity there exists for every $x \in A_*$ and $c \in \mathbb{Q}_+, c < 1$ elements $y, z \in A_*$ such that $\varpi^c x = y^p + \varpi z$. Setting $w := y/\varpi^{c/p}$ we get $w^p = y^p/\varpi^c = y^p \in A_*$.
Claim. *If $w \in R$ with $w^p \in A_*$ then $w \in A_*$.*

Proof. By the injectivity of $\Phi : A_*/\varpi^{1/p} \to A_*/\varpi$ we know that for any $y \in A_*$ with $y^p \in \varpi A_*$ we have $y \in \varpi^{1/p}A_*$. For some $k \geq 1$ we can write $y = \varpi^{k/p}x \in A_*$ and thus $y^p = \varpi^k x^p \in \varpi^k A_*$. Since $y \in \varpi^{1/p}A_*$ we can lower k by one: $\varpi^{k-1/p}x = y\varpi^{-1/p} \in A_*$ and thus deduce the claim by induction. Thus we have shown that $w \in A_*$, i.e. x has a p-th root modulo $\mathfrak{m}$ which proves the equivalence (1).

To prove the equivalence (2) we wil need some deformation theory. We want to show that any perfectoid $K^{\circ a}/\varpi$-algbera deforms uniquely to $K^{\circ a}$.
Let us briefly recall the construction of the cotangent complex. For any map of rings $A \to B$ we can define the cotangent complex $\mathbb{L}_{B/A}$ in the derived category of B-modules, by taking a simplicial resolution $B_\bullet$ of B by free A-algebras and letting $\mathbb{L}_{B/A}$ be the object of the derived category which corresponds to $\Omega_{B_\bullet/A} \otimes_{B_\bullet} A$ via the Dold-Kan correspondence.

Proposition 12. *Let $I \subset B$ be an ideal with $I^2 = 0$ and set $B_0 = B/I$. Let C_0 be a flat B_0-algebra.*
(1.) There exists an obstruction in $\mathrm{Ext}^2_{C_0}(\mathbb{L}_{C_0/B_0}, C_0 \otimes_{B_0} I)$ to the existence of a flat B-algebra C with $C \otimes_B B_0 = C_0$.
(2.) If the obstruction vanishes the set of isomorphism classes of such a lifting forms a torsor under the group $\mathrm{Ext}^1_{C_0}(\mathbb{L}_{C_0/B_0}, C_0 \otimes_{B_0} I)$.
(3.) The group of automorphisms of a lifting is naturally isomorphic to $\mathrm{Ext}^0_{C_0}(\mathbb{L}_{C_0/B_0}, C_0 \otimes_{B_0} I)$.

Proposition 13. *Let B, B_0, I be as above. Suppose we have two flat B-algebras C, C' and a morphism $f_0 : C_0 \to C'_0$ between their reductions mod I.*
(1.) There is an obstruction in $\mathrm{Ext}^1_{C_0}(\mathbb{L}_{C_0/B_0}, C'_0 \otimes_{B_0} I)$ to the existence of a lift $f : C \to C'$ of the morphism f_0.
(2.) If the obstruction vanishes the set of such morphisms forms a torsor under the group $\mathrm{Ext}^0_{C_0}(\mathbb{L}_{C_0/B_0}, C'_0 \otimes_{B_0} I)$.

Proposition 14 . *Let $R \to S$ be a map of rings. Suppose that $\mathbb{F}_p \subset R$. Denote by $R_{(\Phi)}$ (resp. $S_{(\Phi)}$) the ring R with R-algebra structure induced by the Frobenius endomorphism Φ. Assume that the map*

$$R_{(\Phi)} \overset{\mathbf{L}}{\otimes}_R S \to S_{(\Phi)},$$

*is an isomorphism in $\mathbf{D}$(R-**Mod**). Then we have $\mathbb{L}_{S/R} \cong 0$.*

Proof. By considering the canonical resolution of S by free R-algebras we can reduce to the case that S is a polynomial R-algebra, say $S = R[X_i \,|\, i \in I]$. Then the relative Frobenius $\Phi_{S/R}$ maps X_i to X_i^p for every $i \in I$ and $\Phi_{S/R}$ is a quasi-isomorphism by assumption. This induces an isomorphism

$$R_{(\Phi)} \overset{\mathbf{L}}{\otimes}_R \mathbb{L}_{S/R} \overset{\simeq}{\to} \mathbb{L}_{S_{(\Phi)}/R_{(\Phi)}} \cong \mathbb{L}_{S/R},$$

but in the case of a polynomial algebra the cotangent complex has the simple description $\mathbb{L}_{S/R} \cong \Omega_{S/R}[0]$ in $\mathbf{D}$(S-**Mod**) and since $d(X_i^p) = pX_i^{p-1}dX_i = 0$ we get the result. □

Gabber and Ramero have defined the derived category of $K^{\circ a}$-modules and $\mathbb{L}^a_{B/A}$ is an object of $D(B$-**Mod**$)$ which depends only on the morphism $A^a \to B^a$. The following Lemma is a consequence of the "almost version" of Proposition 14.

Lemma 15. *Let $\overline{A}$ be a perfectoid $(K^{\circ a}/\varpi)$-algebra. Then $\mathbb{L}^a_{\overline{A}/(K^{\circ a}/\varpi)} \cong 0$ in $\mathbf{D}(\overline{A}$-**Mod**$)$.*

Now we are ready to prove the equivalence (2): let $\overline{A}$ be a perfectoid $K^{\circ a}/\varpi$-algebra. We want to lift it to a flat $(K^\circ/\varpi^n)^a$-algebra $\overline{A}_n$. By the above results on deformation theory we can do this uniquely if $\mathbb{L}_{\overline{A}_n/(K^\circ/\varpi^n)^a}$ vanishes. By the Lemma this is true for $n = 1$ and we can proceed by induction. By tensoring the short exact sequence

$$0 \to \overline{A} \overset{\varpi^{n-1}}{\to} \overline{A}_n \to \overline{A}_{n-1} \to 0,$$

with $\mathbb{L}^a_{\overline{A}_n/(K^{\circ a}/\varpi^n)}$ we get the triangle

$$\mathbb{L}_{\overline{A}/(K^{\circ a}/\varpi)} \to \mathbb{L}_{\overline{A}_n/(K^\circ/\varpi^n)^a} \to \mathbb{L}_{\overline{A}_{n-1}/(K^\circ/\varpi^{n-1})^a}$$

and can conclude by induction on n.
Thus we obtain a unique system of lifts $\overline{A}_n$ with $\overline{A}_n/\varpi^{n-1} = \overline{A}_{n-1}$ and we denote the inverse limit by A. Then A is a perfectoid $K^{\circ a}$-algebra and we get the result.

We thus have shown the following diagram of equivalences of categories

$$\begin{array}{ccccc} K\text{-}\mathbf{Perf} \xleftrightarrow[(1)]{} K^{\circ a}\text{-}\mathbf{Perf} \xleftrightarrow[(2)]{} (K^{\circ a}/\varpi)\text{-}\mathbf{Perf} & & \\ \| & & \\ (K^{\flat\circ a}/\varpi^\flat)\text{-}\mathbf{Perf} \xleftrightarrow{(3)} K^{\flat\circ a}\text{-}\mathbf{Perf} \xleftrightarrow{(4)} K^\flat\text{-}\mathbf{Perf} \end{array}$$

Now, let $R, A, \overline{A}, \overline{A}^\flat, A^\flat, R^\flat$ be a sequence of perfectoids under the equivalences

$$K\text{-}\mathbf{Perf} \cong K^{\circ a}\text{-}\mathbf{Perf} \cong (K^{\circ a}/\varpi)\text{-}\mathbf{Perf} = (K^{\flat\circ a}/\varpi^\flat)\text{-}\mathbf{Perf} \cong K^{\flat\circ a}\text{-}\mathbf{Perf} \cong K^\flat\text{-}\mathbf{Perf}.$$

We look at the categories of étale almost finitely presented algebras over these perfectoid algebras, i.e. the diagram

$$R\text{-}\mathbf{F\acute{E}t} \overset{(A)}{\leftarrow} A\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} \overset{(B)}{\leftarrow} \overline{A}\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} = \overline{A}^\flat\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} \overset{(C)}{\rightarrow} A^\flat\text{-}\mathbf{\acute{E}t}_{\mathrm{afp}} \overset{(D)}{\rightarrow} R^\flat\text{-}\mathbf{F\acute{E}t},$$

where R-**FÉt** denotes the (usual) category of finite étale algebras over R. First, note that the functors (B) and (C) are equivalences of categories since étale algebras lift uniquely over nilpotents and the property of being almost finitely presented is preserved (Theorem 9).

Proposition 16. *Let $\overline{A}$ be a perfectoid $(K^{\circ a}/\varpi)$-algebra and let $\overline{B}$ be an étale almost finitely presented $\overline{A}$-algebra. Then $\overline{B}$ is a perfectoid $(K^{\circ a}/\varpi)$-algebra.*

Proof. The flatness is clear. We are left to show the Frobenius property. Let $f : A \to B$ be an étale map of $\mathbb{F}_p$-algebras. We need to show that the induced Frobenius map $\Phi : B^{(1)} := B \otimes_{A,\Phi_A} A \to B$ is an isomorphism. A map between étale algebras is étale and on both sides B and $B^{(1)}$ the map Φ factors through powers of the Frobenius.
Claim. *Let $\Phi : R \to S$ be an étale map of $\mathbb{F}_p$-algebras factoring through powers of Frobenius on R and S. Then Φ is an isomorphism.*
Proof. We begin by showing that Φ is faithfully flat, i.e. that $R/I \otimes_R S = 0$ implies $R/I = 0$ for an ideal $I \subset R$. Base changing we reduce to $R = R/I$, i.e. to $S = 0$ implies $R = 0$. This is clear from the factoring assumption.
Note that $\mathbf{1}_B \otimes \Phi : B \to B \otimes_A B \overset{\mu}{\to} B$ also factors through powers of Frobenius and so does μ. Thus μ is an isomorphism and the claim follows by fully faithfulness. □

By Proposition 16 and K-**Perf** $\cong K^{\circ a}$-**Perf**, respectively $K^{\flat\circ a}$-**Perf** $\cong K^\flat$-**Perf** we see that the functors (A) and (D) are fully faithfull. Scholze's *almost purity theorem* states that these functors are euqivalences of categories. At his point we will only prove this in characteristic p.

Theorem 17. *Let K be a perfectoid field of characteristic p. Let R be a perfectoid K-algebra and let S be a finite étale R-algebra. Then S is a perfectoid K-algebra and $S^{\circ a}$ is étale almost finitely presented over $R^{\circ a}$.*

Proof. Let $\eta : R^{\circ a} \to S^{\circ a}$ be a map of perfectoid $K^{\circ a}$-algebras such that $\eta[\varpi^{-1}]$ is finite étale. Since $R \to S$ is unramified there exists an idempotent $e \in (S^\circ \otimes_{R^\circ} S^\circ)[\varpi^{-1}]$. For some $c \geq 0$ we have $\varpi^c e \in S^\circ \otimes_{R^\circ} S^\circ$. By perfectness there exists $\varpi^{c/p^n} e \in S^\circ \otimes_{R^\circ} S^\circ$ and thus $e \in (S^{\circ a} \otimes_{R^{\circ a}} S^{\circ a})_*$. Therefore $R^{\circ a}$ is unramified over $S^{\circ a}$.
By the above we can write εe as $\sum_{i=1}^n a_i \otimes b_i$ with $a_i, b_i \in S^\circ$ for any $\varepsilon \in \mathfrak{m}$. Define the maps $S^\circ \to R^{\circ n}$, $s \mapsto (\mathrm{Tr}_{S/R}(s, b_1), \ldots, \mathrm{Tr}_{S/R}(s, b_n))$ and $R^{\circ n} \to S^\circ$, $(r_1, \ldots, r_n) \mapsto \sum_{i=1}^n a_i r_i$. The composition of the two maps is multiplication by ε which implies that R°/S° is almost projective but almost finitely generated and almost projective modules are also almost finitely presented. □

We will complete the proof of the Theorem in the 0-dimensional case:
Proof of Theorem 5. By the above we know that the functor $K^\flat$-**FÉt** $\to K$-**FÉt** is fully faithful. Let $M = \widehat{\overline{K^\flat}}$ then M is perfectoid and the untilt $M^\sharp$ is an algebraically closed perfectoid field extension of

K. Taking the union $N = \cup_{K^\flat \subset L \subset M} L^\sharp$ over finite extensions $L \subset M$ of $K^\flat$ gives a field N which is dense in $M^\sharp$ ($M^{\sharp\circ}$ is the completion of the colimit of the $L^{\sharp\circ}$). Krasner's lemma implies that N is algebraically closed. Hence any finite extension F/K is contained in M, i.e. there exists some $L^\sharp$ containing F. In particular F is associated to a subgroup H of $\mathrm{Gal}(L^\sharp/K) = \mathrm{Gal}(L/K^\flat)$ and $F^\flat = L^H$ untilts to F. $\square$