Slope Filtrations over Robba Rings for Relative Frobenius

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1 Introduction to ϕ-modules over Robba Rings

Definition 1.1. Let K be a field complete for a discrete valuation with residue field k; let $\mathfrak{o}_K$ denote the valuation subring of K and let $\mathfrak{m}_K$ denote the maximal ideal of $\mathfrak{o}_K$. Suppose π is the uniformizer, $\|\cdot\|$ is the norm and v is the corresponding valuation on it. For $r \in (0,1)$, let $\mathcal{R}^r$ be $\{\sum\limits_{i\in\mathbb{Z}} a_i t^i : v(a_i) + i\rho \to +\infty$ as $i \to \infty, \rho \in (0, -log(r)]\}$, i.e. the Laurent series converging on $(0, -log(r)]$.

Definition 1.2. For $r > 0$, let $\|\cdot\|_r$ to be the r-Gauss norm:

$$\left\|\sum_{i\in\mathbb{Z}} a_i t^i\right\|_r = \sup_{i\in\mathbb{Z}} \|c_i\| r^i.$$

For vectors, we extend the norm by taking the maximum.

Remark 1.3. By Proposition 2.1.2 of [Ked10], we know that the Gauss norm is multiplicative.

Definition 1.4. The Robba ring $\mathcal{R}$ is defined to be $\bigcup\limits_{r\in(0,1)} \mathcal{R}^r$. $\mathcal{R}^{int}$ is defined to be the formal sums in $\mathcal{R}$ with coefficients in $\mathfrak{o}$. $\mathcal{R}^{bd}$ is defined to be $\{\sum\limits_{n\in\mathbb{Z}} a_n t^n \in \mathcal{R} : N \in \mathbb{Z}, \sup\limits_{i\in\mathbb{Z}} \|a_i\| < +\infty\}$.

Remark 1.5. $\mathcal{R}^{int} \subset \mathcal{R}^{bd}$.

Theorem 1.6. $\mathcal{R}^\times = \mathcal{R}^{bd} \setminus \{0\}$.

Proof. In fact, we have $\mathcal{R}^r \cap \mathcal{R}^{bd} \subset B((0, -log(r)])$.

Recall that in Prof. Xiao's lectures, for $f \in B((0, -log(r))])$, we have the typical factorization $f = P_f u_f$ where P_f is a polynomial and $u_f \in B((0, -log(r)))^\times$.

Take a nonzero element $f \in \mathcal{R}^{bd}$. It lies in $\mathcal{R}^r$ for some r. Then we have $f \in B((0, -log(r)])$. Take the typical factorization $P_f u_f$. Since the roots of

P_f is finite, we can chose r' bigger enough such that P_f has no roots on $[r', 1)$. Hence f becomes a unit in $B((0, -log(r')]) \subset \mathcal{R}$. Thus $\mathcal{R}^{bd} \setminus \{0\} \subset \mathcal{R}^{\times}$.

Just using slope argument, we will learn that $\mathcal{R}^{\times} \subset \mathcal{R}^{bd}$. $\square$

Corollary 1.7. *$\mathcal{R}^{bd}$ is equipped with a discrete valuation w where $\mathcal{R}^{int}$ is the integral ring and π is the uniformizer.*

Proof. Just set $w(\sum_{i\in\mathbb{Z}} a_i t^i) := \inf_{i\in\mathbb{Z}} v(a_i)$. $\square$

Remark 1.8. For detailed proof of the above theorem and corollary, you may consult Lemma 15.1.3 of [Ked10].

Theorem 1.9. *$\mathcal{R}$ is a Bézout domain, i.e. every finitely generated ideal is a principal ideal.*

Proof. See [Laz62]. $\square$

Definition 1.10. Fix an integer $q > 1$. A relative q-power Frobenius lift on the Robba ring is a homomorphism $\phi : \mathcal{R} \to \mathcal{R}$ of the form $\sum_{i\in\mathbb{Z}} c_i t^i \to \sum_{i\in\mathbb{Z}} \phi_K(c_i) u^i$ where ϕ_K is an isometric field endomorphism of K and $u \in \mathcal{R}^{int}$ is such that $u - t^q$ is in the maximal ideal of $\mathcal{R}^{int}$. If k has characteristic $p > 0$ and q is a power of p, we define an absolute q-power Frobenius lift as a relative Frobenius lift in which ϕ_K is itself of q-power Frobenius lift.

Remark 1.11. Note that $\left\|\frac{u}{t^q} - 1\right\|_s$ is continuous with respect to s by considering the Newton polygon. Since $\left\|\frac{u}{t^q} - 1\right\|_1 < 1$, there exists $r_0 < 1$ such that $\left\|\frac{u}{t^q} - 1\right\|_s < 1, \forall s \in [r_0, 1]$. Then $\left\|\frac{u^i}{t^{qi}} - 1\right\|_s < 1, \forall s \in [r_0, 1], \forall i \in \mathbb{Z}$. Thus we have $\|\phi(f)\|_{r^{\frac{1}{q}}} = \|f\|_r, \forall r \in [r_0, 1)$

Theorem 1.12. *Let ϕ be a relative Frobenius lift and let A be an $n \times n$ matrix over $\mathcal{R}^{int}$. Then the map $\mathbf{v} \to \mathbf{v} - A\phi(\mathbf{v})$ on column vectors induces a bijection on $(\mathcal{R}/\mathcal{R}^{bd})^n$.*

Proof. (See Proposition 1.2.6 of [Ked08])

In fact we can replace $\mathbf{v}$ and A with $t^m\mathbf{v}$ and $\frac{t^m}{\phi(t^m)}A$ without changing the theorem.

Note that we have $\frac{t^m}{\phi(t^m)} = t^{(1-q)m}(1 + \pi f)^m$ for some $f \in \mathcal{R}^{int}$.

Suppose A is defined on $[r, 1)$. Note that we have

$$\left\|t^{1-q}(1 + \pi f)\right\|_r = r^{1-q} \|1 + \pi f\|_r \geq r^{1-q} > 1.$$

Thus we can pick proper m such that $\left\|\frac{t^m}{\phi(t^m)}A\right\|_r \leq 1$. On the other hand, we have

$$\left\|t^{1-q}(1 + \pi f)\right\|_1 = \|1 + \pi f\|_1 = 1.$$

Thus we learned that $\left\|\frac{t^m}{\phi(t^m)}A\right\|_1 = \|A\|_1$. Above all, by the replacement mentioned above, we may assume that A is defined on $[r,1)$ and entries of A are bounded on $[r,1]$ where r is chosen among $[r_0,1]$.

Injectivity:

Suppose entries of $\mathbf{w} := \mathbf{v} - A\phi(\mathbf{v})$ are in $\mathcal{R}^{bd}$. We may further assume that $\mathbf{w}$ and $\mathbf{v}$ both are defined over $[r,1)$. Then there exists $c>0$ such that $\|\mathbf{w}\|_s \le c, \forall s \in [r,1]$ and $\|\phi(\mathbf{v})\|_s \le c, \forall s \in [r, r^{\frac{1}{q}}]$. Then we have $\|\mathbf{v}\|_s \le c, \forall s \in [r, r^{\frac{1}{q}}]$. Hence we have $\|\phi(\mathbf{v})\|_s \le c, \forall s \in [r^{\frac{1}{q}}, r^{\frac{1}{q^2}}]$. Repeat the above argument, we will get that $\|\mathbf{v}\|_s \le c, \forall s \in [r,1)$. Hence, by the continuity, we know that $\|\mathbf{v}\|_1 \le c$.

Surjectivity:

Pick up $\mathbf{w} \in \mathcal{R}^n$. Suppose that $\mathbf{w}$ is defined on $[r',1)$ where $r' \in [r_0^{\frac{1}{q}}, 1)$. Let's take $r \in (r',1)$. We are going to construct a sequence $\{\mathbf{w}_i\}_{i\in\mathbb{N}}$. Start with $\mathbf{w}_0 = \mathbf{w}$. Given $\mathbf{w}_l = \sum_{i\in\mathbb{Z}} \mathbf{w}_{l,i} t^i$ defined on $[r,1)$, decompose it as follows

$$\mathbf{w}_l = \mathbf{w}_l^+ + \mathbf{w}_l^-, \mathbf{w}_l^+ = \sum_{i>0} \mathbf{w}_{l,i} t^i.$$

Note that $t^{-1}\mathbf{w}_l^+$ is defined on $[0,1)$. We have the following inequality:

$$\left\|\mathbf{w}_l^+\right\|_r \le s^{\frac{q-1}{q}} \left\|\mathbf{w}_l^+\right\|_{r^{\frac{1}{q}}} \le r^{\frac{q-1}{q}} \left\|\mathbf{w}_l\right\|_{r^{\frac{1}{q}}}.$$

Let's put $\mathbf{w}_{l+1}$ to be $\phi(\mathbf{w}_l^+)$. Then we know that

$$\left\|\mathbf{w}_{l+1}\right\|_{r^{\frac{1}{q}}} \le r^{\frac{q-1}{q}} \left\|\mathbf{w}_l\right\|_{r^{\frac{1}{q}}}.$$

Thus we learn that $\mathbf{w}_l^+$ converges to zero with respect to the norm $\|\cdot\|_{r^{\frac{1}{q}}}$ (hence the norms $\|\cdot\|_s, s \le r^{\frac{1}{q}}$). Hence $\sum_{i\in\mathbb{N}} \mathbf{w}_l^+$ can be defined over $[0, r^{\frac{1}{q}}]$.

Similarly, we have $\left\|\mathbf{w}_l^-\right\|_s \le \left\|\mathbf{w}_l^-\right\|_{r^{\frac{1}{q}}} \le \|\mathbf{w}_l\|_{r^{\frac{1}{q}}}, \forall s \in [r^{\frac{1}{q}}, 1)$. Thus we have $\sum_{i\in\mathbb{N}} \mathbf{w}_l^-$ is well defined and bounded over $[r^{\frac{1}{q}}, 1)$. Hence $\sum_{i\in\mathbb{N}} \mathbf{w}_l^-$'s entries lie in $\mathcal{R}^{bd}$.

Let's set $\mathbf{v} := \sum_{l\in\mathbb{N}} \mathbf{w}_l^+$. Then $\mathbf{v}$ is defined over $[0, r^{\frac{1}{q}}]$ and $\phi(\mathbf{v})$ is defined over $[r_0^{\frac{1}{q}}, r^{\frac{1}{q^2}}]$.

Above all, we have the following term $\mathbf{w} - \mathbf{v} + A\phi(\mathbf{v}) = \sum_{i\in\mathbb{N}} \mathbf{w}_l^-$ which can be defined over $[r', r^{\frac{1}{q^2}}]$. Thus we can rewrite $\mathbf{v}$ as $\mathbf{w} - \sum_{i\in\mathbb{N}} \mathbf{w}_l^- + A\phi(\mathbf{v})$ over $[r', r^{\frac{1}{q^2}}]$. Thus $\mathbf{v}$ can be defined over $[0, r^{\frac{1}{q^2}}]$. Repeat the above process, we get that $\mathbf{v}$ can be defined over $[0,1)$. After all, we have constructed $\mathbf{v} \in \mathcal{R}^n$ such that $\mathbf{w} - \mathbf{v} + A\phi(\mathbf{v}) = \sum_{i\in\mathbb{N}} \mathbf{w}_l^-$ are of $\mathcal{R}^{bd}$ coefficients. $\square$

Remark 1.13. We need $q \neq 1$ in this theorem.

Definition 1.14. A ϕ-ring R is just a ring equipped with an endomorphism ϕ. It is called inversive if ϕ is bijective. For a module M over R, we can view it as a bimodule M_ϕ over R where the right multiplication factors through ϕ. A (dualizable) ϕ-module over a ϕ-ring R is just a finite free R-module M with an isomorphism between R-modules $\phi^*M \cong M$ where $\phi^*M = R_\phi \otimes M$. Thus we have a map $\phi_M : M \to M$ such that $\phi_M(am) = \phi(a)\phi_M(m)$.

Remark 1.15. Pick up a basis of M: $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$. Consider the following matrix $A = (A_{i,j})_{1\leq i,j\leq n}$ such that $\phi_M(\mathbf{e}_i) = \sum\limits_{1\leq j\leq n} A_{i,j}\mathbf{e}_j$. In fact, we can identify M as R^n. Then the action of ϕ_M is determined by $\phi_M(x) = A\phi(x), x \in R^n$ where $\phi(x)$ means ϕ acts on each coordinates. In fact, the R-linear isomorphism $\phi^*M \cong M$ is given by A.

Remark 1.16. Usually, we don't require the morphism between $\phi^*M = M$ is an isomorphism. The requirement here is to assure the existence of dual. For the concrete construction, $M^\vee := \mathrm{Hom}_R(M, R)$ with ϕ-action $(\phi_{M^\vee}(f))(c\phi_M(m)) = c\phi(f(m)), c \in R$. The related matrix is $(A^t)^{-1}$

Remark 1.17. Let's consider the twisted polynomial ring $R_\phi\{T\}$ where $Ta = \phi(a)T$. Every ϕ-module over R can be identified as a left module of $R_\phi\{T\}$ which is finite free over R. Without the finite free condition and isomorphic condition, ϕ-modules will become an abelian category.

Remark 1.18. Our ϕ-modules admit tensor products, symmetric products and exterior products.

Definition 1.19. For a positive integer a, the a-pushforward functor $[a]_*$ is just using the automorphism $\phi^*M \cong M$ a times for a ϕ-module M to make it a ϕ^a-module. Or, in a fancier language, view the $R_\phi\{T\}$-module M as an $R_{\phi^a}\{T^a\}$-module through the injection $R_{\phi^a}\{T^a\} \hookrightarrow R_\phi\{T\}$.

Definition 1.20. For a positive integer a, the a-pullback functor $[a]^*$ is just taking a ϕ^a-module to a ϕ-module $R_\phi\{T\} \otimes_{R_{\phi^a}\{T^a\}} M$.

The following statements are facts of $[a]^*$ and $[a]_*$:
1. They are exact functors commuting with duals.
2. The functors $([a]^*, [a]_*)$ and $([a]_*, [a]^*)$ form an adjoint pair.
3. The functor $[a]_*$ commutes with tensor products while $[a] + *$ does not.
4. If M is a ϕ-module and N is a ϕ^a-module, then $M \otimes [a]^*N \cong [a]*([a]_*M \otimes N)$.
5. If M is a ϕ-module, then $\mathrm{rank}([a]_*M) = \mathrm{rank}(M)$.
6. If N is a ϕ^a-module, then $\mathrm{rank}([a]^*N) = a\mathrm{rank}(N)$.
7. If N is a ϕ^a-module, then $[a]_*[a]^*N \cong N \oplus \phi^*(N) \oplus \cdots \oplus (\phi^{a-1})^*(N)$

Definition 1.21. For M a ϕ-module, put

$$H^0(M) = \ker(\phi - 1 : M \to M); H^1(M) = \mathrm{coker}(\phi - 1 : M \to M).$$

Remark 1.22. In the category of ϕ-modules, let's consider Hom and Ext. For ϕ-modules M, N, we have

$$H^0(M^\vee \otimes N) = Hom(M, N); H^1(M^\vee \otimes N) = Ext(M, N).$$

For a ϕ^a-module N, we have natural bijections $H^i(N) = H^i([a]^* N), i = 1, 2$.

2 Preliminary for slope filtrations theorem

In this section, we will introduce degrees, slopes, stability, *é*tale modules and pure modules.

We will assume the following hypothesis(This word is used in [Ked08]. Personally, I prefer 'assumption') in this section:

Hypothesis 2.1. Let $R^{int} \subset R^{bd} \subset R$ be inclusions of Bézout domains such that $R^\times = R^{bd} \setminus \{0\}$. Let ϕ be an injective endomorphism of R which also acts on R^{int} and R^{bd}. Let $w : R^{bd} \to \mathbb{Z}$ be a ϕ-equivariant valuation such that $w(R^\times) = \mathbb{Z}$ and $R^{int} = \{r \in R^{bd} | w(r) \geq 0\}$. Suppose in addition that for any $n \times n$ matrix A over R^{int}, the map $\mathbf{v} - A\phi(\mathbf{v})$ on column vectors induces an injection(weak form) or bijection(strong form) on $(R/R^{bd})^n$. Note that the analogous hypothesis for ϕ^n also holds since one van identify the kernel and cokernel of $\mathbf{v} \to \mathbf{v} - A\phi^a \mathbf{v}$ on $(R/R^{bd})^n$ with the kernel and cokernel of

$$(\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_{a-1}) \to (\mathbf{v}_0 - A\phi(\mathbf{v}_{a-1}), \mathbf{v}_1 - \phi(\mathbf{v}_0), \cdots, \mathbf{v}_{a-1} - \phi(\mathbf{v}_{a-2}))$$

on $(R/R^{bd})^n$.

Example 2.2. By theorem 1.9 and 2.22, we know $\mathcal{R}^{int} \subset \mathcal{R}^{bd} \subset \mathcal{R}$ with the relative Frobenius satisfies the strong hypothesis. We will construct an extended Robba ring $\tilde{\mathcal{R}}$ satisfying the strong hypothesis in proving the slope filtrations theorem.

Definition 2.3. For a ϕ-module M of rank n over R, the exterior power $\bigwedge^n M$ is of rank 1 over R. Let $\mathbf{v}$ be a generator of $\bigwedge^n M$ and write $\phi(\mathbf{v}) = r\mathbf{v}$ for some $r \in R^\times$. Define the degree of M of m by setting $deg(M) = w(r)$. If M is nonzero, the slope of M is defined to be $\mu(M) := \frac{deg(M)}{rank(M)}$.

Remark 2.4. We can view deg as the determinant of A.

Here are some facts about slopes:

1. If $0 \to M_1 \to M \to M_2 \to 0$ is exact, then $deg(M) = deg(M_1) + deg(M_2)$; hence $\mu(M)$ is a weighted average of $\mu(M_1)$ and $\mu(M_2)$.
2. $\mu(M_1 \otimes M_2) = \mu(M_1) + \mu(M_2)$.
3. $\mu(\bigwedge^i M) = i\mu(M)$
4. $deg(M^\vee) = -deg(M), \mu(M) = -\mu(M)$.
5. If M is a ϕ-module, then $\mu([a]_* M) = a\mu(M)$.
6. If N is a ϕ^a-module, then $\mu([a]^* M) = \frac{1}{a}\mu(M)$.

Definition 2.5. We say a ϕ-module M is (module-)semistable if for any nontrivial ϕ-submodule N, we have $\mu(N) \geq \mu(M)$. We say a ϕ-module M is (module-)stable if for any proper non-trivial ϕ-submodule N, we have $\mu(N) > \mu(M)$.

Remark 2.6. Twisted by a rank 1 module doesn't change the semistability or stability.

Theorem 2.7. *Any ϕ-module of rank 1 is stable.*

Proof. We only need the weak hypothesis.

By twisting, it suffices to check the case that $M = R$ and $\phi_M = \phi$. Suppose $N = Rx$ and $\lambda = \frac{\phi(x)}{x} \in R^*$.Thus we have $\mu(N) = w(\lambda)$. If $\mu(N) \leq 0$, then we have $x - \lambda^{-1}\phi(x) = 0$ where $\lambda^{-1} \in R^{int}$. By weak hypothesis, we know that $x \in R^{bd} \setminus \{0\} = R^\times$. Thus we have $N = M$. □

Corollary 2.8. *If $N \subset M$ is an inclusion of ϕ-modules of same rank, then $\mu(N) \geq \mu(M)$ where the equality hold if and only if $N = M$.*

Proof. Just using the above theorem for top wedge product. □

Before going on, let's recall some facts about Bézout domain.

1. Every finitely generated torsion-free module is finite free.(Dedekind Theorem)

2. Every finitely presented module is a direct sum of finite free module and a finitely presented torsion module.(See Proposition 4.9 of [Cre98])

Definition 2.9. Given an inclusion of ϕ-modules $N \subset M$, define the saturation of N to be $N' = M \cap (N \otimes Frac(R))$.

Lemma 2.10. *Given an inclusion of ϕ-modules $N \subset M$, both the saturation N' and M/N' are ϕ-modules.*

Proof. Firstly, by the definition of N', we have $N' = M \cap (N' \otimes Frac(R))$. Thus M/N' is finitely generated torsion free R-module. So M/N' is free.

Secondly, let's show $\phi_M(N') \subset N'$. Note that for any $n' \in N'$, there exists $a \in R \setminus \{0\}$ such that $an \in N$. Thus we have $\phi_M(an') = \phi(a)\phi_M(n')$. Since ϕ is injective, we have $\phi(a) \neq 0$. Inside $M \otimes Frac(R)$, $\phi(a)\phi_M(n') \in N \otimes Frac(R)$ implies that $\phi_M(n') \in N \otimes Frac(R)$. Thus we have $\phi_M(n') \in N'$.

Last but not least, M is isomorphic to $N' \oplus M/N'$ as R-modules. Thus the determinant of ϕ_M is the product of the determinants of ϕ-action on N' and M/N'. By the determinant of ϕ-action on M is invertible, we know that so are N' and M/N'. Thus both M and M/N' are ϕ-modules. □

Remark 2.11. By Corollary 2.8, we have $\mu(N') \leq \mu(N)$.

Theorem 2.12. *Let M be a ϕ-module over R. Then the slopes of nonzero ϕ-submodules of M are bounded below.*

Proof. We will prove it by induction.

When M is of rank 1, it's just from Theorem 2.7.

Let's consider the case that M is of rank greater than 1.

If M has no ϕ-submodules of rank strictly smaller, it just comes from corollary 2.8.

Suppose that M has a ϕ-submodule N of rank strictly smaller. Using Lemma 2.10, we can assume that M/N is also a ϕ-module.

For any ϕ-submodule P of M, we have the following exact sequence ϕ-modules(since $N \cap P$ is invariant under ϕ action and $(P+N)/N$ is free):

$$0 \to N \cap P \to P \to (P+N)/N \to 0.$$

Note that $\mu(P)$ is a weighted average of $\mu(N \cap P)$ and $\mu((P+N)/N)$. By induction, we know that $\mu(N \cap P)$ is bounded below by some constant related to N and $\mu((P+N)/N)$ is bounded below by some constant related to M/N. Thus we have $\mu(P)$ is bounded below since we have already fixed our N at the very beginning. □

Theorem 2.13. *Let M be a nonzero ϕ-module over R. Then there is a largest ϕ-submodule of M of least slope which is module-semistable. Moreover, this largest ϕ-submodule is saturated.*

Proof. By Theorem 2.12, we know that there exists a least slope s for ϕ-submodules of M. Clearly, any ϕ-submodules of M with slope s is semistable. By Remark 2.11, we know that for a ϕ-submodule N of slope s, its saturation is still of slope s.

Moreover, ϕ-submodules of slope s is closed under taking sums:

Suppose we have two ϕ-submodules M_1, M_2. Let's consider the following exact sequence:

$$0 \to K \to M_1 \oplus M_2 \to M_1 + M_2 \to 0.$$

Note that $M_1 + M_2$ is a finitely generated torsion free R-module. So it's free thus finitely presented. Hence we know that the kernel K is finitely generated. Moreover, it's torsion free since it's a submodule of M. We know that K is free, too. So the above exact sequence lies in the category of ϕ-modules. Since K and $M_1 + M_2$ are ϕ-submodules of M, we have $\mu(K), \mu(M_1 + M_2) \geq s$. On the other hand, $\mu(M_1 \oplus M_2)$, as a weighted average of $\mu(M_1) = s$ and $\mu(M_2) = s$, is a weighted average of $\mu(K)$ and $\mu(M_1 + M_2)$. Thus we have $\mu(M_1 + M_2) = s$.

Above all, let's pick up the saturated ϕ-submodules with slope s and maximal rank: $(N_i)_{i \in I}$. As is shown, any ϕ-submodule with slope s is contained in N_i for some $i \in I$. My goal is to show $\#(I) = 1$. For any $i, j \in I$, let's consider N to be the saturation of $N_i + N_j$. As is shown, N is with slope s and maximal rank. By checking the definition, we know that N is also the saturation of N_i. By the uniqueness of saturation, we know that $N = N_i$. Thus we have $i = j$. Above all, $\#(I) = 1$. □

Corollary 2.14. *Let M be a nonzero ϕ-module over R. Then for any positive integer a, M is module-semistable if and only if $[a]_* M$ is module-semistable.*

Proof. It's clear for the 'if' part. Let's check the 'only if' part.

Note that any ϕ-submodule of the least slope is also of the least slope as ϕ^a-module. Suppose $[a]_*M$ is not semistable, then its largest ϕ^a-submodule with least slope M_1 is of strictly lower rank. Note that $\phi(M_1)$ is of the same slope of M_1 as ϕ^a-modules. Thus we have $\phi(M_1) \subset M_1$. Then, as a ϕ-submodule, M_1 is of the least slope. Thus we know that M_1 is the largest one with the least slope as a ϕ-module, which contradicts with the fact that M is semistable as a ϕ-module. □

Let M be a ϕ-module over R. A module-semistable filtration of M is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ by saturated ϕ-submodules such that each quotient M_i/M_{i-1} is module-semistable. A Harder-Narasimhan(HN) filtration is a module-semistable filtration in which

$$\mu(M_1/M_0) < \cdots \mu(M_l/M_{l-1}).$$

Remark 2.15. By Theorem 2.13, the HN filtration always exists.

Definition 2.16. Define the slope multiset of a module-semistable filtration of a ϕ-module of M as the multiset in which each slope of a successive quotient occurs with multiplicity equal to the rank of that quotient. These assemble into the lower boundary of a convex region in the xy-plane as follows: start at $(0, 0)$, then take each slope s in increasing order and append to the polygon a segment with slope s and width equal to the multiplicity of s. The result is called the slope polygon of the filtration. For the HN filtration, we call the result the HN polygon.

Remark 2.17. The HN polygon lies on or above the slope polygon of any module semistable filtration., with the same endpoint

Definition 2.18. A ϕ-module M over R or R^{bd} is said to be *é*tale if it can be obtained by base change from a ϕ-module N over R^{int}. We will call this N as the *é*tale lattice of M.

Remark 2.19. The *é*tale lattice is not unique generally.

Remark 2.20. The dual of an *é*tale ϕ-module is again *é*tale.

Definition 2.21. Define an isogeny ϕ-module over R^{int} to be a finite free R^{int}-module M quipped with an injection $\phi * M \hookrightarrow M$ whose cokernel will be killed by some power of a uniformizer of R^{int}, i.e. the injection will become isomorphism after base change to R^{bd}.

Theorem 2.22. *Let M be an isogeny ϕ-module over R^{int}. Then the natural maps $H^i(M \otimes R^{bd}) \to H^i(M \otimes R)$ for $i = 0$(under weak hypothesis) or $i = 0, 1$(under strong hypothesis) are bijective.*

Proof. Note that ϕ is defined on R^{int}. Thus the above statements come directly from the hypothesis. □

Theorem 2.23. *The base change functor from étale ϕ-modules over R^{bd} to étale ϕ-modules over R is an equivalence of categories.*

Proof. The essential surjectivity just comes from definition. It suffices to check full faithfulness. Using theorem 2.22, for any étale ϕ-modules M_1, M_2 over R^{bd}, we have $Hom(M_1, M_2) = H^0(M_1^\vee \otimes M_2) \to H(M_1^\vee \otimes M_2 \otimes R) = Hom(M_1 \otimes R, M_2 \otimes R)$ is a bijection. $\square$

Lemma 2.24. *Let M be an étale ϕ-module over R^{bd}. Then any finitely generated ϕ-stable R^{int}-submodule of M is a ϕ-module over R^{int}.*

Proof. Let N be finitely generated ϕ-stable R^{int}-submodule of M. Note that R^{int} is a DVR and R^{bd} is the fraction field. We are able to pick up an étale lattice M_0 of M such that $N \subset M_0$. It suffices to check the ϕ transforms a basis of N to another basis of N. By replacing M with $\bigwedge^{rank(N)} M$, we are able to assume that N is of rank 1 with the basis $\mathbf{n}$. Pick up a basis of M_0: $\mathbf{e}_1, \cdots \mathbf{e}_m$. Suppose $\phi_{M_0}(\mathbf{e}_i) = \sum\limits_{1 \le j \le m} a_{i,j}\mathbf{e}_j$. Then $\phi_{M_0}(x) = A\phi(x)$ where $A = (a_{i,j}) \in GL_m(R^{int})$. Thus we will find $w(\mathbf{n}) = w(\phi_{M_0}(\mathbf{n}))$. Hence the map between N and ϕ^*N is a bijection. $\square$

Theorem 2.25. *Let $0 \to M_1 \to M \to M_2 \to 0$ be a short ecsvt sequenvr of ϕ-modules over R. If any two of M_1, M_2, M are étale(except possibly M_1, M_2 in the case of weak hypothesis), then so is the third.*

Proof. First, suppose M and M_2 are *étale*. By Theorem 2.23, we know that the morphism $M \to M_2$ can descend to R^{bd}. By Lemma 2.24, we learn that the kernel the map between étale lattice of M and M_2 produces an étale lattice of M_1.

By duality, we will know if M and M_1 are étale, so is M_2.

Suppose M_1 and M_2 are étale and the strong hypothesis holds. Pick up étale lattices $N_1(N_2)$ of $M_1(M_2)$. Using theorem 2.22, we know that there is a natural bijection $Ext(N_1 \otimes R^{bd}, N_2 \otimes R^{bd}) = H^1(N_1^\vee \otimes N_2 \otimes R^{bd})$ and $H^1(N_1^\vee \otimes N_2 \otimes R) = Ext(N_1 \otimes R, N_2 \otimes R)$. Thus we may descend the exact sequence $0 \to M_1 \to M \to M_2 \to 0$ to an exact sequence of R^{bd}-modules. The following is just analysing the DVR R^{int} and its fraction field R^{bd}. $\square$

Definition 2.26. Let a ϕ-module M is of slope $s = \frac{d}{c}$ where c, d are coprime integers with $d > 0$. We say M is pure of slope s if for some ϕ-module N of rank 1 and degree $-c$, $([d]_*M) \otimes N$ is étale(it is equivalent to hold for any such N since any module with rank 1 and degree 0 is étale).

Remark 2.27. For a ϕ-module M of rank 1, let's tale N to be $M^\vee$. Then we will see that M is pure.

Remark 2.28. A module is pure of slope 0 if and only if it's étale.

Remark 2.29. Suppose M is pure of slope s and N is the corresponding rank 1 module. Note that $M^\vee \otimes N^\vee = (M \otimes N)^\vee$ is étale. Thus $M^\vee$ is pure of slope $-s$.

Lemma 2.30. *Let M be a ϕ-module over R^{bd} or R, and let a be a positive integer. Then M is pure of some slope s if and only if $[a]_*M$ is pure of slope as.*

Proof. See Lemma 1.6.3 of [Ked08] for details. Just reduce to the étale(slope 0 case. ☐

Here are some facts of pure modules(for proof, you may check section 1.6 of [Ked08]):

1. Suppose M is of slope $s = \frac{c}{d}$ where $d > 0$. Then M is pure of slope s if and only if there exists some ϕ-module N of rank 1 and degree $-c$ such that $([d]_*M) \otimes N$ is *étale*.

2. If M_1, M_2 are pure ϕ-modules of slopes s_1, s_2, then $M_1 \otimes M_2$ is pure of slope $s_1 + s_2$

3. For any rational number s, the base change functor from pure ϕ-modules of slope s over R^{bd} to pure ϕ-modules of slope s over R is an equivalence of categories.

4. Let $0 \to M_1 \to M \to M_2 \to 0$ is a short exact sequence of ϕ-modules over R. If any two of M_1, M_2, M are pure of slope s (except possibly M_1, M_2 in the case of weak hypothesis), then so is the third.

5. Let M be a pure ϕ-module over R of positive slope. Then $H^0(M) = 0$.

6. If M and N are pure ϕ-modules over R with $\mu(M) < \mu(N)$, then $Hom(M, N) = 0$.

7. If M is a pure ϕ-module over R of slope s, then M is module semistable. Moreover, if it has a ϕ-submodule N of slope s, then N is saturated and both N and M/N are pure of slope s.

8. $M_1 \oplus M_2$ is pure of slope s if and only if both M_1 and M_2 are pure of slope s.

9. Let M be a ϕ^a-module over R. The M is pure of some slope s if and only if $[a] * M$ is pure of slope $\frac{s}{a}$.

3 Slope filtration theorem

In this section, let's turn back to Robba ring.

Here is the statement of the slope filtration theorem.

Theorem 3.1. *Every module-semistable ϕ-module over the Robba ring $\mathcal{R}$ is pure. In particular, every ϕ-module over $\mathcal{R}$ admits a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ by saturated ϕ-modules whose successive quotients are pure with $\mu(M_1/M_0) < \cdots < \mu(M_l/M_{l-1})$.*

Proof. Here is the sketch of the proof. We will construct an extended Robba ring $\tilde{\mathcal{R}}$ satisfying the strong hypothesis. We will also have the following facts:

1. Every semistable ϕ-module over $\mathcal{R}$ is still semistable after basing change to $\tilde{\mathcal{R}}$.

2. Every semistable ϕ-module over $\tilde{\mathcal{R}}$ is pure.

3. For a ϕ-module M over $\mathcal{R}$, if $M \otimes \tilde{\mathcal{R}}$ is pure, then M is pure. ☐

Let us give the construction of extended Robba ring now.
For technical issues, we will make the following assumptions:
ϕ_K is a field automorphism of K and any *étale* *phi*-module is trivial.

Definition 3.2. For $r > 0$, define $\tilde{\mathcal{R}}^r$ to be set of formal sums $\sum_{i \in \mathbb{Q}} r_i u^i, r_i \in K$ satisfying the following conditions:

1. For each $c > 0$, the set of $i \in \mathbb{Q}$ such that $\|r_i\| \geq c$ is discrete.
2. We have $\|r_i\| e^{-ri} \to 0$ as $i \to -\infty$.
3. For all $s > 0$, we have $\|r_i\| e^{-si} \to 0$, as $i \to +\infty$.

The addition is just pointwise addition and the multiplication is convolution:

$$(\sum_{i \in \mathbb{Q}} r_i u^i)(\sum_{q \in \mathbb{Q}} r'_i u^i) = \sum_{i \in \mathbb{Q}} (\sum_{i' \in \mathbb{Q}} r_{i'} r'_{i-i'}) u^i.$$

Then the extended Robba ring is just $\tilde{\mathcal{R}} := \bigcup_{r>0} \tilde{\mathcal{R}}^r$ with ϕ action as follows:

$$\phi(\sum_{i \in \mathbb{Q}} r_i u^i) := \sum_{i \in \mathbb{Q}} \phi_K(r_i) u^{qi}.$$

References

[Cre98] Richard Crew. Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve. *Annales scientifiques de l'École Normale Supérieure*, Ser. 4, 31(6):717–763, 1998.

[Ked08] Kiran S. Kedlaya. Slope filtrations for relative frobenius. In Berger Laurent, Breuil Christophe, and Colmez Pierre, editors, *Représentations p-adiques de groupes p-adiques I : représentations galoisiennes et (φ, Γ)-modules*, number 319 in Astérisque, pages 259–301. Société mathématique de France, 2008.

[Ked10] Kiran S. Kedlaya. *p-adic Differential Equations.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2010.

[Laz62] Michel Lazard. Les zéros des fonctions analytiques d'une variable sur un corps valué complet. *Inst. Hautes Études Sci. Publ. Math.*, (14):47–75, 1962.