Sen's Theory

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# 1 Introduction

The first goal of this lecture is to study $\mathbb{C}_p$-representations, which we want to reduce to $K_\infty$-representations.

**Theorem 1.** *Let $K$ be a finite extension of $\mathbb{Q}_p$, the inflation-restriction map*

$$H^1(\Gamma_K, \mathrm{Gl_d(K_\infty)}) \longrightarrow \mathrm{H^1(G_K, Gl_d(\mathbb{C}_p))}$$

*is a bijection.*

In another word, any $\mathbb{C}_p$-representation $W$ of $G_K$ has a $K_\infty$-submodule $W_{K,\infty}$ which is $\Gamma_K$-stable, which we call as Sen space $D_{\mathrm{Sen}}(W) := W_{K,\infty}$. One thing we can say about it is to define a $K_\infty$-linear map $\Theta : D_{\mathrm{Sen}} \to D_{\mathrm{Sen}}$ named as the Sen operator. To see it, a group homomorphism $\mathbb{Z}_p \simeq \Gamma_{\mathbb{Q}_p} \to \mathrm{Gl_d(K_\infty)}$ is a one-parameter subgroup of $\mathrm{Gl_d(K_\infty)}$, then we can define $\Theta$ as its derivative in $\mathfrak{gl}_d(K_\infty)$. We will show that its characteristic polynomial actually lies in $K[x]$. The Sen operator shows plentiful properties of a $\mathbb{C}_p$-representations. For example,

**Theorem 2.** *Let $K$ be a finite extension of $\mathbb{Q}_p$, $(\rho, V)$ is a $\mathbb{Q}_p$-representation. Then the followings are equivalent.*

*(1) $(\rho, V)$ is $\mathbb{C}_p$-admissible.*

*(2) $\rho(I_K)$ is finite.*

*(3) $\Theta = 0$.*

By $\mathbb{C}_p$-admissible we mean $\dim_K(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p} V$. We may not have time to introduce the proof in the lecture, but I will write it down in this notes.

Another thing we can say about theorem 1 is to generalize $\mathbb{C}_p$ to other $\mathbb{Q}_p$-algebras. We use the Colmez-Sen-Tate condition, or CST condition for short. Then theorem 1 is still true for those $\mathbb{Q}_p$-algebras after passing to a finite extension of $K$. For example, the overconvergent elements for radius $r$ in $\widetilde{B}$ will form a domain that satisfies the CST condition.

# 2 The Colmez-Sen-Tate condition

Let $\widetilde{\Omega}$ be a $\mathbb{Q}_p$-algebra, with a valuation $\mathrm{val}_\Omega : \widetilde{\Omega} \to \mathbb{R} \cup \{\infty\}$, namely the following holds

1. $\mathrm{val}_\Omega(x) = +\infty$ if and only if $x = 0$.

2. $\mathrm{val}_\Omega(x+y) \geq \min \mathrm{val}_\Omega(x), \mathrm{val}_\Omega(y)$.

3. $\mathrm{val}_\Omega(xy) \geq \mathrm{val}_\Omega(x) + \mathrm{val}_\Omega(y)$.

4. $\mathrm{val}_\Omega(p) > 0$ and $\mathrm{val}_\Omega(px) = \mathrm{val}_\Omega(p) + \mathrm{val}_\Omega(x)$ if $x \in \widetilde{\Omega}$.

We assume $\widetilde{\Omega}$ to be complete with respect to the topology defined by $\mathrm{val}_\Omega$. And suppose $\widetilde{\Omega}$ is equipped by an action of $G_K$ that preserves the valuation. We say that $\widetilde{\Omega}$ satisfies the CST condition if there exists three constants $c_1, c_2, c_3 \in \mathbb{R}_{\geq 0}$ such that the following three conditions hold.

CST1 For every finite extensions $M/L$ of $K$, there exists $\alpha \in \widetilde{\Omega}^{H_M}$ such that $\mathrm{val}_\Omega(\alpha) > -c_1$ and $\mathrm{Tr}_{M_\infty/L_\infty}(\alpha) = 1$.

CST2 For every finite extension $L$ of $K$, there exists $n(L) \in \mathbb{Z}_{\geq 1}$ and an increasing sequence $\{\Omega_{L,n}\}_{n \geq n(L)}$ of closed sub $\mathbb{Q}_p$-algebras of $\widetilde{\Omega}^{H_L}$ along with maps $R_{L,n} : \widetilde{\Omega}^{H_L} \to \Omega_{L,n}$ satisfying the following properties.

1. if $x \in \widetilde{\Omega}^{H_L}$ then $\mathrm{val}_\Omega(R_{L,n}(x)) \geq \mathrm{val}_\Omega(x) - c_2$ and $R_{L,n}(x) \to x$ as $n \to \infty$.
2. if $L_2/L_1$ is finite, then $\Omega_{L_1,n} \subseteq \Omega_{L_2,n}$ and $R_{L_2,n}|_{\Omega_{L_1,n}} = R_{L_1,n}$.
3. $R_{L,n}$ is $\Omega_{L,n}$-linear and is the identity on $\Omega_{L,n}$.
4. if $g \in G_K$ then $g(\Omega_{L,n}) = \Omega_{g(L),n}$ and $g \circ R_{L,n} = R_{g(L),n} \circ g$.

CST3 For every finite extension $L$ of $K$, there exists $m(L) \geq n(L)$ such that if $\gamma \in \Gamma_L$ and $n \geq \max(n(\gamma), m(L))$, then $1-\gamma$ is invertible on $X_{L,n} = (1 - R_{L,n})(\widetilde{\Omega}^{H_L})$ and we have $\mathrm{val}_\Omega((\gamma - 1)^{-1}(x)) \geq \mathrm{val}_\Omega(x) - c_3$ if $x \in X_{L,n}$.

Here $n(\gamma)$ means $\mathrm{val}_p(\chi(\gamma) - 1)$, namely $\gamma$ topologically generates $\Gamma_{L_n}$.

By CST2, we have $\widetilde{\Omega} = \Omega_{L,n} \oplus X_{L,n}$. Let $\Omega_{L,\infty} := \cup_{n \geq 0} \Omega_{L,n}$ so that $\Omega_{L,n}$ is dense in $\widetilde{\Omega}^{H_L}$.

In the previous lectures, we've known that $\mathbb{C}_p$ is a typical example for the CST condition. For CST1, choose any $c_1 > 0$, it follows by proposition 9.2 that $\mathrm{tr}_{M_\infty/L_\infty}(\mathfrak{m}_{M_\infty}) = \mathfrak{m}_{L_\infty}$, then take some $\alpha' \in \mathfrak{m}_{M_\infty}$ such that $\mathrm{val}_p(\mathrm{tr}_{M_\infty/L_\infty}(\alpha')) < c_1$. And then $\alpha := \alpha'/\mathrm{tr}(\alpha')$ will be okay. For CST2 and CST3, it follows from the discussion in §10.

Now we can introduce the main theorem of this section.

**Theorem 3.** *If $\widetilde{\Omega}$ satisfies the CST condition, and $U$ is a cocycle on $G_K$ with values in $\mathrm{Gl_d}(\widetilde{\Omega})$, then there exists a finite extension $L/K$, and a matrix $M \in \mathrm{Gl_d}(\widetilde{\Omega})$ such that the cocycle on $G_L$ defined by $\widetilde{U} : g \mapsto M^{-1} U_g g(M)$ is trivial on $H_L$ and has values in $\mathrm{Gl_d}(\Omega_{L,n})$ for $n \gg 0$.*

**Definition 1.** For $c \in \mathbb{R}_{>0}$ and $R \subseteq \widetilde{\Omega}$ is a subring. Denote

$$\mathrm{Gl_d(c, R)} := \{\mathrm{X} \in \mathrm{Gl_d(R)} \mid \mathrm{val}_\Omega(1 - \mathrm{X}) \geq \mathrm{c}\}$$

Notice that if $X \in \mathrm{Gl_d(c, R)}$ ,then

$$\begin{aligned} X^{-1} &= (1 - (1 - X))^{-1} \\ &= 1 + (1 - X) + (1 - X)^2 + \dots \in \mathrm{Gl_d(c, R)} \end{aligned}$$

Thus $\mathrm{Gl_d(c, R)}$ is an open (and hence closed) subgroup of $\mathrm{Gl_d(R)}$.

**Lemma 1.** If $a > c_1$, $U$ is a cocycle on $H_L$ with values in $\mathrm{Gl_d(a,\widetilde{\Omega})}$, then there exists $M \in \mathrm{Gl_d(a-c_1,\widetilde{\Omega})}$ such that the cocycle $g \mapsto M^{-1}U_g g(M)$ has values in $\mathrm{Gl_d(a+1,\widetilde{\Omega})}$.

*Proof.* Take some finite extension $N/L$ such that $U(H_N) \subseteq \mathrm{Gl_d(a+1+c_1,\widetilde{\Omega})}$. It's possible because $U^{-1}(\mathrm{Gl_d(a+1+c_1,\widetilde{\Omega})})$ is a compact open neighborhood of the identity, and thus contains an open subgroup. Then by CST1, there exisits $\alpha \in \widetilde{\Omega}^{H_N}$ such that $\mathrm{tr}_{N_\infty/L_\infty}(\alpha) = 1, \mathrm{val}_\Omega(\alpha) \geq -c_1$.

Take $Q$ to be a system of representations of the cosets $H_L/H_N$, and define

$$M_Q := \sum_{h\in Q} h(\alpha)U_\alpha \in \mathrm{Gl_d(a-c_1,\widetilde{\Omega})}.$$

Then by the cocycle relation, we have $U_g g(M_Q) = M_{gQ}$. If we take $M$ to be $M_Q$, then $M_Q^{-1}U_g g(M_Q) = M_Q^{-1}M_{gQ} = 1 + M_Q^{-1}(M_{gQ} - M_Q)$. So it suffices to show that $\mathrm{val}_\Omega(M_{gQ} - M_Q) \geq a+1$. Notice that $gQ$ is also a representation of the cosets $H_L/H_N$, for any $h' \in gQ$, write it as $h' = hn$ with $h \in Q, n \in H_N$. Then

$$\begin{aligned} M_{gQ} - M_Q &= \sum_{h'\in gQ} h'(\alpha)U_{h'} - M_Q \\ &= \sum_{h\in Q} h(\alpha)U_h h(U_n) - M_Q \\ &= \sum_{h\in Q} h(\alpha)U_h(h(U_n) - 1) \end{aligned}$$

So we have $\mathrm{val}_\Omega(M_{gQ} - M_Q) \geq a+1$. □

**Corollary 1.** *If $a > c_1$, $U$ a cocycle on $H_L$, then there exists $M \in \mathrm{Gl_d(a-c_1,\widetilde{\Omega})}$ such that $M^{-1}U_g g(M) = 1$.*

*Proof.* Using lemma 1, take $\{M_k\}_{k\in\mathbb{N}}$ inductively, then the cocycle

$$g \mapsto M_k^{-1}M_{k-1}^{-1}\dots M_0^1 U_g g(M_0M_1\dots M_k)$$

has values in $\mathrm{Gl_d(a+k,\widetilde{\Omega})}$.
Since we have $M_k \in \mathrm{Gl_d(a+k-c_1,\widetilde{\Omega})}$, the product $\prod_{k\geq 0} M_k$ exists, and then we can take $M = \prod_{k\geq 0} M_k$. □

**Proposition 1.** *The inflation map*

$$H^1(\Gamma_K, \mathrm{Gl_d}(\widehat{\mathrm{K_\infty}})) \longrightarrow \mathrm{H^1(G_K, Gl_d(\mathbb{C}_p))}$$

*is a bijection.*

*Proof.* By the inflation-restriction exact sequence, it suffices to show $H^1(H_K, \mathrm{Gl_d}(\mathbb{C}_\mathrm{p}))$ is trivial.

For $U \in H^1(H_K, \mathrm{Gl_d}(\mathbb{C}_\mathrm{p}))$, pick some $a > c_1$, and a finite extension $L/K$ such that $U(H_L) \subseteq \mathrm{Gl_d(a,\mathbb{C}_p)}$. By the previous corollary, $U|_{H_L}$ is trivial. Now consider the exact sequence.

$$0 \longrightarrow H^1(H_K/H_L, \mathrm{Gl_d}(\widehat{\mathrm{L_\infty}})) \longrightarrow \mathrm{H^1(H_K, Gl_d(\mathbb{C}_p))} \longrightarrow \mathrm{H^1(H_L, Gl_d(\mathbb{C}_p))}$$

$U$ becomes trivial in $H^1(H_L, \mathrm{Gl_d}(\mathbb{C}_\mathrm{p}))$, then $U$ must come from $H^1(H_K/H_L, \mathrm{Gl_d}(\widehat{\mathrm{L_\infty}}))$, which by Hilbert's 90 is also trivial. Hence $U$ is trivial. □

**Remark 1.** This proof relies on Hilbert's 90, which is only true for the $\mathbb{C}_p$ case. For general $\widetilde{\Omega}$ that satisfies the CST conditions, we have to pass to a finite extension $L$, and the bijection still holds.

**Lemma 2.** If $a \geq c_2 + c_3 + 1$, $b \geq \max\{a + c_2, 2c_2 + 2c_3 + 1\}$, and $\gamma \in \Gamma_L$, $n \geq \max\{n(\gamma), m(L)\}$. Suppose a matrix $U = 1 + U_1 + U_2$ with

$$U_1 \in M_d(\Omega_{L,n}) \quad \mathrm{val}_\Omega(U_1) \geq a.$$
$$U_2 \in M_d(\widetilde{\Omega}^{H_L}) \quad \mathrm{val}_\Omega(U_2) \geq b.$$

Then there exists $M \in \mathrm{Gl_d(b - c_2 - c_3, \widetilde{\Omega}^{H_L})}$ such that $M^{-1}U\gamma(M) = 1 + V_1 + V_2$, with

$$V_1 \in M_d(\Omega_{L,n}) \quad \mathrm{val}_\Omega(V_1) \geq a.$$
$$V_2 \in M_d(\widetilde{\Omega}^{H_L}) \quad \mathrm{val}_\Omega(V_2) \geq b + 1.$$

*Proof.* By CST2, write $U_2 = R_{L,n}(U_2) + X$ where $X \in X_{L,n}$. And by CST3, we may write $X = (1 - \gamma)(V)$, where

$$\mathrm{val}_\Omega(R_{L,n}(U_2)) \geq b - c_2 \geq a.$$
$$\mathrm{val}_\Omega(V) \geq \mathrm{val}_\Omega(X) - c_3 \geq b - c_2 - c_3.$$

Take $M = 1 + V$ and $V_1 = U_1 + R_{L,n}(U_2)$, then

$$\begin{aligned}(1 + V)^{-1}U(1 + \gamma V) &= (1 - V + O(V^2))(1 + V_1 + (1 - \gamma)V)(1 + \gamma V) \\ &= 1 + V_1 + V_1 O(V) + O(V^2)\end{aligned}$$

Then $\mathrm{val}_\Omega(V_1) \geq \max\{\mathrm{val}_\Omega(U_1), \mathrm{val}_\Omega(R_{L,n})(U_2)\} \geq a$;$\mathrm{val}_\Omega(V_1 V) \geq a + b - c_2 - c_3 \geq b + 1$; $\mathrm{val}_\Omega(V_2) \geq 2\mathrm{val}_\Omega(V) \geq 2(b - c_2 - c_3) \geq b + 1$. ☐

**Corollary 2.** *If $b \geq 2c_2 + 2c_3 + 1$ and $U \in \mathrm{Gl_d(b, \widetilde{\Omega}^{H_L})}$, then there exists $M \in \mathrm{Gl_d(b - c_2 - c_3, \widetilde{\Omega}^{H_L})}$ such that $M^{-1}U\gamma(M) \in \mathrm{Gl_d(\Omega_{L,n})}$.*

*Proof.* Exercise. ☐

*Proof of Theorem 3.* By remark 1, we may assume $U \in H^1(\Gamma_L, \widetilde{\Omega}^{H_L})$ for a finite extension $L/K$. Choose some $n \gg 0$ such that for all $\gamma \in \Gamma_L$ that $n(\gamma) \geq n$, then $U_\gamma \in \mathrm{Gl_d(2c_2 + 2c_3 + 1, \widetilde{\Omega}^{H_L})}$.

Fix such a $\gamma$, by corollary 2, there exists $M \in \mathrm{Gl_d(c_2 + c_3 + 1, \widetilde{\Omega})}$ such that $U'_\gamma := M^{-1}U_\gamma\gamma(M) \in \mathrm{Gl_d(\Omega_{L,n})}$. For any other $\sigma \in \Gamma_L$, we also denote $U'_\sigma := M^{-1}U_\sigma\sigma(M)$. Then

$$U'_\gamma\gamma(U'_\sigma) = U'_{\gamma\sigma} = U'_{\sigma\gamma} = U'_\sigma\sigma(U'_\gamma)$$

shows $U'^{-1}_\gamma U'_\sigma\sigma(U'_\gamma) = \gamma(U'_\sigma)$. It suffices to prove the following lemma.

**Lemma 3.** If $\gamma \in \Gamma_L$ and $n \geq \max\{n(\gamma), m(L)\}$, with three matrices

$$M_1 \in \mathrm{Gl_d(> c_3, \Omega_{L,n})}$$
$$M_2 \in \mathrm{Gl_d(> c_3, \Omega_{L,n})}$$
$$B \in M_{d_1 \times d_2}(\widetilde{\Omega}^{H_L})$$

If $M_1 B M_2 = \gamma(B)$, then $B \in M_{d_1 \times d_2}(\Omega_{L,n})$.

*Proof of Lemma 3.* Let $C = B - R_{L,n}(B) \in M_{d_1\times d_2}(X_{L,n})$, then we also have $M_1CM_2 = \gamma(C)$. Then

$$\begin{aligned}(\gamma-1)C &= \gamma(C) - C \\ &= (M_1-1)CM_2 + M_1C(M_2-1) + (M_1-1)C(M_2-1)\end{aligned}$$

Hence $\mathrm{val}_\Omega((\gamma-1)C) > \mathrm{val}_\Omega(C) + c_3$. But by CST3, it shows $\mathrm{val}_\Omega(C) = +\infty$, namely $C = 0$, and $B \in M_{d_1\times d_2}(\Omega_{L,n})$. □

Back to theorem 2, then we have $U'_\sigma \in \mathrm{Gl_d}(\Omega_{\mathrm{L,n}})$ for all $\sigma \in \Gamma_L$. □

Now we are able to prove proposition 2, and together with proposition 1, theorem 1 follows.

**Proposition 2.** *Let $K$ be a finite extension of $\mathbb{Q}_p$, then the restriction map*

$$H^1(H_K, \mathrm{Gl_d}(\Omega_{\mathrm{K},\infty})) \longrightarrow \mathrm{H^1(H_K, Gl_d(\widetilde{\Omega}^{H_K}))}$$

*is a bijection.*

*Proof.* In the proof of theorem 3, we have proved the surjectivity. Now for the injectivity, if $U, U'$ be two cocycles that become cohomologous in $\mathrm{Gl_d}(\widetilde{\Omega}^{\mathrm{H_K}})$, namely there exists $M \in \mathrm{Gl_d}(\widetilde{\Omega}^{\mathrm{H_K}})$, such that $M^{-1}U_\gamma\gamma(M) = U'_\gamma$ for all $\gamma \in \Gamma_K$. Choose $\gamma$ sufficiently close to 1, then $U_\gamma, U'_\gamma \in \mathrm{Gl_d}(c_3+1, \Omega_{\mathrm{K},\infty})$. Apply lemma 3, we have $M \in \mathrm{Gl_d}(\Omega_{\mathrm{K},\infty})$. Then $U$ and $U'$ are also cohomologous in $\mathrm{Gl_d}(\Omega_{\mathrm{K},\infty})$. □

If we translate the language of cohomology into Galois representations, we then get the following theorem.

**Theorem 4.** *If $W$ is a free $\widetilde{\Omega}$-module of rank $d$ with an action of $G_K$, then for $n \gg 0$ there exists a finite extension $L/K$ and a $\Omega_{L,n}$-submodule $W_{L,n} \subseteq W^{H_L}$ which is free of rank $d$ and stable under $\Gamma_L$, and such that $W = \widetilde{\Omega} \otimes_{\Omega_{L,n}} W_{L,n}$. Moreover, taking $\Omega_{L,\infty} := \Omega_{L,n} \otimes_{L_n} L_\infty$, if $X_{L,\infty}$ is an $\Omega_{L,\infty}$-submodule of $W^{H_L}$ which is free of rank $d$ and stable under $\Gamma_L$ then $X_{L,\infty} \subseteq W_{L,\infty}$.*

*Proof.* The existence of $W_{L,n}$ just follows from theorem 3. For the second part of the theorem, fix a basis of $W_{L,\infty}$ and a basis of $X_{L,\infty}$. Let $B$ be the matrix of the basis of $X_{L,\infty}$ under the basis of $W_{L,\infty}$. Then for any $\gamma \in \Gamma_L$, consider the matrix of $\gamma$ we have

$$B^{-1}\mathrm{Mat_W}(\gamma)\gamma(\mathrm{B}) = \mathrm{Mat_X}(\gamma)$$

Choose $\gamma$ close enough to 1 to satisfy the condition of lemma 3, we have $B \in \Omega_{L,n}$ for $n \gg 0$. Then $X_{L,\infty} \subseteq W_{L,\infty}$. □

This theorem shows that $W_{L,\infty}$ is canonical, although the choice of $W_{L,n}$ may not be canonical.

## 3 Sen's Operator

In this section we set $\widetilde{\Omega} = \mathbb{C}_p$. And in theorem 4 we need not pass to a finite extension $L$. In this case, we denote $D_{\text{Sen}}(W) := W_{K,\infty}$.

**Definition 2.** For a $\mathbb{C}_p$-representation of $G_K$, then a vector $v \in W^H$ is $K$-finite if the set $\Gamma_K.v$ generates a finite $K$-subspace of $W$.

For example, choose a $W_{K,n}$ for $n \gg 0$, then elements of $W_{K,n}$ must be all $K$-finite. Denote the set of finite elements by $W_0$, then we have $D_{\text{Sen}}(W) = W_{K,n} \otimes K_\infty \subseteq W_0 \subseteq D_{\text{Sen}}(W)$. Hence $D_{\text{Sen}}(W)$ is exactly the finite vectors in $W^H$.

As a corollary, taking $D_{\text{Sen}}$ is left exact. And then by dimensional analysis, it is also exact.

Fix a basis $\{e_1, e_2, \ldots, e_d\}$ of $D_{\text{Sen}}(W)$, by theorem 3 it corresponds to a cocycle $U : \Gamma_K \to \mathrm{Gl_d(K_n)}$ for some $n \gg 0$. Then on $\Gamma_{K_n}$ the map $U$ is actually a group homomorphism, and is hence $\mathbb{Z}_p$-linear. We may also assume on $\Gamma_{K_n}$, $\log \circ U$ is well defined, for example, $U(\Gamma_{K_n}) \subseteq \mathrm{Gl_d(1, K_n)}$. Then there exist a unique linear endomorphism $\Theta$ of $D_{\text{Sen}}(W)$ defined by

$$\mathrm{Mat}(\Theta) = \frac{\log \mathrm{U}_\gamma}{\log_\mathrm{p} \chi(\gamma)}, \quad \gamma \in \Gamma_{\mathrm{K_n}}.$$

which is independent on the choice of $\gamma$ since $U$ is $\mathbb{Z}_p$-linear on $\Gamma_{K_n}$. In another word, we have

$$\gamma.v = \exp(\log(\chi(\gamma)) \cdot \Theta).v$$

for any $\gamma \in \Gamma_{K_n}$ and $v \in D_{\text{Sen}}(W)$. We can also rewrite the definition of $\Theta$

$$\Theta(v) = \frac{1}{\log_p \chi(\gamma)} \lim_{\substack{t \in \mathbb{Z}_p \\ t \to 0}} \frac{\gamma^t.v - v}{t}, \quad \text{for } t \in D_{\text{Sen}}(W).$$

in the above formula it's easy to see $\Theta$ commutes with the whole $\Gamma_K$.

**Definition 3.** The $\Theta : D_{\text{Sen}}(W) \to D_{\text{Sen}}(W)$ defined above is called the **Sen's operator** of $W$, and its eigenvalues are called **Sen weights**.

**Proposition 3.** *If $W$ is a $\mathbb{C}_p$-representation of $G_K$, then the characteristic polynomial of $\Theta_W$ has coefficients in $K$.*

*Proof.* Since $\Theta$ commutes with any element $\gamma \in \Gamma_K$, then for any $\gamma \in \Gamma_K$, we have

$$\mathrm{Mat}(\Theta) \cdot \mathrm{U}_\gamma = \mathrm{Mat}(\Theta \circ \gamma) = \mathrm{Mat}(\gamma \circ \Theta) = \mathrm{U}_\gamma \cdot \gamma(\mathrm{Mat}(\Theta)).$$

Hence $\gamma(\mathrm{Mat}(\Theta))$ is similar to $\mathrm{Mat}(\Theta)$ for all $\gamma \in \Gamma_K$. It follows that the characteristic polynomial of $\mathrm{Mat}(\Theta)$ is invariant under $\Gamma_K$, namely it lies in $K[X]$. □

**Example 1.** *Take $K = \mathbb{Q}_p(\zeta_p)$, then $\chi(G_K) = (1 + p\mathbb{Z}_p)^\times$ on which $\log_p$ is convergent. Then for any $\lambda \in \mathbb{Z}_p$, define $\chi^\lambda(\cdot) := \exp(\log_p(\chi(\cdot)) \cdot \lambda)$, then we can take $W := \mathbb{C}_p(\lambda)$. The Sen's operator $\Theta$ is multiplication by $\lambda$, then $W$ has the weight $\lambda$.*

# 4 Hodge-Tate representations

We set $B_{\mathrm{HT}} := \bigoplus_{i\in\mathbb{Z}} \mathbb{C}_p(i)$, then we can talk about the $B_{\mathrm{HT}}$-admissible representations.

**Definition 4.** Suppose $V$ is a $\mathbb{Q}_p$-representation of $G_K$, then $W$ is **Hodge-Tate** or **$B_{\mathrm{HT}}$-admissible** if the following holds

$$\dim_{\widehat{K_\infty}} (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K} = \dim_{\mathbb{Q}_p} V.$$

where we denote $D_{\mathrm{HT}} := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{G_K}$.

A strict definition for $B$-admissible representations can be found in many notes. It's a powerful method to detect properties of a representation. For example, in following lectures we will introduce the domain $B_{\mathrm{dR}}$, then $V$ is a de Rham representation iff it is $B_{\mathrm{dR}}$-admissible. We will not discuss it here, and just take two examples. In the next section we will talk about $\mathbb{C}_p$-admissible representations.

**Definition 5.** Note that if $V$ is Hodge-Tate, then $D_{\mathrm{HT}} = \bigoplus_{i\in\mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(i))^{G_K}$. The $i$ for which $\dim(V \otimes \mathbb{C}_p(-i))^{G_K} > 0$ are called the Hodge-Tate weights of $V$, and in this case $\dim(V \otimes \mathbb{C}_p(-i))^{G_K}$ is called the multiplicity of $i$.

**Theorem 5.** *Suppose $W$ is a $\mathbb{C}_p$-representation of $G_K$, then the following three things are equivalent.*

*1) $W$ is isomorphic to $\mathbb{C}_p(h_1) \oplus \ldots \oplus \mathbb{C}_p(h_d)$.*

*2) $\Theta$ is semisimple on $D_{\mathrm{Sen}}(W)$ with eigenvalues $h_1, h_2, \ldots, h_d \in \mathbb{Z}$.*

*Proof.* 1) obviously implies 2). Now for 2) $\Rightarrow$ 1), we can write

$$D_{\mathrm{Sen}}(W) = \bigoplus_{h\in\mathbb{Z}} D_{\mathrm{Sen}}(W)^{\Theta=h}$$

where each summand is stable under $\Gamma_K$. For each $h \in \mathbb{Z}$, $\Theta = h$ on $D_{\mathrm{Sen}}(W)^{\Theta=h}$. Thus there exists some open subgroup $\Gamma_0 \subseteq \Gamma_K$ such that $\gamma.v = \chi(\gamma)^h(v)$ for each $\gamma \in \Gamma_0$ and $v \in D_{\mathrm{Sen}}(W)^{\Theta=h}$. Use Hilbert's 90 we can see it's true on the whole $\Gamma_K$. Hence $D_{\mathrm{Sen}}(W)^{\Theta=h}$ is a direct sum of $K_\infty(h)$, which implies 1). □

So a $\mathbb{Q}_p$-representation $V$ is Hodge-Tate if $W = V \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ satisfies the conditions in theorem 5. Example 1 shows that semi-simple is not enough to force a $\mathbb{Q}_p$-representation to be Hodge-Tate.

**Corollary 3.** *If there is an exact sequence of $G_K$ representations*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

*and if $U, W$ are Hodge-Tate with no weights in common, then $V$ is Hodge-Tate.*

**Example 2.** *If $U$ and $W$ has common weights, then corollary 3 may not be true. Consider a two-dimensional $\mathbb{Q}_p$-representation with*

$$\rho(g) = \begin{pmatrix} 1 & \alpha(g) \\ 0 & 1 \end{pmatrix}, \quad \alpha : G_K \to (\mathbb{Q}_p, +)$$

*where $\alpha$ is such that $\alpha(I_K)$ is infinite, which is possible since $I(K^{\mathrm{ab}}/K) \simeq \mathcal{O}_K^*$. Then we get an exact sequence*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow V \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

*But if $V$ is Hodge-Tate, then its weights must be $0, 0$, and so $\Theta$ must be $0$, and by theorem 2, $I_K$ must have finite image.*

**Proposition 4.** *If $W$ satisfies the condition in theorem 5, then if $X \subseteq D_{\mathrm{Sen}}(W)$ is a finite dimensional $K_n$ vector space stable under $\Gamma_K$ and has a basis on which $\Gamma_{K_n}$ acts by integer powers of $\chi$, then $X \subseteq W_{K,n}$.*

*Proof.* Choose a basis $\{e_1, e_2, \ldots, e_d\}$ of $D_{\mathrm{Sen}}(W)$ that $\Gamma_K$ acts on it by $\chi^{h_i}$. Suppose $e_{d+1} \in D_{\mathrm{Sen}}(W)$ such that $g(e_{d+1}) = \chi(g)^{h_{i+1}} e_{d+1}$ for all $g \in \Gamma_{K_n}$. The elements $e_1, e_2, \ldots, e_{d+1}$ are linear dependant in $D_{\mathrm{Sen}}(W)$ so that we can write

$$\sum_{i=1}^{d+1} \lambda_i e_i = 0.$$

Suppose that this relation has minimal length and then by letting $\Gamma_{K_n}$ acts, we get

$$\sum_{i=1}^{d+1} g(\lambda_i)\chi(g^{h_i})e_i = 0.$$

then for nonzero $\lambda_i, \lambda_j$ we have

$$\frac{g(\lambda_i)}{\lambda_i}\chi(g)^{h_i - 1} = \frac{g(\lambda_j)}{\lambda_j}\chi(g)^{h_j - 1}.$$

namely

$$\lambda_i/\lambda_j \in K_\infty^{\Gamma_{K_n} = \chi^{h_j - h_i}} = \begin{cases} 0 & \text{if } h_i \neq h_j. \\ K_n & \text{if } h_i = h_j. \end{cases}$$

So the relation can have coefficients in $K_n$, which means $e_{d+1} \in W_{K,n}$. □

**Example 3.** *A typical example for Hodge-Tate representations is the Tate module of Tate curves, which is an elliptic curve $E/K$ that is isomorphic to $K^*/q^{\mathbb{Z}}$ for some $|q| < 1$.*

*Looking at the group structure of $\overline{K}^*$, we have an exact sequence*

$$1 \longrightarrow \mu_{p^n}(\overline{K}) \longrightarrow (\overline{K}^*/q^{\mathbb{Z}})[p^n] \longrightarrow q^{1/p^n\mathbb{Z}}/q^{\mathbb{Z}} \longrightarrow 1.$$

*Then by taking limits, we have an exact sequence*

$$1 \longrightarrow \varprojlim \mu_{p^n}(\overline{K}) \longrightarrow T_p(E_q) \longrightarrow \varprojlim q^{1/p^n\mathbb{Z}}/q^{\mathbb{Z}} \longrightarrow 1.$$

$$0 \longrightarrow \mathbb{Z}_p(1) \longrightarrow T_p(E_q) \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

*So after tensoring with $\mathbb{Q}_p$, we have*

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow V_p(E_q) \longrightarrow \mathbb{Q}_p \longrightarrow 0.$$

*So its Tate module is Hodge-Tate with weights $0$ and $1$.*

There are two big generalizations of example 3.

**Theorem 6** (Raynaud)**.** *Any abelian variety over $K$ of dimension $g$ is Hodge-Tate with weights $0, 1$, each of multiplicity $g$.*

**Theorem 7** (Faltings)**.** *Let $X/K$ be a proper and smooth variety of dimension $d$, then for $0 \leq n \leq 2d$, $H^n_{\text{et}}(X, \mathbb{Q}_p)$ is Hodge-Tate.*

# 5 A sketch to the proof of theorem 2

I'm not going to talk about this section in class. If you have interest, then you are free to have a look at this section and its reference.

Firstly, to show (1) $\Leftrightarrow$ (3), it suffices to show the stronger result.

**Proposition 5.** *If $W$ is a $\mathbb{C}_p$-representation of $G_K$. The kernel of $\Theta$ is the $\mathbb{C}_p$-subspace of $W$ generated by the elements invariant under $G$. Namely $W^G \otimes_K \mathbb{C}_p = \ker\Theta$.*

*Proof.* Denote $\ker\Theta$ by $X$ .Obviously $W^G \subseteq X = \ker\Theta$. So it suffices to show that $W^G$ actually generates $X$, in another word, to find a $K_\infty$-basis $\{e_1, e_2, \ldots, e_n\}$ of $D_{\text{Sen}}(X)$ such that each $e_i$ is fixed by $\Gamma_K$. But since $\Theta(e_i) = 0$, by the formula

$$\Theta(v) = \frac{1}{\log_p \chi(\gamma)} \lim_{\substack{t \in \mathbb{Z}_p \\ t \to 0}} \frac{\gamma^t.v - v}{t}, \quad \text{for } t \in D_{\text{Sen}}(W).$$

the $\Gamma_K$-orbit of $e_i$ is finite. Hence $e_i$ is fixed by some open subgroup $\Gamma_i$. Take $\Gamma' := \bigcap_i \Gamma_i$, the basis $\{e_1, \ldots e_n\}$ is fixed by the open subgroup $\Gamma'$. Using Hilbert's 90, there exists a basis of $D_{\text{Sen}}(X)$ fixed by the whole $\Gamma$. ☐

**Corollary 4.** *In theorem 2, (1) and (3) are equivalent.*

To see (2) $\Leftrightarrow$ (3), we need the following theorem. The proof for it is much more complicated, you can find it in [1].

**Theorem 8.** *Let $(\rho, V)$ be any $\mathbb{Q}_p$-representation, and let $G := \rho(I_K) \subseteq \mathrm{Gl_d}(\mathbb{Q}_\mathrm{p})$. $G$ is compact since $I_K$ is compact and thus is a closed subgroup of a Lie group, so is itself a Lie group. Let its Lie algebra be $\mathfrak{g}$. Thus*

$$\mathfrak{g} = \{M \in \mathfrak{gl}_d(\mathbb{Q}_p) \mid \exp(tM) \in G, t \to 0\}$$

*This is a subspace of $\mathfrak{gl}_d(\mathbb{Q}_p)$, and $\dim G = \dim \mathfrak{g}$. Then $\mathfrak{g}$ is the smallest subspace of $\mathfrak{gl}_d(\mathbb{Q}_p)$ such that $\Theta \in \mathfrak{gl}_d(\mathbb{C}_p)$ lies in $\mathfrak{g} \otimes \mathbb{C}_p \subseteq \mathfrak{gl}_d(\mathbb{C}_p)$.*

*Proof.* See [1], p100. ☐

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# References


[1] Theory of $p$-adic Galois Representations, Jean-Marc Fontaine, Yi Ouyang.

[2] Galois Representations, Joel Bellaiche.