IWASAWA COHOMOLOGY

1. Euler-Poincaré characteristic

1.1. $D^{\psi=1}$ and $D/(\psi-1)$.

Lemma 1.1. *Let D be an étale φ-module over $E_{\mathbb{Q}_p}$. Then :*

1. *$D^{\psi=1}$ is compact.*
2. *$\dim_{\mathbb{F}_p}(D/(\psi-1)) < +\infty$.*

Proof. Choose a basis $e_1, \dots, e_d$ of D. Then by definition $\varphi(e_1), \dots, \varphi(e_d)$ is a basis, too. Let (a_{ij}) be the matrix relating the two bases, and let $c = \inf v_E(a_{ij})$. For $x \in D$, write $x = \sum_i x_i \varphi(e_i)$ for $x_i \in E_{\mathbb{Q}_p}^+$ by étaleness. Let $v_E(x) := \inf_i v_E(x_i)$. Then, from

$$\psi(x) = \sum_i \psi(x_i) e_i,$$

we have

$$v_E(\psi(x)) \geq c + \inf_i v_E(\psi(x_i)).$$

Since x_i are in $E_{\mathbb{Q}_p}^+$, and ψ preseves $E_{\mathbb{Q}_p}^+$ and $\psi(\bar{\pi}^{p^k} x_i) = \bar{\pi}^k \psi(x_i)$, we get $v_E(\psi(x_i)) \geq [v_E(x_i)/p]$. Therefore,

$$v_E(\psi(x)) \geq c + [v_E(x)/p] \tag{1}$$

implies that if $v_E(x) < \frac{p(c-1)}{p-1}$, then $v_E(\psi(x)) > v_E(x)$. So $D^{\psi=1}$ is a subset of the set

$$M := \{x : v_E(x) \geq \frac{p(c-1)}{p-1}\} \subset \sum_{i=1}^{d} \bar{\pi}^k \mathbb{F}_p[[\bar{\pi}]].\varphi(e_i)$$

for appropriate k. This set is compact, and $D^{\psi=1}$ is a closed set.

Above proof shows that $\psi - 1$ is bijective on D/M. Thus, it suffices to prove that $(\psi-1)D$ contains $\{x : v_E(x) \geq c'\}$ for some c' to prove (2). Let $\varphi(x_i) = \sum_{j=1}^d b_{ij} e_j$ and $c_0 = \inf_{i,j} v_E(b_{ij})$. Then we have $x = \sum_{j=1}^d y_j e_j$ for $y_j = \sum_{i=1}^d x_i b_{ij}$ and thus $v_E(y_j) \geq c_0 + v_E(x)$. Then, $\varphi(x) = \sum_{i=1}^d \varphi(y_i)\varphi(e_i)$ shows that

$$v_E(\phi(x)) = p \inf v_E(y_j) \geq p v_E(x) + p c_0. \tag{2}$$

So, if $v_E(x) \geq \frac{-pc_0}{p-1} + 1$, then $v_E(\varphi^n(x)) \geq p^n$, which implies that $y = \sum_{i=1}^{\infty} \varphi^i(x)$ converges in D. Then, $(\psi-1)(y) = x$ shows that $(\psi-1)D$ contains the set $\{x : v_E(x) \geq \frac{-pc_0}{p-1} + 1\}$. □

Proposition 1.2. *Let D is an étale φ-module over A_K (resp. over B_K). Then :*

1. *$D^{\psi=1}$ is compact (resp. locally compact).*
2. *$D/\psi-1$ is finitely generated over $\mathbb{Z}_p$ (resp. over $\mathbb{Q}_p$).*

Proof. Observe first that we can assume $K = \mathbb{Q}_p$ because A_K is a finite free module over $A_{\mathbb{Q}_p}$. Also, the statement for B_K follows from the statement for A_K. So we only need to consider D over $A_{\mathbb{Q}_p}$.

Then note that $D^{\psi=1} \cong \varprojlim (D/p^n D)^{\psi=1}$. Thus, it suffices to prove the statement of compactness at each finite level, since then we get a closed subset of a product of compact spaces. We have proven the result for $n = 1$ in the previous lemma. The argument is completed by induction which is to compute that by the same analysis of action of ψ that we have done, we can express each $(D/p^N)^{\psi=1}$ as a closed subset of $\pi^{-k_n}(D/p^{N-1})$ for a suitable (positive) choice of k_n.

Similarly, we know that $(D/p)/\psi - 1$ is finite dimensional over $\mathbb{F}_p$. It suffices to prove that $D/\psi - 1$ does not contain p-divisible elements, i.e. if there exists x such that there exist y_n with $x = (\psi - 1)y_n + p^n\mathbb{Z}_p$ for all n, then in fact $x = (\psi - 1)D$. If $m \geq n$, then $y_m - y_n \in (D/p^n)^{\psi=1}$, and this is compact. Hence, there is a subsequence of the sequence y_m which converges modulo p^n. Doing this for each n, we can get a diagonal subsequence that converges modulo p^n for all n and which has a limit y. By passing to the limit, we get $x = (\psi - 1)y$. □

1.2. **The Γ-action.** If $p \neq 2$, we let $\Gamma_0 = \Gamma_{\mathbb{Q}_p} \cong \mathbb{Z}_p^*$. Let $\Gamma_n \subset \Gamma_0$ with $\Gamma_n \cong 1 + p^n\mathbb{Z}_p$ for $n \geq 1$. Then $\Gamma_0 \cong \Delta \times \Gamma_1$ where $\Delta = \mu_{p-1}$ and $\Gamma_n \cong \varprojlim \Gamma_n/\Gamma_{n+m}$. Then we define

$$\mathbb{Z}_p[[\Gamma_n]] := \varprojlim \mathbb{Z}_p[\Gamma_n/\Gamma_{n+m}].$$

Then, for $n \geq 1$, let γ_n be a topological generator of Γ_n, so that $\Gamma_n \cong \gamma_n^{\mathbb{Z}_p}$. We have isomorphisms

$$\mathbb{Z}_p[[\Gamma_n]] \xleftarrow{\sim} \mathbb{Z}_p[[T]] \xrightarrow{\sim} A^+_{\mathbb{Q}_p}$$

obtained by

$$\gamma_n \leftarrow T \rightarrow \pi.$$

Then we have $\mathbb{Z}_p[[\Gamma_0]] \cong \mathbb{Z}_p[\Delta] \times \mathbb{Z}_p[[\Gamma_1]]$. Furthermore, for $n \geq 1$ we define

$$\mathbb{Z}_p\{\{\Gamma_n\}\} := (\mathbb{Z}_p[[\Gamma_n]][(\gamma_n - 1)^{-1}])^\wedge$$

and this is isomorphic as a ring to $A_{\mathbb{Q}_p}$. Finally, $\mathbb{Z}_p\{\{\Gamma_0\}\} \cong \mathbb{Z}_p[\Delta] \times \mathbb{Z}_p\{\{\Gamma_1\}\}$. We can also go modulo p to get $\mathbb{F}_p\{\{\Gamma_n\}\} \cong E_{\mathbb{Q}_p}$ as a ring.

If $M \cong M/M_i$ is a topological $\mathbb{Z}_p$-module with a continuous action of Γ_n. Then the group algebra $\mathbb{Z}_p[[\Gamma_n]]$ acts continuously on M. Furthermore, if the element $\gamma_n - 1$ has a continuous inverse, then $\mathbb{Z}_p\{\{\Gamma_n\}\}$ also acts continuously on M.

Lemma 1.3. *(1) If $n \geq 1, v_E(\gamma_n(\bar{\pi}) - \bar{\pi}) = p^n v_E(\bar{\pi})$.*
(2) For all $x \in E_{\mathbb{Q}_p}, v_E(\gamma_n(x) - x) \geq v_E(x) + (p^n - 1)v_E(\bar{\pi})$.

Proof.

$$\gamma_n(\bar{\pi}) - \bar{\pi} = \gamma_n(1 + \bar{\pi}) - (1 + \bar{\pi})$$

and since γ_n acts by the cyclotomic character, we get

$$\gamma_n(\bar{\pi}) - \bar{\pi} = (1 + \bar{\pi})((1 + \bar{\pi})^u - 1)^{p^n}$$

for some p-adic unit u. Thus (1) follows.

In general, if $x \in E_{\mathbb{Q}_p}$ equals $\sum_{k=k_0}^{\infty} a_k \bar{\pi}^k$, then $v_E(x) = k_0 v_E(\bar{\pi})$ and RHS of (2) becomes $p^n v_E(\bar{\pi}) + (k_0 - 1)v_E(\bar{\pi})$. We have

$$\frac{\gamma_n(x) - x}{\gamma_n(\bar{\pi}) - \bar{\pi}} = \sum_{k=k_0}^{\infty} a_k \frac{\gamma_n(\bar{\pi})^k - \bar{\pi}^k}{\gamma_n(\bar{\pi}) - \bar{\pi}}.$$

Thus, the result follows from (1) and the observation that

$$\frac{\gamma_n(\bar{\pi})^k - \bar{\pi}^k}{\gamma_n(\bar{\pi}) - \bar{\pi}} \geq (k-1)v_E(\bar{\pi}).$$

□

Proposition 1.4. *Let D be an étale (φ, Γ)-module of dimension d over $E_{\mathbb{Q}_p}$. Assume $n \geq 1, (i,p) = 1$. Then :*

(1) $\gamma \in \Gamma$ induces an isomorphism $\epsilon^i \varphi^n(D) \cong \epsilon^{\chi(\gamma)i}\varphi^n(D)$.

(2) $\gamma_n - 1$ admits a continuous inverse on $\epsilon^i\varphi^n(D)$. Moreover, if $\{e_1, \dots, e_d\}$ is a basis of D, then

$$\mathbb{F}_p\{\{\Gamma_n\}\}^d \xrightarrow{\sim} \varphi^n(D)$$

given by

$$(\lambda_1, \dots, \lambda_d) \to \lambda_1 * \epsilon^i\varphi^n(e_1) + \dots + \lambda_d * \epsilon^i\varphi^n(e_d)$$

is a topological isomorphism. ($*$ denotes the group element acting.)

Remark 1. This is the most delicate point of all the computations.

Proof. (1) follows from the action of Γ. For (2), we claim that if the assertion is true for $(n+1)$, then it is true for n. This is so, because we have direct sum decompositions

$$\epsilon^i\varphi^n(D) \cong \epsilon^i\varphi^n(\oplus_{j=0}^{p-1}\epsilon^j\varphi(D)) = \oplus_{j=0}^{p-1}\epsilon^{i+p^n j}\varphi^{n+1}(D)$$

and

$$\mathbb{F}_p\{\{\Gamma_n\}\} \cong \mathbb{F}_p\{\{\Gamma_{n+1}\}\} \oplus \dots \oplus \gamma_n^{p-1}\mathbb{F}_p\{\{\Gamma_{n+1}\}\}$$

along with the identification $\frac{1}{\gamma_n - 1} = \frac{1}{\gamma_{n+1}-1}(1 + \gamma_n + \dots + \gamma_n^{p-1})$. So we can and will assume n to be sufficiently large.

Recall $v_E(x) = \inf_i v_E(x_i)$ for $x = \sum_i x_i e_i$. We can assume, because n is large, $v_E(\gamma_n(e_i) - e_i) \geq 2v_E(\bar{\pi})$ by an induction argument, this implies $v_E(\gamma_n(x) - x) \geq v_E(x) + 2v_E(\bar{\pi})$ by the previous lemma. Now, we have $\chi(\gamma_n) = 1 + p^n u$ as before, with $u \in \mathbb{Z}_p^*$. Hence, we have

$$\gamma_n(\epsilon^i\varphi^n(x)) - \epsilon^i\varphi^n(x) = \epsilon^i(\epsilon^{ip^n u}\varphi^n(\gamma_n(x)) - \varphi^n(x)) = \epsilon^i\varphi^n(\epsilon^{iu}\gamma_n(x) - x).$$

Thus, it suffices to prove

$$x \to f(x) = \epsilon^{iu}\gamma_n(x) - x$$

has a continuous inverse on D and D is a $\mathbb{F}_p\{\{f\}\}$-module with basis $\{e_1, \dots, e_d\}$.

Let $\alpha = \epsilon^{iu} - 1, iu \in \mathbb{Z}_p^*$. Then $v_E(\alpha) = v_E(\bar{\pi})$. Hence,

$$v_E\left(\frac{f}{\alpha}(x) - x\right) \geq v_E(x) + v_E(\bar{\pi}).$$

This implies that the sum $g = \sum_{n=0}^{\infty}(1 - \frac{f}{\alpha})^n$ converges, and thus g is an inverse for $\frac{f}{\alpha}$ with $v_E(g(x) - x) \geq v_E(x) + v_E(\bar{\pi})$. Therefore, f has an inverse $g(\frac{x}{\alpha})$ and $v_E(f^{-1}(x) - \frac{x}{\alpha}) \geq v_E(x)$.

By induction, for all $k \in \mathbb{Z}$, we have

$$v_E(f^k(x) - \alpha^k x) \geq v_E(x) + (k+1)v_E(\bar{\pi}).$$

Let $M = E^+_{\mathbb{Q}_p}e_1 \oplus \dots \oplus E^+_{\mathbb{Q}_p}e_d$, then f^k induces

$$M/\bar{\pi}M \cong \alpha^k M/\alpha^{k+1}M \cong \bar{\pi}^k M/\bar{\pi}^{k+1}M.$$

Therefore, $f^k\mathbb{F}_p\{\{f\}\}e_1 \oplus \ldots \oplus f^k\mathbb{F}_p\{\{f\}\}e_d$ is dense in $\bar{\pi}^k M$ and hence is equal to it by compactness.

$\square$

Corollary 1.5. *$\gamma - 1$ has a continuous inverse on $D^{\psi=0}$, and $D^{\psi=0}$ is a free $\mathbb{F}_p\{\{\Gamma_0\}\}$-module with basis $\{\epsilon\varphi(e_1), \ldots, \epsilon\varphi(e_d)\}$.*

Proof. The proof is the same as (2) for $n+1$ implies (2) for n in the previous proposition for $n = 0$ using $\gamma_1 = \gamma_0^{p-1}$. $\square$

Proposition 1.6. *If D is an étale (φ, Γ)-module over A_K or B_K, then $\gamma - 1$ has a continuous inverse on $D^{\psi=0}$.*

Proof. The statement for B_K follows from the statement for A_K. We further observe that

$$(\mathrm{Ind}_K^{\mathbb{Q}_p} D)^{\psi=0} = (\mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\mathbb{Z}[\Gamma_{\mathbb{Q}_p}], D))^{\psi=0} = \mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\mathbb{Z}[\Gamma_{\mathbb{Q}_p}], D^{\psi=0}).$$

Since $\mathbb{Z}[\Gamma_{\mathbb{Q}_p}$ is a finite free $\mathbb{Z}[\Gamma_K]$-module, the statement of the theorem for D is equivalent to that for $\mathrm{Ind}_K^{\mathbb{Q}_p} D$. Thus we reduce to the case $K = \mathbb{Q}_p$.

Since $D^{\psi=0} \to (D/p)^{\psi=0}$ is surjective, we have the exact sequence

$$0 \to (pD)^{\psi=0} \to D^{\psi=0} \to (D/p)^{\psi=0} \to 0.$$

Since everything is p-adically complete, it suffices to verify the result modulo p, which is the previous corollary. $\square$

1.3. **Compuation of Galois cohomology groups.**

Proposition 1.7. *Let $C_{\psi,\gamma}$ be the complex*

$$0 \to D(V) \xrightarrow{(\psi-1,\gamma-1)} D(V) \oplus D(V) \xrightarrow{(\gamma-1)pr_1-(\psi-1)pr_2} D(V) \to 0.$$

Then we have a commutative diagram of complexes

$$\begin{array}{ccccccccccc}
C_{\varphi,\gamma}: 0 & \longrightarrow & D(V) & \longrightarrow & D(V) & \oplus & D(V) & \longrightarrow & D(V) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle Id} & & \Big\downarrow{\scriptstyle -\psi} & & \Big\downarrow{\scriptstyle Id} & & \Big\downarrow{\scriptstyle -\psi} & & \\
C_{\psi,\gamma}: 0 & \longrightarrow & D(V) & \longrightarrow & D(V) & \oplus & D(V) & \longrightarrow & D(V) & \longrightarrow & 0
\end{array}$$

that induces an isomorphism on cohomology.

Proof. The diagram commutes since $(-\psi)(\varphi - 1) - \psi - 1$ and ψ commutes with γ. ψ is surjective, hence the cokernel complex is 0. The kernel complex is given by

$$0 \to 0 \to D(V)^{\psi=0} \xrightarrow{\gamma-1} D(V)^{\psi=0} \to 0,$$

and by the previous proposition, its cohomology vanishes. $\square$

Theorem 1.8. *If V is a $\mathbb{Z}_p$ or $\mathbb{Q}_p$-representation of G_K, then the complex $C_{\psi,\gamma}(K, V)$ computes the Galois cohomology of V :*

(1) $H^0(G_K, V) = D(V)^{\psi=1,\gamma=1} = D(V)^{\varphi=1,\gamma=1}$.
(2) $H^2(G_K, V) \cong \frac{D(V)}{(\psi-1,\gamma-1)}$.
(3) There is an exact sequence

$$0 \to \frac{D(V)^{\psi=1}}{\gamma - 1} \to H^1(C_{\psi,\gamma}(K, V)) \to \left(\frac{D(V)}{\psi - 1}\right)^{\gamma=1} \to 0.$$

We further express one of the terms in (3) more explicitly, this helps in the Euler-Poincaré computation later.

Definition 1.9. Define $C(V) = (\varphi - 1)D^{\psi-1} \subset D^{\psi=0}$.

The exact sequence

$$0 \to D(V)^{\varphi=1} \to D(V)^{\psi=1} \to C(V) \to 0$$

induces an exact sequence

$$0 \to \frac{D(V)^{\varphi=1}}{\gamma - 1} \to \frac{D(V)^{\psi=1}}{\gamma - 1} \to \frac{C(V)}{\gamma - 1} \to 0$$

since $C(V)^{\gamma=1} \subset (D^{\psi=0})^{\gamma=1} = 0$, since $\gamma - 1$ has an inverse.

Proposition 1.10. *If D is an étale (φ, Γ)-module of dimension d over $E_{\mathbb{Q}_p}$, then $C = (\varphi - 1)D^{\psi=1}$ is a free $\mathbb{F}_p[[\Gamma_0]]$-module of rank d.*

Proof. It suffices to prove that C contains $\{\epsilon\varphi(e_1), \ldots, \epsilon\varphi(e_d)\}$, for some basis $\{e_1, \ldots, e_d\}$ of D over $E_{\mathbb{Q}_p}$. (By earlier propositions) This can be extracted from any basis $\{f_1, \ldots, f_d\}$ using properties of φ and ψ. $\square$

Theorem 1.11. *If V is a finite $\mathbb{Z}_p$-representation of G_K, then*

$$\chi(V) = \prod_{i=0}^{2} \mid H^i(G_K, V) \mid^{(-1)^i} = \mid V \mid^{-[K:\mathbb{Q}_p]} .$$

Proof. By Shapiro's Lemma, we have

$$H^i(G_K, V) \cong H^i(G_{\mathbb{Q}_p}, \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}_p}} V).$$

Since $\mid \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}_p}} V \mid = \mid V \mid^{[K:\mathbb{Q}_p]}$, we can assume $K = \mathbb{Q}_p$. By multiplicativity of dimensions in exact sequences, we further reduce to the case that V is an $\mathbb{F}_p$-representation of $G_{\mathbb{Q}_p}$. Then we have :

$$\mid H^0 \mid = \mid D(V)^{\varphi=1.\gamma=1} \mid;$$

$$\mid H^1 \mid = \mid \frac{D(V)^{\varphi=1}}{\gamma - 1} \mid . \mid \frac{C(V)}{\gamma - 1} \mid . \mid \left(\frac{D(V)^{\gamma-1}}{\psi - 1} \right) \mid;$$

$$\mid H^2 \mid = \mid \frac{D(V)}{\psi - 1, \gamma - 1} \mid .$$

Then, we get that $|H^0||H^2||H^1|^{-1} = |\frac{C(V)}{[\gamma-1]}|^{-1}$, because $D(V)^{\varphi=1}$ and $\frac{D(V)}{\psi-1}$ are finite groups since the ranks are d. And for finite groups M, $|M^{\gamma=1}| = |\frac{M}{\gamma-1}|$. So we have to prove that $|\frac{C(V)}{\gamma-1}| = |V|$. But these two are $\mathbb{F}_p$-vector spaces of the same dimension. Hence, done. $\square$

2. Tate Duality

Let M be a finite $\mathbb{Z}_p$-module. Then Tate's duality constructs a perfect pairing

$$H^i(G_K, M) \times H^{2-i}(G_K, M^\wedge(1)) \to \mathbb{Q}_p/\mathbb{Z}_p.$$

Here, $M^\wedge(1)$ is a certain Tate twist of M. Using Shapiro's Lemma as before, we may assume $K = \mathbb{Q}_p$. We write a precise version of Tate duality first.

Theorem 2.1. *Let V be a $G_{\mathbb{Q}_p}$-representation that is p-torsion, and $n \in \mathbb{N}$ such that $p^n V = 0$. Put $V^\wedge(1) := \mathrm{Hom}(V, \mu_{p^n})$. Then there is a canonical isomorphism from $H^2(G_{\mathbb{Q}_p}, \mu_{p^n})$ to $\mathbb{Z}/p^n$ and a perfect pairing given by the cup product*

$$H^i(G_{\mathbb{Q}_p}, V) \times H^{2-i}(G_{\mathbb{Q}_p}, V^\wedge(1)) \xrightarrow{\cup} H^2(G_{\mathbb{Q}_p}, \mu_{p^n}) \cong \mathbb{Z}/p^n.$$

Sketch. (1) **Beginning :** A fundamental and starting step is a computation of $D(\mu_{p^n})$. In particular, if Ω^1 is the module of (continuous) differential forms of $A_{\mathbb{Q}_p}$ over $W(\bar{k}$, then after fixing π a lift of uniformizer, this module is generated by the symbol $d\pi$. So that for any $x = \sum_{k\in\mathbb{Z}} a_k \pi^k \in B_{\mathbb{Q}_p}$, we can consider the differential form xdz and define its residue $\mathrm{res}(xdz) := a_{-1}$.

Ω^1 has an étale (φ, Γ)-module structure by the formulas

$$\varphi(\lambda d\pi) := \frac{1}{p}\varphi(\lambda)d(\varphi(\pi)),\ \gamma(\lambda d\pi) := \gamma(\lambda)d(\gamma(\pi)).$$

A key fact is that there is a natural isomorphism of (φ, Γ)-modules between $D(\mu_{p^n})$ and the reduction Ω^1_n of Ω^1 modulo p^n.

(2) **Pontryagin duality and topological arguments :** We have $\tilde{D} := D(V^\wedge(1)) = \mathrm{Hom}(D(V), \Omega^1_n)$. By composing the residue map with trace map, we can get a surjective and continuous map Tr_n from $D(V)$ to $\mathbb{Z}/p^n$. Using this map, we can explicitly describe Pontryagin dual of $D(V)$.

(3) **Pontryagin duality implies local duality :** We can dualize the cohomology complex using Pontryagin duality to get a duality as required. All that remains is -

(4) **To show that $H^2(\Omega^1_n)$ is $\mathbb{Z}/p^n$ and that duality we got is actually gotten from the cup product :** Both of these can be deduced by explicit (computational) methods. Choices involved cancel each other out to give canonical maps.

See Herr's paper in Math. Ann. for details. □

3. (φ, Γ)-modules and Iwasawa theory

3.1. **Iwasawa modules.** Let K be a finite extension of $\mathbb{Q}_p$ and G_K is the absolute Galois group of K. Then $K_n = K(\mu_{p^n})$ and $\Gamma_n = \mathrm{Gal}(K_\infty/K_n) = \gamma_n^{\mathbb{Z}_p}$ if $n \geq 1$ (For $p = 2$, if K contains $\mathbb{Q}_2(\mu_4)$), otherwise for $n \geq 2$) where γ_n is a topological generator of Γ_n. We choose γ_n such that $\gamma_n = \gamma_1^{p^{n-1}}$. (Similar for $p = 2$.) The *Iwasawa algebra* $\mathbb{Z}_p[[\Gamma_K]]$ is isomorphic to $\mathbb{Z}_p[[T]]$ with the (p, T)-adic topology by sending T to $\gamma - 1$. We have

$$\mathbb{Z}_p[[\Gamma_K]]/(\gamma_n - 1) = \mathbb{Z}_p[\mathrm{Gal}(K_n/K)].$$

Furthermore, $\mathbb{Z}_p[[\Gamma_K]]$ is a G_K-module via the quotient. Similarly for $\mathbb{Z}_p[\mathrm{Gal}(K_n/K)]$.

Using Shapiro's Lemma, we get for M a $\mathbb{Z}_p[G_K]$-module,

$$H^i(G_{K_n}, M) \xrightarrow{\sim} H^i(G_K, \mathbb{Z}_p[\mathrm{Gal}(K_n/K)] \otimes M)$$

with the inverse map given by

$$\left((\sigma_1, \ldots, \sigma_i) \to \sum_{g\in\mathrm{Gal}(K_n/K)} g \otimes C_g(\sigma_1, \ldots, \sigma_i)\right) \to ((\sigma_1, \ldots, \sigma_i) \to C_{id}(\sigma_1, \ldots, \sigma_i)).$$

Thus, we have a commutative diagram

$$\begin{array}{ccc} H^i(G_{K_{n+1}}, M) & \xrightarrow{\sim} & H^i(G_K, \mathbb{Z}_p[\mathrm{Gal}(K_{n+1}/K)] \otimes M) \\ \downarrow{\scriptstyle cor} & & \downarrow \\ H^i(G_{K_n}, M) & \xrightarrow{\sim} & H^i(G_K, \mathbb{Z}_p[\mathrm{Gal}(K_n/K)] \otimes M) \end{array}$$

It can be checked that the second vertical map is induced by the natural map $\mathrm{Gal}(K_{n+1}/K) \to \mathrm{Gal}(K_n/K)$.

Definition 3.1. (i) If T is a $\mathbb{Z}_p$-representation of G_K, define

$$H^i_{Iw}(K, T) := \varprojlim_n H^i(G_{K_n}, T).$$

(ii) If V is a $\mathbb{Q}_p$-representation of G_K, choose T a stable $\mathbb{Z}_p$-lattice in V, then define

$$H^i_{Iw}(K, V) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i_{Iw}(K, T).$$

Note that we can always assume $n >> 1$.

3.2. **Description of Iwasawa cohomology in terms of $D(V)$.**

Lemma 3.2. *Let $\tau_n = \frac{\gamma_n - 1}{\gamma_{n-1} - 1} = 1 + \gamma_{n-1} + \ldots + \gamma_{n-1}^{p-1} \in \mathbb{Z}_p[[\Gamma_K]]$. Then the diagram*

$$\begin{array}{llllllllllll} C_{\psi,\gamma_n}(K_n, V) : 0 & \longrightarrow & D(V) & \longrightarrow & D(V) & \oplus & D(V) & \longrightarrow & D(V) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \\ C_{\psi,\gamma_{n-1}}(K_{n-1}, V) : 0 & \longrightarrow & D(V) & \longrightarrow & D(V) & \oplus & D(V) & \longrightarrow & D(V) & \longrightarrow & 0 \end{array}$$

is commutative and induces corestrictions on cohomology via

$$H^i(C_{\psi,\gamma_n}(K_n, V)) \xrightarrow{\sim} H^i(G_{K_n}, V).$$

Proof. τ_n is a cohomological functor and it induces $Tr_{K_n/K_{n-1}}$ on H^0, hence it induces corestrictions on H^i. □

Theorem 3.3. *Let V be a $\mathbb{Z}_p$ or $\mathbb{Q}_p$-representation of G_K. Then we have :*

(i) $H^i_{Iw}(K, V) = 0$, if $i \neq 1, 2$.
(ii) $H^1_{Iw}(K, V) \cong D(V)^{\psi=1}$, $H^2_{Iw}(K, V) \cong \frac{D(V)}{\psi - 1}$, and the isomorphisms are canonical.

Before proving the theorem, let us state a lemma.

Lemma 3.4. *If M is compact with continuous action of Γ_K, then*

$$M \cong \varprojlim_n (M/\gamma_n - 1).$$

Proof of Theorem. It is clear that $H^i_{Iw}(K, V)$ vanishes if $i \geq 3$ for V a $\mathbb{Z}_p$-representation and the case of $\mathbb{Q}_p$ follows.

For $i = 0$, by definition,

$$H^0_{Iw}(K, V) = \varprojlim_{Tr} V^{G_{K_n}}.$$

Since V has finite rank over $\mathbb{Z}_p$ and $V^{G_{K_n}}$ is an increasing sequence of submodules, it stabilizes for $n \geq n_0$. Then Tr_{K_{n+1}/K_n} is multiplication by p for $n \geq n_0$, but V does not contain p-divisible elementws, so that we get the required vanishing.

For $i = 2$: $H^2(G_{K_n}, V) = \frac{D(V)}{(\psi-1,\gamma_n-1)}$ by previous section. The corestriction map is induced by Id on $D(V)$. Thus,

$$H^2_{Iw}(K,V) = \varprojlim \frac{D(V)}{\psi-1}/(\gamma_n - 1) = \frac{D(V)}{\psi-1}$$

by previous lemma, as $D(V)/\psi - 1$ is compact.

For $i = 1$: We have commutative diagrams -

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{D(V)^{\psi=1}}{\gamma_n-1} & \longrightarrow & H^1(G_{K_n}, V) & \longrightarrow & \left(\frac{D(V)}{\psi-1}\right)^{\gamma_n=1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle cor} & & \downarrow{\scriptstyle \tau_n} & & \\ 0 & \longrightarrow & \frac{D(V)^{\psi=1}}{\gamma_{n-1}-1} & \longrightarrow & H^1(G_{K_{n-1}}, V) & \longrightarrow & \left(\frac{D(V)}{\psi-1}\right)^{\gamma_{n-1}=1} & \longrightarrow & 0 \end{array}$$

where p_1 denotes the projection onto second coordinate and p_2 denotes the projection onto first coordinate. Applying the functor $\varprojlim$, we get

$$0 \to \varprojlim \frac{D(V)^{\psi=1}}{\gamma_n - 1} \to \varprojlim H^1(G_{K_n}, V) \to \varprojlim \left(\frac{D(V)}{\psi-1}\right)^{\gamma_n=1}.$$

The first term is $D(V)^{\psi=1}$, so it suffices to prove that the last term vanishes. This is the same argument which was used to show the vanishing of H^0_{Iw}.

□

3.3. **Structure of $H^1_{Iw}(K,V)$.** Recall that we proved that if D is an étale (φ,Γ)-module of dimension d over $E_{\mathbb{Q}_p}$, then $C = (\varphi - 1)D^{\psi=1}$ is a free $\mathbb{F}_p[[\Gamma_{\mathbb{Q}_p}]]$-module of rank d. The same proof shows that if $n \geq 1$ and $i \in \mathbb{Z}_p^*$, $C \cap \epsilon\varphi^n(D)$ is free of rank d over $\mathbb{F}_p[[\Gamma_n]]$.

Corollary 3.5. *If D is an étale (φ,Γ)-module of dimension d over E_K, then C is a free $\mathbb{F}_p[[\Gamma_K]]$-module of rank $d.[K : \mathbb{Q}_p]$.*

Proposition 3.6. *Let V be a free $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representation of rank d of G_K. Then,*

(i) $D(V)^{\varphi=1}$ is a torsion sub-$\mathbb{Z}_p[[\Gamma_K \cap \Gamma_1]]$-module of $D(V)^{\psi=1}$.

(ii) We have an exact sequence

$$0 \to D(V)^{\varphi=1} \to D(V)^{\psi=1} \to C(V) \to 0.$$

$C(V)$ is free of rank $d.[K : \mathbb{Q}_p]$ over $\mathbb{Z}_p[[\Gamma_K]]$. (or over $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma_K]]$.)

Proof. The fact that $D(V)^{\varphi=1} = V^{H_K}$ is torsion follows from (ii) since it is finitely generated over $\mathbb{Z}_p$. To prove (ii), we have to prove that $C(V)/p$ is free of rank $d.[K : \mathbb{Q}_p]$ over $\mathbb{F}_p[[\Gamma_K]]$.

Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D(V)^{\varphi=1} & \longrightarrow & D(V)^{\psi=1} & \xrightarrow{\varphi-1} & C(V) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (D(V)/p)^{\varphi=1} & \longrightarrow & (D(V)/p)^{\psi=1} & \xrightarrow{\varphi-1} & C(V/p) & \longrightarrow & 0 \end{array}$$

for our modules. Using the exact sequence

$$0 \to pV \to V \to V/p \to 0$$

and using the snake lemma, we get the cokernel complex

$$\frac{D(V)}{(\varphi-1)}[p] \to \frac{D(V)}{(\psi-1)}[p] \to \frac{C(V/p)}{C(V)/p} \to 0.$$

Then since the middle term is a finite dimensional $\mathbb{F}_p$-vector space, $\frac{C(V/p)}{C(V)/p}$ is, too. Therefore, $C(V)/p$ is a $\mathbb{F}_p[[\Gamma_K]]$-lattice of $C(V/p)$ but $C(V/p)$ is a free $\mathbb{F}_p[[\Gamma_K]]$-module of rank $d.[K:\mathbb{Q}_p]$ and we conclude. □

Remark 2. (i) The sequence

$$0 \to D(V)^{\varphi=1} \to D(V)^{\psi=1} \to C(V) \to 0.$$

is the inflation-restriction exact sequence

$$0 \to H^1(\Gamma_K, \Lambda \otimes V^{H_K}) \to H^1(G_K, \Lambda \otimes V) \to H^1(H_K, \Lambda \otimes V)^{\Gamma_K} \to 0.$$

(ii) Let $0 \to V_1 \to V \to V_2 \to 0$ be an exact sequence. Then from snake lemma we get

$$0 \to D(V_1)^{\psi=1} \to D(V)^{\psi=1} \to D(V_2)^{\psi=1} \to \frac{D(V_1)}{\psi-1} \to \frac{D(V)}{\psi-1} \to \frac{D(V_1)}{\psi-1} \to 0.$$

This is just the sequence of H^1_{Iw} and H^2_{Iw} for respective modules. It can also be obtained from the long exact sequence of cohomology from the exact sequence

$$0 \to \mathbb{Z}_p[[\Gamma_K]] \otimes V_1 \to \mathbb{Z}_p[[\Gamma_K]] \otimes V \to \mathbb{Z}_p[[\Gamma_K]] \otimes V_2 \to 0.$$

Corollary 3.7. *Let V be a free $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representation of rank d of G_K. Then the torsion sub-$\mathbb{Z}_p[[\Gamma_K \cap \Gamma_1]]$-module of $H^1_{Iw}(K,V)$ is $D(V)^{\varphi=1} = V^{H_K}$, and $H^1_{Iw}(K,V)/V^{H_K}$ is free of rank $d.[K:\mathbb{Q}_p]$ over $\mathbb{Z}_p[[\Gamma_K]]$.*