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$K3$ CATEGORIES, ONE-CYCLES ON CUBIC FOURFOLDS, AND THE BEAUVILLE–VOISIN FILTRATION

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Abstract We explore the connection between $K3$ categories and 0-cycles on holomorphic symplectic varieties. In this paper, we focus on Kuznetsov's noncommutative $K3$ category associated to a nonsingular cubic 4-fold.

By introducing a filtration on the CH_1-group of a cubic 4-fold Y, we conjecture a sheaf/cycle correspondence for the associated $K3$ category $\mathcal{A}_Y$. This is a noncommutative analog of O'Grady's conjecture concerning derived categories of $K3$ surfaces. We study instances of our conjecture involving rational curves in cubic 4-folds, and verify the conjecture for sheaves supported on low degree rational curves.

Our method provides systematic constructions of (a) the Beauville–Voisin filtration on the CH_0-group and (b) algebraically coisotropic subvarieties of a holomorphic symplectic variety which is a moduli space of stable objects in $\mathcal{A}_Y$.

Keywords: $K3$ categories; cubic fourfolds; O'Grady filtration; holomorphic symplectic varieties; Beauville–Voisin conjectures

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Contents

0. Introduction

0.1. $K3$ categories and the Beauville–Voisin filtration

The purpose of this paper is to study the interactions between $K3$ categories and the Beauville–Voisin conjecture for holomorphic symplectic varieties.

A triangulated category is called a $K3$ category if it has the same Serre functor and Hochschild homology as the derived category of coherent sheaves on a $K3$ surface. New examples of $K3$ categories are constructed using the derived categories of certain Fano varieties and semiorthogonal decompositions; see [17, 19, 21].

Let $\mathcal{A}$ be a $K3$ category. If M is a nonsingular projective moduli space of stable objects with respect to a stability condition [10] on $\mathcal{A}$, then it is a holomorphic symplectic variety. The nondegenerate holomorphic 2-form is given by the Serre functor and the Mukai pairing,

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E},\mathcal{E}) \to \mathrm{Ext}^2_{\mathcal{A}}(\mathcal{E},\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathbb{C}.$$

The Beauville–Voisin conjecture [6, 39, 41] predicts that the Chow group $\mathrm{CH}_0(M)$ admits an increasing filtration

$$S_0\mathrm{CH}_0(M) \subset S_1\mathrm{CH}_0(M) \subset \cdots \subset S_{\frac{1}{2}\dim M}\mathrm{CH}_0(M) = \mathrm{CH}_0(M) \tag{0.1}$$

which is opposite to the conjectural Bloch–Beilinson filtration. Let $K_0(\mathcal{A})$ be the Grothendieck group of the triangulated category $\mathcal{A}$ and let

$$p_{\mathcal{A}} : \mathcal{A} \to K_0(\mathcal{A})$$

be the natural map. We have the following speculation on the structure of $\mathcal{A}$.

Speculation 0.1. *Let $\mathcal{A}$ be a $K3$ category.*

(a) *There exists an increasing filtration $S_\bullet(\mathcal{A})$ on $K_0(\mathcal{A})$ which governs the Beauville–Voisin filtration (0.1) for any moduli space M as above. More precisely, the i-th piece $S_i\mathrm{CH}_0(M)$ is spanned by the classes of $\mathcal{E} \in M$ with $p_{\mathcal{A}}(\mathcal{E}) \in S_i(\mathcal{A})$.*

(b) *For every object* $\mathcal{E} \in \mathcal{A}$, *we have*

$$p_{\mathcal{A}}(\mathcal{E}) \in S_{d(\mathcal{E})}(\mathcal{A})$$

with $d(\mathcal{E}) = \frac{1}{2} \dim \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{E}, \mathcal{E})$.

Speculation 0.1(b) can be viewed as a sheaf/cycle correspondence for the $K3$ category $\mathcal{A}$. For a nonsingular projective moduli space M, Speculation 0.1(b) implies exactly that

$$S_{\frac{1}{2}\dim M}\mathrm{CH}_0(M) = \mathrm{CH}_0(M).$$

0.2. O'Grady's conjecture

The first evidence of Speculation 0.1 is the case $\mathcal{A} = D^b(X)$ where X is a $K3$ surface. In [31], O'Grady introduced a filtration $S_\bullet(X)$ on the Chow group $\mathrm{CH}_0(X)$,

$$S_0(X) \subset S_1(X) \subset \cdots \subset S_i(X) \subset \cdots \subset \mathrm{CH}_0(X).$$ [1]

Here $S_i(X)$ is the union of $[z] + \mathbb{Z} \cdot [o_X]$ with z an effective 0-cycle of length i and $[o_X] \in \mathrm{CH}_0(X)$ the Beauville–Voisin canonical class [8].

The following generalized version of O'Grady's conjecture [31] is proven in [34], based on earlier results of Huybrechts, O'Grady, and Voisin in [15, 31, 40].

Theorem 0.2. *For any object* $\mathcal{E} \in D^b(X)$, *we have*

$$c_2(\mathcal{E}) \in S_{d(\mathcal{E})}(X).$$

Theorem 0.2 established a sheaf/cycle correspondence for $D^b(X)$. Moreover, O'Grady's filtration is indeed expected to govern the Beauville–Voisin filtration for any nonsingular moduli space M of stable objects in $D^b(X)$; see [34] for further details.

0.3. Cubic fourfolds and one-cycles

In this paper, we discuss Speculation 0.1 for $K3$ categories other than the derived categories of $K3$ surfaces.

Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic hypersurface. Kuznetsov constructed in [17] a $K3$ category $\mathcal{A}_Y$ as a full subcategory of $D^b(Y)$,

$$\mathcal{A}_Y = \{\mathcal{E} \in D^b(Y) : \operatorname{Ext}^*_{D^b(Y)}(\mathcal{O}_Y(i), \mathcal{E}) = 0 \text{ for } i = 0, 1, 2\}. \tag{0.2}$$

If Y is very general, then $\mathcal{A}_Y$ is not equivalent to $D^b(X)$ of a $K3$ surface X.[2] Hence $\mathcal{A}_Y$ is viewed as a noncommutative $K3$ surface.

Our first result introduces a filtration on the Chow group $\mathrm{CH}_1(Y)$, which serves as a candidate of the filtration in Speculation 0.1.[3] We briefly describe the construction below; see § 1 for more details.

[1]The pull-back via the map $K_0(X) \xrightarrow{c_2} \mathrm{CH}_0(X)$ induces a filtration on $K_0(X)$ as in Speculation 0.1(a).

[2]In fact, the category $\mathcal{A}_Y$ for a very general cubic 4-fold Y is not equivalent to the derived category of twisted sheaves on a $K3$ surface.

[3]Again, the filtration on $K_0(\mathcal{A}_Y)$ is obtained by pulling back via the natural maps $K_0(\mathcal{A}_Y) \to K_0(Y) \xrightarrow{c_3} \mathrm{CH}_1(Y)$.

Let F denote the Fano variety of lines in Y. We fix a uniruled divisor D on F,

$$\begin{array}{ccc} D & \hookrightarrow & F, \\ \downarrow{\scriptstyle q} & & \\ B & & \end{array}$$

where q is a rational map whose general fibers are rational curves.

We call a line $l \subset Y$ *special* (with respect to the uniruled divisor D) if the 0-cycle class $[l] \in \mathrm{CH}_0(F)$ is represented by a point on D. A line l is called *canonical* if it satisfies

$$3[l] = [H]^3 \in \mathrm{CH}_1(Y),$$

where $[H] \in \mathrm{CH}^1(Y)$ is the hyperplane class.

We define a filtration $S_\bullet(Y)$ on $\mathrm{CH}_1(Y)$,

$$S_0(Y) \subset S_1(Y) \subset \cdots \subset S_i(Y) \subset \cdots \subset \mathrm{CH}_1(Y),$$

where $S_i(Y)$ is the union of $[l_1 + l_2 + \cdots + l_i] + \mathbb{Z} \cdot [l_0]$ with l_k $(k > 0)$ special lines and l_0 a canonical line. It is shown in Lemma 1.1 that the filtration $S_\bullet(Y)$ does not depend on the choice of D and is "intrinsic" to Y.

We propose the following conjecture relating the $K3$ category $\mathcal{A}_Y$ to the filtration $S_\bullet(Y)$.

Conjecture 0.3. *For any object $\mathcal{E} \in \mathcal{A}_Y$, we have*

$$\mathrm{c}_3(\mathcal{E}) \in S_{d(\mathcal{E})}(Y).$$

Here c_3 is the composition of the inclusion $\mathcal{A}_Y \subset D^b(Y)$ and

$$\mathrm{c}_3 : D^b(Y) \to \mathrm{CH}_1(Y).$$

See also Remark 2.4 for an equivalent formulation of Conjecture 0.3.

Comparing to the derived category of a $K3$ surface, one advantage of studying the $K3$ category $\mathcal{A}_Y$ is that cubic 4-folds have a 20-dimensional moduli space. Hence our filtration provides a candidate of the Beauville–Voisin filtration of certain holomorphic symplectic varieties of $K3^{[n]}$ type in 20-dimensional families.[4]

0.4. Rational curves

We study the interplay between Conjecture 0.3 and the geometry of rational curves in nonsingular cubic 4-folds [7, 16, 22, 24].

Let

$$\iota^* : D^b(Y) \to \mathcal{A}_Y$$

be the left adjoint functor of the natural inclusion $\iota_* : \mathcal{A}_Y \hookrightarrow D^b(Y)$. The following theorem concerns low degree rational curves in Y.

[4]Stability conditions and moduli spaces of stable objects related to $\mathcal{A}_Y$ are explored in [4, 5, 22, 25].

Theorem 0.4. *Let $C \subset Y$ be a nonsingular connected rational curve of degree $\leqslant 4$. If $\mathcal{E}$ is a 1-dimensional sheaf supported*[5] *on C, then Conjecture 0.3 holds for $\iota^*\mathcal{E}$.*

$\deg C$	1	2	3	4
$\min d(\iota^*\mathcal{E})$	2	2	4	5

For a nonsingular connected rational curve C of degree $\leqslant 4$, we list in the table above the minimal possible values of

$$d(\iota^*\mathcal{E}) = \tfrac{1}{2}\dim \mathrm{Ext}^1_{\mathcal{A}_Y}(\iota^*\mathcal{E}, \iota^*\mathcal{E})$$

for all $\mathcal{E}$. These numbers are related to the maximal rationally connected (MRC) fibration on the moduli space of rational curves in Y; see § 2 for further discussions.

0.5. Algebraically coisotropic subvarieties

Let M be a holomorphic symplectic variety of dimensional $2d$. Following [41, Definition 0.6], a closed subvariety $Z_i \subset M$ of codimension i is called algebraically coisotropic if there exists a diagram

$$\begin{array}{ccc} Z_i & \hookrightarrow & M, \\ \downarrow{\scriptstyle q} & & \\ B_i & & \end{array}$$

such that the general fibers of q are i-dimensional, and the restriction of the holomorphic 2-form on M coincides with the pull-back of a holomorphic 2-form on B_i.

Voisin [41, Conjecture 0.4] conjectured that for every $i \leqslant d$, there exists an algebraically coisotropic subvariety $Z_i \dashrightarrow B_i$ of codimension i whose general fibers are constant cycle subvarieties of M.[6]

This conjecture was addressed in [34] when M is a moduli space of stable objects in the derived category of a $K3$ surface. We discuss in § 3 the connection between Conjecture 0.3 and Voisin's conjecture for the moduli spaces of stable objects in $\mathcal{A}_Y$; see Theorem 3.2. The crucial geometric input is the construction in Lemma 1.8 of a special uniruled divisor on the Fano variety F.

0.6. Conventions

Throughout, we work over the complex numbers $\mathbb{C}$. All varieties are assumed to be (quasi-)projective. Morphisms between triangulated categories are $\mathbb{C}$-linear.

1. A filtration on $\mathrm{CH}_1(Y)$

Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic 4-fold and let F be the Fano variety of lines in Y. In this section, we present some basic properties of the filtration $S_\bullet(Y)$ introduced in § 0.3.

[5]Here we mean the reduced support of the sheaf $\mathcal{E}$ is C.

[6]A constant cycle subvariety is a subvariety whose points all share the same class in the CH_0-group of the ambient variety.

Our filtration on $\mathrm{CH}_1(Y)$, which is analogous to O'Grady's filtration on the CH_0-group of a $K3$ surface, relies heavily on the geometry of the Fano variety F.

1.1. Uniruled divisors

Uniruled divisors on F play an important role in the definition of the filtration $S_\bullet(Y)$. Note that there exist uniruled divisors on the Fano variety of lines in any nonsingular cubic 4-fold. Below is a geometric construction.

In [38], Voisin constructed a self-rational map

$$\varphi : F \dashrightarrow F \tag{1.1}$$

sending a general line $l \subset Y$ to its residual line with respect to the unique plane $\mathbb{P}^2 \subset \mathbb{P}^5$ tangent to Y along l. The exceptional locus of φ then gives a uniruled divisor on F; see [41, Proposition 4.4].

The following lemma asserts that the filtration $S_\bullet(Y)$ does not depend on the choice of the uniruled divisor.

Lemma 1.1. *If a line $l \subset Y$ is special with respect to one uniruled divisor $D \subset F$, then it is special with respect to any uniruled divisor of F.*

Proof. We may assume that D is irreducible. Let $D' \subset F$ be another irreducible uniruled divisor. We need to show that every point on D is rationally equivalent to a point on D'.

Let q_F denote the Beauville–Bogomolov quadratic form on $H^2(F, \mathbb{Z})$. From the proof of [11, Theorem 5.1], we see that either

$$q_F(D, D') \neq 0$$

or there exists a sequence of irreducible uniruled divisors D_i $(0 \leqslant i \leqslant m)$ with $D_0 = D$ and $D_m = D'$ satisfying

$$q_F(D_i, D_{i+1}) \neq 0, \quad i = 0, 1, \dots, m-1.$$

In the first case, by [11, Lemma 5.2] the intersection number of every rational curve in the ruling of D and the divisor D' is nonzero. Hence any point on D is rationally equivalent to a point on D', and Lemma 1.1 follows. In the second case we can use D_i $(1 \leqslant i \leqslant m-1)$ as transitions. □

1.2. Zero-cycles on F

We discuss the relationship between the class of a line in $\mathrm{CH}_1(Y)$ and the corresponding point class in $\mathrm{CH}_0(F)$.[7]

Let $P = \{(l, x) \in F \times Y : x \in l\} \subset F \times Y$ be the incidence variety, which induces a morphism

$$[P]_* : \mathrm{CH}_0(F) \to \mathrm{CH}_1(Y). \tag{1.2}$$

A result of Paranjape [32] says that $[P]_*$ is surjective. The following fact is noted for later reference.

[7] By abuse of notation, for a line $l \subset Y$ we write both $[l] \in \mathrm{CH}_1(Y)$ and $[l] \in \mathrm{CH}_0(F)$.

Lemma 1.2. *The Chow groups* $\mathrm{CH}_0(F)$ *and* $\mathrm{CH}_1(Y)$ *are torsion-free.*

Proof. The statement for $\mathrm{CH}_0(F)$ follows from Roĭtman's theorem [33]. For $\mathrm{CH}_1(Y)$, since the morphism $[P]_*$ in (1.2) is surjective, it suffices to show that the kernel of $[P]_*$ is divisible. This is done by Shen and Vial in [36, Theorem 20.5] and the proof of [36, Lemma 20.6]. □

We also show that special lines are sufficient to span $\mathrm{CH}_1(Y)$.

Proposition 1.3. *Let* $j : D \hookrightarrow F$ *be a uniruled divisor. Then* $[P]_*$ *induces a natural isomorphism*

$$\mathrm{Im}(j_* : \mathrm{CH}_0(D) \to \mathrm{CH}_0(F)) \xrightarrow{\sim} \mathrm{CH}_1(Y).$$

Proof. By [11, Theorem 5.1], the image

$$\mathrm{Im}(j_* : \mathrm{CH}_0(D) \to \mathrm{CH}_0(F)) \subset \mathrm{CH}_0(F)$$

does not depend on the choice of the uniruled divisor $D \subset F$. Hence we can choose D as the exceptional locus of (1.1). Then [36, Proposition 19.5 and Theorem 20.5] imply that

$$\mathrm{Im}(j_* : \mathrm{CH}_0(D) \to \mathrm{CH}_0(F)) \simeq \mathrm{CH}_0(F)/\mathrm{Ker}([P]_*) \simeq \mathrm{CH}_1(Y). \qquad \square$$

By [36, 39], the Chow group $\mathrm{CH}_0(F)$ carries a canonical 0-cycle class $[o_F]$ of degree 1 which can be taken as any point lying on a constant cycle surface in F. Moreover, all 0-dimensional intersections of divisor classes and Chern classes of F are multiples of $[o_F]$.

Recall that a line $l \subset Y$ is canonical if

$$3[l] = [H]^3 \in \mathrm{CH}_1(Y)$$

where $[H] \in \mathrm{CH}^1(Y)$ is the hyperplane class. The following lemma shows the existence of canonical lines in Y and provides a complete criterion.

Lemma 1.4. *A line* $l \subset Y$ *is canonical if and only if*

$$[l] = [o_F] \in \mathrm{CH}_0(F).$$

Proof. By the proof of [39, Lemma 3.2], there exists a surface $\Sigma \subset F$ such that the class of every point on Σ is $[o_F] \in \mathrm{CH}_0(F)$. We first choose a line $l_0 \subset Y$ lying on $\Sigma \subset F$ such that there exists a plane $\mathbb{P}^2_{l_0} \subset \mathbb{P}^5$ tangent to Y along l_0. In particular, we have

$$[l_0] = [o_F] \in \mathrm{CH}_0(F).$$

Let l_0' be the residual line of l_0 with respect to the plane $\mathbb{P}^2_{l_0}$,

$$\mathbb{P}^2_{l_0} \cdot Y = 2l_0 + l_0'.$$

By definition, we have $[o_F] = \varphi_*([l_0]) = [l_0'] \in \mathrm{CH}_0(F)$. It follows that

$$[H]^3 = [P]_*(2[l_0] + [l_0']) = 3[P]_*[o_F] \in \mathrm{CH}_1(Y).$$

Hence by Lemma 1.2, a line $l \subset Y$ is canonical if and only if

$$[P]_*[l] = [P]_*[o_F] \in \mathrm{CH}_1(Y). \tag{1.3}$$

It suffices to show that (1.3) is equivalent to $[l] = [o_F] \in \mathrm{CH}_0(F)$. Let

$$[l] = [o_F] + [l]_{(2)} + [l]_{(4)} \in \mathrm{CH}_0(F)$$

be the motivic decomposition of the point class $[l] \in \mathrm{CH}_0(F)$ constructed in [36, Part 3]. By [36, Theorem 20.5], the condition (1.3) is equivalent to $[l]_{(2)} = 0$, which implies $[l] = [o_F]$ after [34, Theorem 3.4]. □

Example 1.5. Let $Y \subset \mathbb{P}^5$ be a nonsingular cubic 4-folds which contains a plane. Then there is a uniruled divisor

$$\begin{array}{ccc} D & \overset{j}{\hookrightarrow} & F \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}$$

over a $K3$ surface X; see [17] and [34, Section 3.2] for the construction. We identify the Chow groups $\mathrm{CH}_0(D)$ and $\mathrm{CH}_0(X)$ via the push-forward q_*. By [34, Theorem 3.6], the embedding $j : D \hookrightarrow F$ induces an injective morphism

$$j_* : \mathrm{CH}_0(X) \simeq \mathrm{CH}_0(D) \hookrightarrow \mathrm{CH}_0(F).$$

Applying Proposition 1.3, we find an isomorphism

$$[P]_* j_* : \mathrm{CH}_0(X) \xrightarrow{\sim} \mathrm{CH}_1(Y). \tag{1.4}$$

We know from Lemma 1.1 that a line $l \subset Y$ is special if and only if the class $[l] \in \mathrm{CH}_1(Y)$ corresponds to a point class $[x] \in \mathrm{CH}_0(X)$ under the isomorphism (1.4). Lemma 1.4 and [34, Theorem 3.6] further imply that a line in Y is canonical if and only if its corresponding point class on X is the Beauville–Voisin class $[o_X] \in \mathrm{CH}_0(X)$.

In conclusion, our filtration on $\mathrm{CH}_1(Y)$ coincides with O'Grady's filtration on $\mathrm{CH}_0(X)$ under the isomorphism (1.4).

1.3. Generalities on the filtration $S_\bullet(Y)$

We prove that $S_\bullet(Y)$ is a filtration into "cones" for any nonsingular cubic 4-fold Y. This is parallel to [31, Corollary 1.7] in the $K3$ surface case.

Proposition 1.6. *Let $\alpha, \alpha' \in \mathrm{CH}_1(Y)$.*

(a) *If $\alpha \in S_i(Y)$ and $\alpha' \in S_{i'}(Y)$, then $\alpha + \alpha' \in S_{i+i'}(Y)$.*

(b) *If $\alpha \in S_i(Y)$, then $m\alpha \in S_i(Y)$ for any $m \in \mathbb{Z}$.*

(c) *We have*

$$\bigcup_{i \geqslant 0} S_i(Y) = \mathrm{CH}_1(Y).$$

Statement (a) is immediate, and (c) follows from (b) and Proposition 1.3. The proof of (b) requires the following lemmas.

Lemma 1.7. *Let $\mathcal{Y} \to T$ be a smooth family of cubic 4-folds over a nonsingular variety T, and let $\alpha \in \mathrm{CH}^3(\mathcal{Y})$. If the restriction $\alpha|_{\mathcal{Y}_t} \in \mathrm{CH}_1(\mathcal{Y}_t)$ lies in $S_i(\mathcal{Y}_t)$ for a very general point $t \in T$, then the same holds for every point $t \in T$.*

Proof. Let $\mathcal{F} \to T$ be the relative Fano variety of lines associated to the family $\mathcal{Y} \to T$. Since the construction of uniruled divisors in § 1.1 works universally over the moduli space of nonsingular cubic 4-folds, we can find a relative uniruled divisor

$$\begin{array}{ccc} \mathcal{D} & \hookrightarrow & \mathcal{F} \\ & & \downarrow \\ & & T \end{array} \tag{1.5}$$

whose restriction to every fiber gives a uniruled divisor.

Let $\mathcal{D}^{(i)} \to T$ denote the i-th relative symmetric product of $\mathcal{D}$. Consider the locus

$$Z = \left\{ \sum_{k=1}^{i} l_{t,k} \in \mathcal{D}^{(i)} : \alpha|_{\mathcal{Y}_t} = \sum_{k=1}^{i} [l_{t,k}] + m[l_{t,0}] \in \mathrm{CH}_1(\mathcal{Y}_t) \right\} \subset \mathcal{D}^{(i)}$$

with $l_{t,0} \subset \mathcal{Y}_t$ a canonical line. By the assumption that $\alpha|_{\mathcal{Y}_t} \in S_i(\mathcal{Y}_t)$ for a very general $t \in T$, the locus Z dominates the base T. A standard argument using Hilbert schemes shows that Z is a countable union of Zariski closed subsets of $\mathcal{D}^{(i)}$. Hence there exists a component $Z' \subset Z$ which dominates T via the natural projection $Z' \to T$. The restriction of Z' to every fiber of $\mathcal{D}^{(i)} \to T$ represents $\alpha|_{\mathcal{Y}_t}$ as

$$\alpha|_{\mathcal{Y}_t} = \sum_{k=1}^{i} [l_{t,k}] + m[l_{t,0}] \in \mathrm{CH}_1(\mathcal{Y}_t)$$

with $l_{t,k}$ $(k \geqslant 1)$ special and $l_{t,0}$ canonical in $\mathcal{Y}_t$. □

Lemma 1.8. *Let Y be a general nonsingular cubic 4-fold and let F be its Fano variety of lines. There exists a uniruled divisor $j : D \hookrightarrow F$ such that for every point $x \in D$ and $m \in \mathbb{Z}$, we can find $y \in D$ satisfying*

$$m[x] = [y] + \alpha \in \mathrm{CH}_0(D)$$

with $j_*\alpha = (m-1)[o_F] \in \mathrm{CH}_0(F)$.

Proof. We first construct the uniruled divisor $D \subset F$.[8] Let

$$\check{\mathbb{P}}^5 = \mathbb{P}H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(1))$$

be the projective space parametrizing hyperplanes $H \subset \mathbb{P}^5$. For $1 \leqslant e \leqslant 5$, let B_e denote the closure of the locus formed by $H \subset \mathbb{P}^5$ such that the cubic 3-fold $H \cap Y$ has e nodes. Since Y is general, the locus $B_e \subset \check{\mathbb{P}}^5$ is nonempty and of codimension e. Consider the incidence variety

$$W = \{(l, H) \subset F \times B_4 : l \subset H \cap Y\} \subset F \times B_4,$$

together with the natural projections

$$\begin{array}{ccc} W & \xrightarrow{p} & F. \\ \downarrow{\scriptstyle q} & & \\ B_4 & & \end{array}$$

Note that the fiber $q^{-1}(H)$ is given by the Fano variety of lines in the cubic 3-fold $H \cap Y$.

[8]We learned this construction from a talk by Kieran O'Grady.

If a cubic 3-fold $H \cap Y$ contains a node, a standard fact [12] says that the Fano variety of lines in $H \cap Y$ is birational to the symmetric product $C_H^{(2)}$ of a genus 4 curve C_H formed by lines passing through the node. In our situation, the cubic 3-fold $H \cap Y$ contains 4 nodes for every $H \in B_4$, and each extra node creates a node on the curve C_H. Hence the fiber $q^{-1}(H)$ is birational to $C_H^{(2)}$ such that the normalization E_H of the curve C_H has genus $\leqslant 1$. It follows that every fiber of $q : W \to B_4$ is birational to a $\mathbb{P}^1$-fibration over E_H, and the 3-fold W is uniruled. We define the uniruled divisor $j : D \hookrightarrow F$ to be the image $p(W) \subset F$.

Claim. For any $H \in B_4$, consider the composition

$$f : q^{-1}(H) \hookrightarrow W \xrightarrow{p} F$$

which induces a morphism of Chow groups

$$f_* : \mathrm{CH}_0(q^{-1}(H)) \to \mathrm{CH}_0(F).$$

Then there exists a point $a_H \in q^{-1}(H)$ such that

$$f_*[a_H] = [o_F] \in \mathrm{CH}_0(F).$$

Proof of the Claim. Let H_0 be a hyperplane lying in $B_5 \subset \check{\mathbb{P}}^5$. Then the fiber $q^{-1}(H_0)$ is a rational surface, and the class of every point on $q^{-1}(H_0)$ is $[o_F] \in \mathrm{CH}_0(F)$; see [39, Lemma 3.2] or [41, Proposition 4.5].

Let $F_H \subset F$ denote the subvariety of lines contained in the cubic 3-fold $H \cap Y$. It suffices to show that

$$F_H \cap F_{H_0} \neq \emptyset. \tag{1.6}$$

For general hyperplanes H_1 and H_2, the intersection number $[F_{H_1}] \cdot [F_{H_2}]$ counts lines in the nonsingular cubic surface $H_1 \cap H_2 \cap Y$. Hence

$$[F_H] \cdot [F_{H_0}] = [F_{H_1}] \cdot [F_{H_2}] = 27,$$

which proves (1.6). □

For $H \in B_4$, consider the canonical isomorphism

$$\mathrm{CH}_0(q^{-1}(H)) \simeq \mathrm{CH}_0(E_H^{(2)}). \tag{1.7}$$

By resolution of singularities and the argument of [34, Lemma 2.2], any point class $[x] \in \mathrm{CH}_0(q^{-1}(H))$ corresponds to a point class $[x'] \in \mathrm{CH}_0(E_H^{(2)})$ under the isomorphism (1.7). Let $[a'_H] \in \mathrm{CH}_0(E_H^{(2)})$ denote the point class corresponding to $[a_H] \in \mathrm{CH}_0(q^{-1}(H))$ as in the Claim. Since E_H has genus $\leqslant 1$, there is an isomorphism

$$\mathrm{CH}_0(E_H^{(2)}) \simeq \mathrm{CH}_0(E_H).$$

Then the group law of elliptic curves gives a point $y' \in E_H^{(2)}$ satisfying

$$m[x'] - [y'] = (m-1)[a'_H] \in \mathrm{CH}_0(E_H^{(2)})$$

for $x' \in E_H^{(2)}$ and $m \in \mathbb{Z}$. Again, via the isomorphism (1.7), we find $y \in q^{-1}(H)$ such that

$$m[x] = [y] + (m-1)[a_H] \in \mathrm{CH}_0(q^{-1}(H)).$$

This proves the lemma. □

Remark 1.9. In the argument above, we have constructed a uniruled divisor $D \subset F$ over an elliptic surface for a general cubic 4-fold. This special uniruled divisor will also be used in § 3 for the connection between Conjecture 0.3 and Voisin's conjecture [41, Conjecture 0.4].

Proof of Proposition 1.6(b). It suffices to show that if $l \subset Y$ is special, then for any $m \in \mathbb{Z}$ there exists a special line $l' \subset Y$ satisfying

$$m[l] = [l'] + (m-1)[l_0] \in \mathrm{CH}_1(Y). \tag{1.8}$$

Here $[l_0] \in \mathrm{CH}_1(Y)$ is the class of a canonical line.

First, we consider when Y is general. By Lemma 1.1, we can assume that the special line $l \subset Y$ lies in the uniruled divisor constructed in Lemma 1.8. Hence there exists a special line l' such that

$$m[l] = [l'] + (m-1)[o_F] \in \mathrm{CH}_0(F).$$

We deduce (1.8) by Lemma 1.4 and by applying the correspondence

$$[P]_* : \mathrm{CH}_0(F) \to \mathrm{CH}_1(Y).$$

Next, we prove Proposition 1.6(b) for every nonsingular cubic 4-fold. We take T to be the moduli space of nonsingular cubic 4-folds with $\mathcal{Y} \to T$ the universal family. Consider the relative uniruled divisor $\mathcal{D} \to T$ as in (1.5). Assume that the cubic 4-fold Y is given by the fiber $\mathcal{Y}_{t_0}$ over $t_0 \in T$. A special line $l \subset Y = \mathcal{Y}_{t_0}$ can be chosen from a point lying on the uniruled divisor $\mathcal{D}_{t_0}$. After taking a finite base change, we may assume that the family $\mathcal{D} \to T$ admits a section $s : T \to \mathcal{D}$ passing through $l \in \mathcal{D}_{t_0}$. The section s gives a special line l_t for every cubic 4-fold $\mathcal{Y}_t$. Since Proposition 1.6(b) is proven for a general cubic 4-fold, we have

$$m[l_t] \in S_1(\mathcal{Y}_t)$$

for a general fiber $\mathcal{Y}_t$. Applying Lemma 1.7, we find

$$m[l] \in S_1(Y),$$

which proves (1.8). □

Using the uniruled divisor constructed in Lemma 1.8, we actually obtain the following stronger result.

Proposition 1.10. *Let $\alpha \in \mathrm{CH}_1(Y)$ and let m be a nonzero integer. We have $\gamma \in S_i(Y)$ if and only if $m\gamma \in S_i(Y)$.*

Proof. We only need to prove the 'only if' part. By Lemma 1.7 and an argument similar to the proof of Proposition 1.6(b), we may assume Y to be general. Let l be any line lying in the uniruled divisor $D \subset F$ constructed in Lemma 1.8. Then the group law of elliptic curves ensures that there exists a line $l' \in D$ such that

$$m[l'] = [l] + (m-1)[o_F] \in \mathrm{CH}_0(F).$$

This proves the proposition. □

2. Rational curves in cubic fourfolds

Let Y be a nonsingular cubic 4-fold. In this section, we prove Theorem 0.4 and discuss its connection to the moduli spaces of rational curves in Y.

2.1. The $K3$ category $\mathcal{A}_Y$

The $K3$ category $\mathcal{A}_Y$ has been introduced by Kuznetsov via the semiorthogonal decomposition of the derived category of a cubic 4-fold [17–19]. We first review some basic properties of $\mathcal{A}_Y$.

Following the notation in [17], let

$$D^b(Y) = \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle$$

denote the semiorthogonal decomposition of the derived category $D^b(Y)$ with respect to the exceptional collection $\mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \in D^b(Y)$. The induced component $\mathcal{A}_Y$ given by (0.2) satisfies the following lemma.

Lemma 2.1 ([20, § 4]). *Let $\mathcal{E}, \mathcal{F} \in \mathcal{A}_Y$.*

(a) *For $i \geqslant 3$, we have $\operatorname{Ext}^i_{D^b(Y)}(\mathcal{E}, \mathcal{F}) = 0$.*

(b) *For $i = 0, 1, 2$, there are canonical isomorphisms*

$$\operatorname{Ext}^i_{D^b(Y)}(\mathcal{E}, \mathcal{F}) \simeq \operatorname{Ext}^{2-i}_{D^b(Y)}(\mathcal{F}, \mathcal{E})^\vee.$$

(c) *We have*

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_{i=0}^{2} (-1)^i \dim \operatorname{Ext}^i_{D^b(Y)}(\mathcal{E}, \mathcal{F}).$$

Let $\mathcal{E}, \mathcal{F} \in \mathcal{A}_Y$. Since

$$\operatorname{Ext}^i_{D^b(Y)}(\mathcal{E}, \mathcal{F}) = \operatorname{Ext}^i_{\mathcal{A}_Y}(\mathcal{E}, \mathcal{F}),$$

Lemma 2.1 yields

$$2 \dim \operatorname{Hom}_{\mathcal{A}_Y}(\mathcal{E}, \mathcal{E}) - \dim \operatorname{Ext}^1_{\mathcal{A}_Y}(\mathcal{E}, \mathcal{E}) = \chi(\mathcal{E}, \mathcal{E}). \tag{2.1}$$

The natural inclusion $\iota_* : \mathcal{A}_Y \hookrightarrow D^b(Y)$ admits a left adjoint functor

$$\iota^* : D^b(Y) \to \mathcal{A}_Y,$$

which is the 'projection' from $D^b(Y)$ to the $K3$ category $\mathcal{A}_Y$.

Lemma 2.2. *Let $[H] \in \mathrm{CH}^1(Y)$ be the hyperplane class, and let $[l_0] \in \mathrm{CH}_1(Y)$ be the class of a canonical line. For any $\alpha \in \mathrm{CH}_2(Y)$, we have*

$$[H] \cdot \alpha \in \mathbb{Z} \cdot [l_0] \subset \mathrm{CH}_1(Y).$$

Proof. Consider the following morphisms induced by $j : Y \hookrightarrow \mathbb{P}^5$,

$$\mathrm{CH}_2(Y) \xrightarrow{j_*} \mathrm{CH}_2(\mathbb{P}^5) \xrightarrow{j^*} \mathrm{CH}_1(Y).$$

Since $\mathrm{CH}_2(\mathbb{P}^5) = \mathbb{Z}\cdot[H]^2$, the class

$$j^* j_* \alpha = 3[H]\cdot\alpha$$

is proportional to $[H]^3$. Hence the lemma follows from Lemma 1.2. □

Corollary 2.3. *For any $\mathcal{E} \in D^b(Y)$, we have $c_3(\mathcal{E}) \in S_i(Y)$ if and only if $c_3(\iota^*\mathcal{E}) \in S_i(Y)$.*

Proof. Any $\mathcal{E} \in D^b(Y)$ fits into a distinguished triangle

$$\mathcal{G} \to \mathcal{E} \to \iota_*\iota^*\mathcal{E} \to \mathcal{G}[1] \tag{2.2}$$

with $\mathcal{G} \in \langle \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle$. The corollary follows directly from (2.2) and Lemma 2.2. □

Remark 2.4. As a consequence of Corollary 2.3, Conjecture 0.3 is equivalent to the following: for any $\mathcal{E} \in D^b(Y)$, we have

$$c_3(\mathcal{E}) \in S_{d(\iota^*\mathcal{E})}(Y).$$

Recall the Mukai lattice on $\mathcal{A}_Y$ introduced in [2, Section 2]. Let $K_{\mathrm{top}}(Y)$ denote the topological K-theory [3] of the cubic 4-fold Y, which is endowed with the Mukai vector

$$v : K_{\mathrm{top}}(Y) \to H^*(Y, \mathbb{Q})$$

and the Euler pairing $\chi(-,-)$. The Mukai lattice of $\mathcal{A}_Y$ is defined to be the abelian group

$$K_{\mathrm{top}}(\mathcal{A}_Y) = \{\kappa \in K_{\mathrm{top}}(Y) : \chi([\mathcal{O}_Y(i)], \kappa) = 0 \text{ for } i = 0, 1, 2\},$$

to which a weight 2 Hodge structure is associated; see [2, Definition 2.2].

Let

$$\mathrm{pr} = \mathrm{pr}_0 \circ \mathrm{pr}_1 \circ \mathrm{pr}_2 : K_{\mathrm{top}}(Y) \to K_{\mathrm{top}}(\mathcal{A}_Y)$$

be the projection map with

$$\mathrm{pr}_i(\kappa) = \kappa - \chi([\mathcal{O}_Y(i)], \kappa)\cdot[\mathcal{O}_Y(i)].$$

For any $\mathcal{E} \in D^b(Y)$, we have

$$\mathrm{pr}[\mathcal{E}] = [\iota^*\mathcal{E}] \in K_{\mathrm{top}}(\mathcal{A}_Y).$$

We define the Mukai pairing on $K_{\mathrm{top}}(\mathcal{A}_Y)$ to be the nondegenerate symmetric bilinear form $-\chi(-,-)$, and we write κ^2 for the self-pairing (κ,κ). Then (2.1) implies

$$\dim \mathrm{Ext}^1_{\mathcal{A}_Y}(\mathcal{E},\mathcal{E}) = [\mathcal{E}]^2 + 2\dim \mathrm{Hom}_{\mathcal{A}_Y}(\mathcal{E},\mathcal{E}) \geqslant [\mathcal{E}]^2 + 2 \tag{2.3}$$

for any $\mathcal{E} \in \mathcal{A}_Y$.

Note also that for a line $l \subset Y$, the special classes

$$\lambda_i = [\iota^*\mathcal{O}_l(i)] = \mathrm{pr}[\mathcal{O}_l(i)] \in K_{\mathrm{top}}(\mathcal{A}_Y), \quad i = 1, 2,$$

span an A_2-lattice

$$A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \subset K_{\mathrm{top}}(\mathcal{A}_Y)$$

with respect to the Mukai pairing on $K_{\mathrm{top}}(\mathcal{A}_Y)$.

2.2. One-dimensional sheaves

Let $\mathcal{E}$ be a 1-dimensional sheaf supported on a nonsingular connected rational curve $C \subset Y$ of degree $e > 0$. The class $[\mathcal{E}] \in K_0(Y)$ can be expressed in terms of line bundles on C. In particular, there exist (uniquely determined) integers $r > 0$ and m such that

$$[\mathcal{E}] = re[O_l(1)] + m[\mathbb{C}_p] \in K_{\mathrm{top}}(Y),$$

where $\mathbb{C}_p$ is the skyscraper sheaf of a point $p \in Y$. On the other hand, by [14, Example 15.3.1], we have

$$c_3(\mathcal{E}) = 2r[C] \in \mathrm{CH}_1(Y).$$

The following proposition gives the lower bound for

$$d(\iota^*\mathcal{E}) = \tfrac{1}{2}\dim \mathrm{Ext}^1_{\mathcal{A}_Y}(\iota^*\mathcal{E}, \iota^*\mathcal{E}).$$

Proposition 2.5. *With the notation above,*

(a) *if $e = 2k$, then*

$$d(\iota^*\mathcal{E}) \geqslant k^2 + 1;$$

(b) *if $e = 2k+1$, then*

$$d(\iota^*\mathcal{E}) \geqslant k^2 + k + 2.$$

Note that the bounds above match the table in § 0.4 for $e \leqslant 4$.

Proof. We have

$$\mathrm{pr}[\mathbb{C}_p] = \lambda_2 - \lambda_1 \in K_{\mathrm{top}}(\mathcal{A}_Y).$$

Hence

$$[\iota^*\mathcal{E}] = re[\iota^*\mathcal{O}_l(1)] + m[\iota^*\mathbb{C}_p] = (re-m)\lambda_1 + m\lambda_2 \in K_{\mathrm{top}}(\mathcal{A}_Y).$$

By the inequality (2.3), we find

$$\begin{aligned} 2d(\iota^*\mathcal{E}) &\geqslant [\iota^*\mathcal{E}]^2 + 2 \\ &= ((re-m)\lambda_1 + m\lambda_2)^2 + 2 \\ &= 2(3m^2 - 3mre + r^2e^2) + 2. \end{aligned}$$

When $e = 2k$, we have

$$d(\iota^*\mathcal{E}) \geqslant 3(m-rk)^2 + (r^2k^2+1) \geqslant k^2 + 1.$$

When $e = 2k+1$, we have

$$d(\iota^*\mathcal{E}) \geqslant 3(m-rk)(m-rk-1) + (r^2k^2 + rk + 2) \geqslant k^2 + k + 2. \qquad \square$$

We write $b(e)$ for the bounds above,

$$b(e) = \begin{cases} k^2+1 & \text{if } e = 2k, \\ k^2+k+2 & \text{if } e = 2k+1. \end{cases}$$

To deduce Theorem 0.4, it suffices to prove the following statement for $e \leqslant 4$:

(†) For any nonsingular connected rational curve $C \subset Y$ of degree e, we have

$$[C] \in S_{b(e)}(Y).$$

Indeed, assuming (†) and applying Proposition 1.6(b), we find

$$c_3(\mathcal{E}) = 2r[C] \in S_{b(e)}(Y).$$

Theorem 0.4 then follows from Proposition 2.5 since $d(\iota^*\mathcal{E}) \geqslant b(e)$.

We prove (†) for $e \leqslant 4$ in Sections 2.3 and 2.4. In [13], it is shown that the moduli space of rational curves of a fixed degree e in Y is irreducible. By Lemma 1.7, the filtration $S_\bullet(Y)$ is preserved under specialization. Hence we only need to consider general rational curves $C \subset Y$.

2.3. Lines, conics, and twisted cubics

Let $g \in \mathrm{CH}^1(F)$ be the polarization class given by the Plücker embedding of $\mathrm{Gr}(2, 6)$. We also fix a uniruled divisor $D \subset F$ in the class ag for some $a > 0$.

Proposition 2.6. *For a general line $l \subset Y$, there exists a plane $\mathbb{P}^2_l \subset \mathbb{P}^5$ and special lines $l_1, l_2 \in D$ such that*

$$\mathbb{P}^2_l \cdot Y = l + l_1 + l_2.$$

Proof. Given a line $l \subset Y$, we write S_l for the surface in F formed by lines meeting l. When l is general, the surface S_l is nonsingular by [37]. There is an involution

$$\tau_l : S_l \to S_l$$

defined as follows. If $l' \in S_l$ is a line other than l, then $\tau_l(l')$ is the residual line of the pair (l, l'). If $l' = l$, then $\tau_l(l') = \varphi(l)$. For a point $x \in l$, there is a curve $C_x \subset S_l$ formed by lines passing through x. The following intersections on S_l are computed in [37]:

$$[C_x]^2 = [l], \quad [\tau_l(C_x)]^2 = [\varphi(l)], \quad g|_{S_l} = [C_x] + 2[\tau_l(C_x)]. \tag{2.4}$$

By [36, Lemma 18.2], the intersection number of $g^2|_{S_l}$ is

$$g^2|_{S_l} = g^2 \cdot [S_l] = 21.$$

Comparing with (2.4), we find

$$[C_x] \cdot [\tau_l(C_x)] = 4,$$

which implies

$$g|_{S_l} \cdot \tau_{l*}(g|_{S_l}) = ([C_x] + 2[\tau_l(C_x)]) \cdot ([\tau_l(C_x)] + 2[C_x]) = 24.$$

Consider the curve $D_l \subset S_l$ given by the intersection of S_l and the uniruled divisor D. To prove the proposition, it suffices to show that

$$D_l \cap \tau_l(D_l) \neq \emptyset.$$

This is achieved by computing the intersection number

$$[D_l] \cdot [\tau_l(D_l)] = a^2 \cdot (g|_{S_l} \cdot \tau_{l*}(g|_{S_l})) = 24a^2 > 0.$$ □

Now we prove Theorem 0.4 in degrees $e \leqslant 3$.

Proof of (†) for $e \leqslant 3$. Let $l_0 \subset Y$ be a canonical line. For a general line $l \subset Y$, Proposition 2.6 shows the existence of lines $l_1, l_2 \in D$ satisfying

$$[l]+[l_1]+[l_2]=[H]^3 \in \mathrm{CH}_1(Y).$$

By Proposition 1.6, we have

$$[l]=-[l_1]-[l_2]+3[l_0] \in S_2(Y).$$

Next, let $C \subset Y$ be a general conic. Then there is a plane $\mathbb{P}^2_C \subset \mathbb{P}^5$ containing C. Let l be the residual line of the conic with respect to the plane $\mathbb{P}^2_C$,

$$\mathbb{P}^2_C \cdot Y = C + l.$$

We find

$$[C]=-[l]+3[l_0] \in S_2(Y).$$

Finally, let $C \subset Y$ be a general twisted cubic, which is contained in a unique projective space $\mathbb{P}^3_C \subset \mathbb{P}^5$. The intersection

$$Y_C = \mathbb{P}^3_C \cap Y$$

is a nonsingular cubic surface. By [41, Proposition 4.8], there exists a pair of lines $l_1, l_2 \subset Y_C$ such that C lies in the linear system $|\mathcal{O}_{Y_C}(l_1 - l_2 + H \cap Y_C)|$, where $H \subset \mathbb{P}^5$ is a hyperplane. This yields

$$[C]=[l_1]-[l_2]+3[l_0] \in S_4(Y). \qquad \square$$

2.4. Quartics and intermediate Jacobians

Let $C \subset Y$ be a general quartic rational curve. Then C is contained in a unique hyperplane $H \subset \mathbb{P}^5$, whose intersection with Y is a nonsingular cubic 3-fold

$$V = H \cap Y.$$

The intermediate Jacobian of V is a principally polarized abelian 5-fold

$$J_V = H^{2,1}(V)^*/H_3(V,\mathbb{Z}).$$

Let S be the Fano surface of lines in V, and let $\mathrm{Alb}(S)$ be the Albanese variety of S. By [12], the Abel–Jacobi map induces a canonical isomorphism

$$\mathrm{Alb}(S) \xrightarrow{\sim} J_V. \tag{2.5}$$

We fix a very ample uniruled divisor $D \subset F$ as in § 2.3. Consider the curve $R = D \cap S$ with $R' \to R$ the normalization. The composition

$$j : R' \to R \hookrightarrow S$$

induces a morphism

$$u : \mathrm{Jac}(R') \to \mathrm{Alb}(S),$$

where $\mathrm{Jac}(R')$ is the Jacobian of the nonsingular curve R'.

Lemma 2.7. *The morphism* $u : \mathrm{Jac}(R') \to \mathrm{Alb}(S)$ *is surjective.*

Proof. It suffices to show that the morphism

$$j^* : H^1(S, \mathbb{Q}) \to H^1(R', \mathbb{Q})$$

is injective. Suppose this does not hold. Then the projection formula would imply that the bilinear form

$$H^1(S, \mathbb{Q}) \times H^1(S, \mathbb{Q}) \to \mathbb{Q}$$
$$\langle \alpha, \beta \rangle = \int_S \alpha \cdot \beta \cdot [R]$$

is degenerate. This contradicts the ampleness of R. □

We fix a point $x_0 \in R$ and write $l_{x_0} \subset V$ for the corresponding line. Let $x_0' \in R'$ be a point in the preimage of x_0. For any $k > 0$, there is a morphism from the symmetric product $R'^{(k)}$ to $\mathrm{Jac}(R')$ with respect to x_0',

$$f_k : R'^{(k)} \to \mathrm{Jac}(R'), \quad f_k\left(\sum_i x_i'\right) = \mathcal{O}_{R'}\left(\sum_i x_i' - kx_0'\right).$$

Let

$$h_k : R'^{(k)} \to J_V$$

be the composition of $f_k : R'^{(k)} \to \mathrm{Jac}(R')$, $u : \mathrm{Jac}(R') \to \mathrm{Alb}(S)$, and the isomorphism (2.5).

Corollary 2.8. *The morphism* h_5 *is surjective.*

Proof. Let g be the genus of the curve R'. Then the morphism

$$f_g : R'^{(g)} \to \mathrm{Jac}(R')$$

is surjective, and Lemma 2.7 implies that h_g is also surjective. In particular, we have $g \geqslant \dim J_V = 5$.

We show by induction that h_k is surjective for any integer k in the range

$$5 \leqslant k \leqslant g.$$

The base case is $k = g$. Now assume the surjectivity of h_{k+1}. Consider the closed embedding

$$R'^{(k)} \hookrightarrow R'^{(k+1)} \tag{2.6}$$

given by $\sum_i x_i' \mapsto \sum_i x_i' + x_0'$.

To show the surjectivity of the composition

$$h_k : R'^{(k)} \hookrightarrow R'^{(k+1)} \xrightarrow{h_{k+1}} J_V,$$

it suffices to prove that the divisor $R'^{(k)} \subset R'^{(k+1)}$ in (2.6) is ample. Let

$$\sigma_{k+1} : R'^{k+1} \to R'^{(k+1)}$$

be the natural quotient map. The pull-back of $O_{R'^{(k+1)}}(R'^{(k)})$ via σ_{k+1} is the ample line bundle

$$\mathcal{O}_{R'}(x_0') \boxtimes \mathcal{O}_{R'}(x_0') \boxtimes \cdots \boxtimes \mathcal{O}_{R'}(x_0').$$

Since π_{k+1} is finite, we obtain the ampleness of $R'^{(k)} \subset R'^{(k+1)}$. □

We finish the proof of Theorem 0.4.

Proof of (†) for $e = 4$. First, note that there always exists a canonical line $l_0 \subset Y$ contained in V. This can be deduced from the Claim in the proof of Lemma 1.8 and specialization.

The Abel–Jacobi map

$$\mathrm{AJ} : \mathrm{CH}_1(V)_{\mathrm{hom}} \to J_V$$

of the cubic 3-fold V is an isomorphism of abelian groups; see [35, Theorem 5.6] and the references therein. Given the quartic $C \subset V$, consider

$$\mathrm{AJ}([C] + [l_0] - 5[l_{x_0}]) \in J_V.$$

By Corollary 2.8, there exist 5 special lines $l_1, \ldots, l_5$ such that

$$\mathrm{AJ}([C] + [l_0] - 5[l_{x_0}]) = \mathrm{AJ}\left(\sum_{i=1}^{5}[l_i] - 5[l_{x_0}]\right).$$

Hence we have

$$[C] = \sum_{i=1}^{5}[l_1] - [l_0] \in S_5(Y).$$ □

The argument above essentially proves the following result.

Corollary 2.9. *For any $\alpha \in \mathrm{CH}_1(Y)$ supported on a general hyperplane section $H \cap Y$, we have*

$$\alpha \in S_5(Y).$$

2.5. Another ten-dimensional example

Markushevich and Tikhomirov studied in [27, 28] the moduli space $\mathcal{M}_{\mathrm{MT}}$ of rank 2 vector bundles supported on nonsingular hyperplane sections $H \cap Y$ with $c_1 = 0$ and $c_2 = 2[l]$.

The (noncompact) moduli space $\mathcal{M}_{\mathrm{MT}}$ is nonsingular and holomorphic symplectic of dimension 10. Moreover, every object in $\mathcal{M}_{\mathrm{MT}}$ lies in the $K3$ category $\mathcal{A}_Y$ by [20, Lemma 7.2].

As a consequence of Corollary 2.9, we have the following proposition.

Proposition 2.10. *Conjecture 0.3 holds for any $\mathcal{E} \in \mathcal{M}_{\mathrm{MT}}$, i.e.,*

$$c_3(\mathcal{E}) \in S_5(Y).$$

Remark 2.11. Every object $\mathcal{E} \in \mathcal{M}_{\mathrm{MT}}$ is obtained from an extension

$$0 \to \mathcal{O}_V \to \mathcal{E}(H) \to \mathcal{I}_{E/V}(2H) \to 0,$$

where $V = H \cap Y$ is a nonsingular hyperplane section and E is a nonsingular quintic elliptic curve. The noncanonical part of $c_3(\mathcal{E})$ comes from the 1-cycle class of E.

2.6. Moduli of rational curves

Theorem 0.4 is closely related to holomorphic symplectic varieties constructed via the moduli spaces of rational curves in a cubic 4-fold, which we now discuss.

For convenience, we assume Y to be a general cubic 4-fold. Let $\mathcal{M}_e$ denote the moduli space of nonsingular connected rational curves of degree e in Y. By [13, 16], the variety $\mathcal{M}_e$ is irreducible of dimension $3e+1$. For $e \leqslant 4$, there is a MRC fibration

$$\pi_e : \mathcal{M}_e \dashrightarrow \mathcal{M}'_e \tag{2.7}$$

such that

(a) the base $\mathcal{M}'_e$ is a holomorphic symplectic variety;

(b) $\dim(\mathcal{M}'_e) = b(e)$.

We briefly review the geometry of the map (2.7). When $e = 1$, the variety $\mathcal{M}'_1$ is the Fano variety F of lines and (2.7) is an isomorphism. When $e = 2$, we still have $\mathcal{M}'_2 = F$ and the map (2.7) sends a conic to its residual line. Hence

$$\dim(\mathcal{M}'_1) = \dim(\mathcal{M}'_2) = 4.$$

When $e = 3$, the map (2.7) is constructed by Lehn, Lehn, Sorger, and van Straten in [24], and the holomorphic symplectic 8-fold $\mathcal{M}'_3$ is shown in [1] to be of $K3^{[4]}$ type. Finally, the case $e = 4$ is related to the recent work of Laza, Saccà, and Voisin [23, 42]. The variety $\mathcal{M}'_4$ is a holomorphic symplectic compactification of the (twisted) intermediate Jacobian fibration associated to Y, which is deformation equivalent to O'Grady's 10-dimensional variety [30].

In all four cases above, we expect[9] that a birational model of the holomorphic symplectic variety $\mathcal{M}'_e$ can be realized as a moduli space of stable objects in the $K3$ category $\mathcal{A}_Y$. Furthermore, for a general rational curve $C \in \mathcal{M}_e$ with $\mathcal{E}_C = \pi_e([C]) \in \mathcal{A}_Y$, there should exist integers $k \neq 0$ and m such that

$$c_3(\mathcal{E}_C) = k[C] + m[l_0] \in \mathrm{CH}_1(Y). \tag{2.8}$$

Here $l_0 \subset Y$ is a canonical line.

Theorem 0.4 says that for $e \leqslant 4$ and $C \in \mathcal{M}_e$, we have

$$[C] \in S_{\frac{1}{2}\dim \mathcal{M}'_e}(Y),$$

which is optimal in view of (2.8).

For $e \geqslant 5$, de Jong and Starr studied in [16] the canonical holomorphic 2-form on a nonsingular projective model of the moduli space $\mathcal{M}_e$. Inspired by [16, Theorem 1.2], we make the following speculation: for every $e \geqslant 5$, there exists an algebraically coisotropic subvariety of a holomorphic symplectic variety M,

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & M, \\ {\scriptstyle q}\,\downarrow & & \\ B & & \end{array}$$

[9]This was verified for the Fano variety of lines in [5] and the Lehn–Lehn–Sorger–van Straten 8-fold in [22].

which satisfies a list of properties.

(a) The variety M (or its birational model) can be realized as a moduli space of stable objects in $\mathcal{A}_Y$.

(b) The general fibers of q are constant cycle subvarieties of M.

(c) For a general point $z \in Z$ with $\mathcal{E}_z = j(z) \in \mathcal{A}_Y$, there exists a rational curve $C \in \mathcal{M}_e$ and integers $k \neq 0$ and m such that

$$c_3(\mathcal{E}_z) = k[C] + m[l_0] \in \mathrm{CH}_1(Y).$$

Here $l_0 \subset Y$ is a canonical line.

(d) The dimension of B is $2\mathbf{b}(e)$, where

$$\mathbf{b}(e) = \begin{cases} \dfrac{3e}{2} & e \text{ even,} \\ \dfrac{3e+1}{2} & e \text{ odd.} \end{cases}$$

When e is odd, de Jong and Starr showed that the canonical holomorphic 2-form on $\mathcal{M}_e$ is nondegenerate. Hence we expect $B \simeq \mathcal{M}_e$.

The speculation above, together with Voisin's proposal [41] and Speculation 0.1, suggests the following optimal bound for the classes of rational curves of degree $\geqslant 5$ with respect to the filtration $S_\bullet(Y)$.

Conjecture 2.12. *For any nonsingular connected rational curve $C \subset Y$ of degree $e \geqslant 5$, we have*

$$[C] \in S_{\mathbf{b}(e)}(Y).$$

Remark 2.13. For $e > 5$, the bound $\mathbf{b}(e)$ grows linearly with e, and clearly we have

$$\mathbf{b}(e) < b(e).$$

Since $\mathcal{M}_e$ is expected to govern only the point classes on an algebraically coisotropic subvariety in a holomorphic symplectic variety, the quadratic bound $b(e)$ should not be optimal for the classes of curves $C \in \mathcal{M}_e$.[10]

Indeed, the following proposition provides a (nonoptimal) linear bound for any curve in Y.[11]

Proposition 2.14. *For any curve $C \subset Y$ of degree e, we have*

$$[C] \in S_{42e}(Y).$$

[10] When $e = 5$, the two bounds $\mathbf{b}(5)$ and $b(5)$ agree. It is possible that $\mathcal{M}_5$ is birational to a holomorphic symplectic variety.

[11] We thank Claire Voisin for suggesting this.

Proof. We prove that there exist integers $k > 0$ and m such that

$$k[C] = -\big([l_1 + l_2 + \cdots + l_{21e}]\big) + m[l_0] \in \mathrm{CH}_1(Y), \tag{2.9}$$

where $l_i \subset Y$ are lines and $l_0 \subset Y$ is a canonical line. The proposition follows immediately from Proposition 1.10 and the $e = 1$ case of (†).

Recall that $P = \{(l, x) \in F \times Y : x \in l\}$ is the incidence variety with natural projections

$$p_F : P \to F, \quad p_Y : P \to Y.$$

Let D be a nonsingular divisor in the polarization class $g \in \mathrm{CH}^1(F)$. We have the following diagram

$$\begin{array}{ccc} P_D & \xrightarrow{f} & Y, \\ \downarrow{\scriptstyle p_D} & & \\ D & & \end{array}$$

where p_D is the restriction of p_F to $D \subset F$, and f is the composition of the inclusion $P_D \hookrightarrow P$ and p_Y. Then f is a finite morphism such that

$$\deg f \cdot [C] = f_* f^*[C] \in \mathrm{CH}_1(Y). \tag{2.10}$$

Since P_D is a projective bundle over D, the class $f^*[C]$ can be uniquely expressed as

$$f^*[C] = p_D^*\alpha_0 + p_D^*\alpha_1 \cdot f^*[H] \in \mathrm{CH}_1(P_D) \tag{2.11}$$

with $\alpha_i \in \mathrm{CH}_i(D)$ and $[H] \in \mathrm{CH}^1(Y)$ the hyperplane class. A direct calculation (as in the proof of [7, Proposition 6]) yields

$$\alpha_0 = -g|_D \cdot \alpha_1, \quad \alpha_1 = p_{D*} f^*[C].$$

In particular, we know that $-\alpha_0$ is an effective 0-cycle class on D. Combining (2.10) and (2.11), we find

$$\deg f \cdot [C] = f_*(p_D^*\alpha_0) + (f_* p_D^*\alpha_1) \cdot [H] \in \mathrm{CH}_1(Y). \tag{2.12}$$

Lemma 2.2 implies that $(f_* p_D^*\alpha_1) \cdot [H]$ is proportional to the class of a canonical line. The degree of the effective class $-\alpha_0$ is calculated by the intersection number

$$g|_D \cdot \alpha_1 = g|_D \cdot p_{D*} f^*(e[l]) = e(g^2 \cdot [S_l]) = 21e.$$

Here recall that $S_l \subset F$ is the surface formed by lines passing through a given line $l \subset Y$. The last equality above is given by [36, Lemma 18.2]. Hence (2.12) gives the required expression (2.9). □

3. Algebraically coisotropic subvarieties

Let M be a holomorphic symplectic variety of dimension $2d$. In [41], Voisin proposed the following conjecture.[12]

[12]It is clear that Conjecture 3.1 implies [41, Conjecture 0.4]. The converse is proven in [41, Theorem 1.3].

Conjecture 3.1 [41, Conjecture 0.4]. *For $0 \leqslant i \leqslant d$, there is a codimension i algebraically coisotropic subvariety*

$$\begin{array}{ccc} Z_i & \hookrightarrow & M, \\ \downarrow{\scriptstyle q} & & \\ B_i & & \end{array}$$

such that the general fibers of q are i-dimensional constant cycle subvarieties of M.

The following theorem is the main result of this section, which shows that the sheaf/cycle correspondence for $\mathcal{A}_Y$ can produce algebraically coisotropic varieties as in Conjecture 3.1 for all holomorphic symplectic moduli spaces of stable objects in $\mathcal{A}_Y$.

Theorem 3.2. *Conjecture 0.3 implies Conjecture 3.1 if the holomorphic symplectic variety M is a moduli space of stable objects in $\mathcal{A}_Y$ for a nonsingular cubic 4-fold Y.*

3.1. $K3$ surfaces

Let X be a $K3$ surface. Theorem 3.2 is parallel to [34, Theorem 0.5(i)] which proves Conjecture 3.1 when M is a moduli space of stable objects in $D^b(X)$. We briefly review the main steps of the proof of [34, Theorem 0.5(i)].

For the moment, assume that the holomorphic symplectic variety M is a $2d$-dimensional moduli space of stable objects in $D^b(X)$.

Step 1. Let $X^{(d)}$ be the symmetric product. Consider the incidence

$$R = \{(\mathcal{E}, \xi) \in M \times X^{(d)} : c_2(\mathcal{E}) = [\xi] + m[o_X] \in \mathrm{CH}_0(X)\},$$

which is a countable union of Zariski closed subsets of $M \times X^{(d)}$. We denote the natural projections by

$$p_M : R \to M, \quad p_{X^{(d)}} : R \to X^{(d)}.$$

By a result by Marian and Zhao [26], all points on the same fiber of $p_{X^{(d)}}$ (resp. p_M) have the same class in $\mathrm{CH}_0(M)$ (resp. $\mathrm{CH}_0(X^{(d)})$).

Step 2. O'Grady's conjecture [31], which was proven in full generality in [34], implies that both p_M and $p_{X^{(d)}}$ are surjective. Hence we can choose a component $R_0 \subset R$ dominating M and $X^{(d)}$,

$$\begin{array}{ccccc} & & R_0 & & \\ & \swarrow^{p_M} & & \searrow^{p_{X^{(d)}}} & \\ M & & & & X^{(d)}. \end{array} \tag{3.1}$$

Moreover, both morphisms p_M and $p_{X^{(d)}}$ in the diagram above are generically finite.

Step 3. For $i \leqslant d$, the codimension i algebraically coisotropic subvarieties with constant cycle fibers are dense in $X^{(d)}$. Hence we can always find an algebraically coisotropic subvariety $Z \subset X^{(d)}$ such that the morphism $p_{X^{(d)}}$ in (3.1) is generically finite over Z, and that the restriction of p_M to $p^{-1}_{X^{(d)}}(Z)$ is also generically finite. Then

$$Z' = p_M(p^{-1}_{X^{(d)}}(Z))$$

is a codimension i algebraically coisotropic subvariety of M which satisfies the condition in Conjecture 3.1.

The main difficulty of the proof of Theorem 3.2 is the absence of the $K3$ surface, which breaks down all three steps above. We show how to overcome this issue using the geometry of cubic 4-folds and their Fano varieties of lines.

3.2. Step 1

From now on, we take the holomorphic symplectic variety M to be a $2d$-dimensional moduli space of stable objects in the $K3$ category $\mathcal{A}_Y$. First, we modify Step 1 in § 3.1 by the following construction.

Let $D \subset F$ be a uniruled divisor over a surface B,

$$\begin{array}{ccc} D & \overset{j}{\hookrightarrow} & F. \\ {\scriptstyle q}\,\vdots & & \\ B & & \end{array}$$

We identify the Chow groups $\mathrm{CH}_0(D)$ and $\mathrm{CH}_0(B)$ via the isomorphism

$$q_* : \mathrm{CH}_0(D) \xrightarrow{\sim} \mathrm{CH}_0(B).$$

For $k > 0$, the embedding $j : D \hookrightarrow F$ induces a morphism

$$j_*^{(k)} : \mathrm{CH}_0(B^{(k)}) \to \mathrm{CH}_0(F).$$

We call $W \subset B^{(k)}$ an F-constant cycle subvariety if $j_*^{(k)}[w]$ is constant in $\mathrm{CH}_0(F)$ for every point $w \in W$.

Lemma 3.3. *There is a uniruled divisor D on F,*

$$\begin{array}{ccc} D & \overset{j}{\hookrightarrow} & F, \\ {\scriptstyle q}\,\vdots & & \\ B & & \end{array}$$

such that the surface B contains infinity many F-constant cycle curves $\{C_i\}$ whose union is Zariski dense in B.

Proof. First, we assume Y to be a general cubic 4-fold. In the proof of Lemma 1.8, we have constructed a uniruled divisor $q : D \dashrightarrow B$ such that B admits a fibration

$$g : B \to T$$

whose general fibers are elliptic curves. Moreover, the Claim in the proof of Lemma 1.8 implies that there exists a multi-section $C \subset B$ of the morphism g which is an F-constant cycle curve.

The required density is provided by the torsion structure of the elliptic curves on B. More precisely, all irreducible components of the locus

$$D_i = \{x \in B : i[x-y] = 0 \in \mathrm{CH}_0(g^{-1}(t)),\, y \in C, t \in T\}$$

give the curves C_i for $i > 0$.

Since specializations preserve uniruled divisors and (possibly singular) elliptic curves, we obtain the lemma for any cubic 4-fold. □

The elliptic surface B in Lemma 3.3 plays the role of the $K3$ surface X in § 3.1. Consider the following incidence

$$R = \{(\mathcal{E}, \xi) \in M \times B^{(d)} : c_3(\mathcal{E}) = [P]_* j_*^{(d)}[\xi] + m[l_0] \in \mathrm{CH}_1(Y)\},$$

where $[P]_*$ is the correspondence in (1.2) and l_0 is a canonical line. There are the two projections

$$p_M : R \to M, \quad p_{B^{(d)}} : R \to B^{(d)}.$$

The argument in [26] gives the following result.

Proposition 3.4. *Two objects $\mathcal{E}_1$, $\mathcal{E}_2 \in M$ satisfy*

$$[\mathcal{E}_1] = [\mathcal{E}_2] \in \mathrm{CH}_0(M)$$

if and only if

$$c_3(\mathcal{E}_1) = c_3(\mathcal{E}_2) \in \mathrm{CH}_1(Y). \tag{3.2}$$

Proof. We observe that the cycle class map

$$\mathrm{CH}^i(Y) \to H^{2i}(Y, \mathbb{Z})$$

is injective when $i \neq 3$. The statement for $i = 0$, 1, and 4 is immediate, and the $i = 2$ case follows from [9, Theorem 1(i) and (ii)]. Then, by Lemma 1.2, the condition (3.2) is equivalent to

$$\mathrm{ch}(\mathcal{E}_1) = \mathrm{ch}(\mathcal{E}_2) \in \mathrm{CH}^*(Y)_{\mathbb{Q}}.$$

The rest of the proof is the same as in [26] via the (quasi-)universal family over $M \times Y$. □

As a consequence of Proposition 3.4, all points on the same fiber of $p_{B^{(d)}}$ have the same class in $\mathrm{CH}_0(M)$. Moreover, by Proposition 1.3, every component of a fiber of p_M is an F-constant cycle subvariety.

3.3. Steps 2 and 3

We modify Steps 2 and 3 in § 3.1, and complete the proof of Theorem 3.2. Conjecture 0.3 now plays the role of O'Grady's conjecture for $K3$ surfaces.

The following proposition is parallel to [31, Proposition 1.3] and [40, Corollary 3.4].

Proposition 3.5. *Assuming Conjecture 0.3, there is a component $R_0 \subset R$ with the following diagram,*

$$\begin{array}{ccc} & R_0 & \\ {\scriptstyle p_M}\swarrow & & \searrow{\scriptstyle p_{B^{(d)}}} \\ M & & B^{(d)}, \end{array}$$

such that both morphisms p_M and $p_{B^{(d)}}$ are dominant and generically finite.

Proof. Conjecture 0.3 implies that $R \to M$ is dominant. Hence we can choose a component $R_0 \subset R$ such that $R_0 \to M$ is also dominant. Now it suffices to show that the other projection $p_{B^{(d)}} : R_0 \to B^{(d)}$ is also dominant.

Note that there is a nondegenerate[13] 2-form ω_B on B satisfying

$$j^*\sigma = q^*\omega_B,$$

where $\sigma \in H^0(F, \Omega_F^2)$ is the holomorphic symplectic form on F. The 2-form ω_B further induces a nondegenerate 2-form $\omega_B^{(d)}$ on $B^{(d)}$. We only need to prove that the pull-back of $\omega_B^{(d)}$ via $p_{B^{(d)}} : R_0 \to B^{(d)}$ coincides with the pull-back of the holomorphic symplectic form on M via p_M (up to scaling). This is deduced from the fact that fibers of p_M are F-constant cycle subvarieties in $B^{(d)}$ and from Mumford's theorem [29]. □

This gives the required modification of Step 2. Finally, the density result of Lemma 3.3 plays the role of [34, Lemma 2.4], and the proof of Theorem 3.2 is identical to that of [34, Theorem 0.5(i)].

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[13]Here we mean that the 2-form is nondegenerate on the regular locus.

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