MOTIVIC DECOMPOSITIONS FOR THE HILBERT SCHEME OF POINTS OF A K3 SURFACE

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ABSTRACT. We construct an explicit, multiplicative Chow–Kühneth decomposition for the Hilbert scheme of points of a K3 surface. We further refine this decomposition with respect to the action of the Looijenga–Lunts–Verbitsky Lie algebra.

1. Introduction

This note is a continuation of [27] by the second author. We study the motivic aspects of the Looijenga–Lunts–Verbitsky (LLV for short) Lie algebra action on the Chow ring of the Hilbert scheme of points of a K3 surface. Using a special element of the LLV algebra and formulas of [22] by Maulik and the first author, we construct an explicit Chow–Kühneth decomposition for the Hilbert scheme, prove its multiplicativity, and show that all divisor classes and Chern classes lie in the correct component of the decomposition. This confirms expectations of Beauville [2] and Voisin [33]. We also obtain a refined motivic decomposition for the Hilbert scheme by taking into account the LLV algebra action.

Both results parallel the case of an abelian variety, which we shall briefly review.

1.1. Abelian varieties. Let $X$ be an abelian variety of dimension $g$. Recall the classical result of Deninger–Murre on the decomposition of the Chow motive $\mathfrak{h}(X)$.

Theorem 1.1 ([5]). There is a unique, multiplicative Chow–Kühneth decomposition

$$
\mathfrak{h}(X) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i(X)
$$

such that for all $N \in \mathbb{Z}$, the multiplication $[N] : X \to X$ acts on $\mathfrak{h}^i(X)$ by $[N]^* = N^i$.

The decomposition (1) specializes to the Kühneth decomposition in cohomology (hence the name Chow–Kühneth), and to the Beauville decomposition [4] in Chow. The latter takes the form

$$
A^*(X) = \bigoplus_{i,s} A^i(X)_s
$$

with

$$
A^i(X)_s = A^i(\mathfrak{h}^{2i-s}(X)) = \{ \alpha \in A^i(X) \mid [N]^* \alpha = N^{2i-s}\alpha, \text{ for all } N \in \mathbb{Z} \}.
$$

The multiplicativity of (1) stands for the fact that the cup product $\cup : \mathfrak{h}(X) \otimes \mathfrak{h}(X) \to \mathfrak{h}(X)$
respects the grading, in the sense that
\[ U : \mathfrak{h}^i(X) \otimes \mathfrak{h}^j(X) \to \mathfrak{h}^{i+j}(X) \]
for all \( i, j \in \{0, ..., 2g\} \). This can be seen by simply comparing the actions of \([N]^*\).

As a result, the bigrading in (2) is multiplicative, i.e., compatible with the ring structure of \( A^*(X) \).

The Beauville decomposition is expected to provide a multiplicative splitting of the conjectural Bloch–Beilinson filtration on \( A^*(X) \). A difficult conjecture of Beauville (and consequence of the Bloch–Beilinson conjecture) predicts the vanishing \( A^s(X) = 0 \) for \( s < 0 \) and the injectivity of the cycle class map \( \text{cl} : A^0(X) \to H^*(X) \).

Further, any symmetric ample class \( \alpha \in A^1(X)_0 \) induces an \( \mathfrak{sl}_2 \)-triple \((e_\alpha, f_\alpha, h)\) acting on \( A^*(X) \). A Lefschetz decomposition of \( h(X) \) with respect to the \( \mathfrak{sl}_2 \)-action was obtained by K"unnemann [10], refining [1]. More generally, Moonen [23] constructed an action of the Néron–Severi part of the Looijenga–Lunts [20] Lie algebra \( \mathfrak{g}_{NS} \) on \( A^*(X) \), which contains all possible \( \mathfrak{sl}_2 \)-triples above (he actually considered the slightly larger Lie algebra \( \mathfrak{sp}(X \times X^\vee) \); see [23, Section 6]). He then obtained a refined motivic decomposition with respect to the \( \mathfrak{g}_{NS} \)-action.

**Theorem 1.2** ([23]). There is a unique decomposition
\[ h(X) = \bigoplus_{\psi \in \text{Irrep}(\mathfrak{g}_{NS})} h_\psi(X) \]
where \( \psi \) runs through all isomorphism classes of finite-dimensional irreducible representations of \( \mathfrak{g}_{NS} \), and \( h_\psi(X) \) is \( \psi \)-isotypic under \( \mathfrak{g}_{NS} \).

Here being \( \psi \)-isotypic means that \( h_\psi(X) \) is stable under \( \mathfrak{g}_{NS} \) and that for any Chow motive \( M \), the \( \mathfrak{g}_{NS} \)-representation \( \text{Hom}(M, h_\psi(X)) \) is isomorphic to a direct sum of copies of \( \psi \).

Again (3) specializes to refined decompositions in cohomology and in Chow.

**1.2. Chow–Künneth.** We switch to the Hilbert scheme case. Let \( S \) be a projective K3 surface and let \( X = \text{Hilb}_n(S) \) be the Hilbert scheme of \( n \) points on \( S \).

In [27], the action of the Néron–Severi part of the LLV algebra \( \mathfrak{g}_{NS} \) was lifted from cohomology to Chow. In particular, there is an explicit grading operator
\[ h \in A^{2n}(X \times X) \]
which appears in every \( \mathfrak{sl}_2 \)-triple \((e_\alpha, f_\alpha, h)\) in \( \mathfrak{g}_{NS} \). We normalize \( h \) so that it acts on \( H^{2i}(X) \) by multiplication by \( i - n \).

We regard \( h \) as a natural replacement for the operator \([N]^*\) in the abelian variety case. Our first result decomposes the Chow motive \( h(X) \) into eigen-submotives of \( h \).

**Theorem 1.3.** There is a unique Chow–Künneth decomposition
\[ h(X) = \bigoplus_{i=0}^{2n} h^{2i}(X) \]
such that \( h \) acts on \( h^{2i}(X) \) by multiplication by \( i - n \).
The mutually orthogonal projectors in the decomposition (4) are written explicitly in terms of the Heisenberg algebra action [15, 25]. We also show that (4) agrees with the Chow–Künneth decomposition obtained by de Cataldo–Migliorini [4] and Vial [32].

As before (4) specializes to a decomposition in Chow
\[ A^*(X) = \bigoplus_{i,s} A^i(X)_{2s} \]
with
\[ A^i(X)_{2s} = A^i(h^{2i-2s}(X)) = \{ \alpha \in A^i(X) \mid h(\alpha) = (i - s - n)\alpha \}. \]

1.3. Multiplicativity. In the seminal paper [2], Beauville raised the question whether hyper-Kähler varieties behave similarly to abelian varieties in the sense that the conjectural Bloch–Beilinson filtration also admits a multiplicative splitting. As a test case, he conjectured that for a hyper-Kähler variety, the cycle class map is injective on the subring generated by divisor classes.

For the Hilbert scheme of points of a K3 surface, Beauville’s conjecture was recently proven in [22]; see also [27] for a shorter proof. But the ultimate goal remains to find the multiplicative splitting. Meanwhile, Shen and Vial [29, 30] introduced the notion of a multiplicative Chow–Künneth decomposition, upgrading Beauville’s question from Chow groups to the level of correspondences/Chow motives.

The main result of this paper confirms that (4) provides a multiplicative Chow–Künneth decomposition for the Hilbert scheme.

**Theorem 1.4.** Let \( S \) be a projective K3 surface and let \( X = \text{Hilb}_n(S) \).

(i) The Chow–Künneth decomposition (4) is multiplicative, i.e., the cup product
\[ \cup : h(X) \otimes h(X) \rightarrow h(X) \]
respects the grading, in the sense that
\[ \cup : h^{2i}(X) \otimes h^{2j}(X) \rightarrow h^{2i+2j}(X) \]
for all \( i, j \in \{0, ..., 2n\} \). As a result, the bigrading in (5) is multiplicative.

(ii) All divisor classes and Chern classes of \( X \) belong to \( A^*(X)_{0} \).

Theorem 1.4 (ii) is related to the Beauville–Voisin conjecture [33], which predicts that for a hyper-Kähler variety, the cycle class map is injective on the subring generated by divisor classes and Chern classes. In the Hilbert scheme case, one may further ask the vanishing \( A^*(X)_{2s} = 0 \) for \( s < 0 \) and the injectivity of the cycle class map
\[ \text{cl} : A^*(X)_0 \rightarrow H^*(X). \]
We do not tackle these questions in the present paper.

The key to the proof of Theorem 1.4 is the compatibility between the grading operator \( h \) and the cup product. For example, at the level of Chow groups, we show that the operator
\[ \tilde{h} = h + n\Delta_X \in A^{2n}(X \times X) \]
acts on \( A^*(X) \) by derivation, i.e.,
\[ \tilde{h}(x \cdot x') = \tilde{h}(x) \cdot x' + x \cdot \tilde{h}(x') \]
for all $x, x' \in A^*(X)$. We achieve this by explicit calculations using the Chow lifts [22] of the well-known machinery for the Heisenberg algebra action [18, 19]. In fact, our argument yields [9] at the level of correspondences; see Section 4. Once the compatibility is established, Theorem 1.4 is deduced by simply comparing the eigenvalues.

1.4. Previous work. Theorem 1.4 was previously obtained by Vial [32] based on Voisin’s announced result [34, Theorem 5.12] on universally defined cycles. A second proof, also relying on Voisin’s theorem, was given by Fu and Tian [9]. They interpreted Theorem 1.4 (i) as the motivic incarnation of Ruan’s crepant resolution conjecture [28]. Our proof has the advantage of being explicit and unconditional at the moment.

We note that multiplicative Chow–Künneth decompositions, for both hyper-Kähler and non-hyper-Kähler varieties, have been studied in [6, 7, 8, 10, 11, 12, 17].

1.5. Refined decomposition. We further obtain a refined decomposition of the Chow motive $h(X)$ with respect to the action of the Néron–Severi part of the LLV algebra $g_{NS}$. Both the statement and the proof parallel the abelian variety case.

**Theorem 1.5.** Let $S$ be a projective K3 surface and let $X = \text{Hilb}_n(S)$. There is a unique decomposition

$$h(X) = \bigoplus_{\psi \in \text{Irrep}(g_{NS})} h_\psi(X)$$

where $\psi$ runs through all isomorphism classes of finite-dimensional irreducible representations of $g_{NS}$, and $h_\psi(X)$ is $\psi$-isotypic under $g_{NS}$.

As before (7) specializes to refined decompositions in cohomology and in Chow.

We may also consider a Cartan subalgebra $t \subset g_{NS}$ and note that the decomposition (7) in terms of the irreducible representations of $g_{NS}$ implies a similar decomposition in terms of the irreducible representations (i.e., characters) of $t$.

More precisely, recall the weight decomposition

$$g_{NS} = g_{NS,-2} \oplus g_{NS,0} \oplus g_{NS,2}, \quad g_{NS,0} = \overline{g}_{NS} \oplus \mathbb{Q} \cdot h$$

where $\overline{g}_{NS}$ is the Néron–Severi part of the reduced LLV algebra (terminology taken from [14]). Let $\bar{t} \subset \overline{g}_{NS}$ be a Cartan subalgebra and write

$$\bar{t} = \tilde{t} \oplus \mathbb{Q} \cdot \tilde{h}.$$ 

(so $\tilde{t}$ differs from $t$ in that the element $h$ was replaced by $\tilde{h} = h + n\Delta_X$). Hence there is a motivic decomposition

$$h(X) = \bigoplus_{\lambda \in \bar{t}^*} h_\lambda(X).$$

We expect (8) to also be multiplicative, in the sense that the cup product sends

$$\cup : h_\lambda(X) \otimes h_\mu(X) \to h_{\lambda+\mu}(X).$$

To this end, it would suffice to prove the analogue of (9) when $\tilde{h}$ is replaced by any element in $\overline{g}_{NS}$. The generators of $\overline{g}_{NS}$ are denoted by

$$h_{\alpha\beta} \in A^{2n}(X \times X)$$

and indexed by $\alpha \wedge \beta$ in $\wedge^2(A^1(X))$. For $n > 1$, we have

$$A^1(X) \cong A^1(S) \oplus \mathbb{Q} \cdot \delta$$
where $\delta$ corresponds to the exceptional divisor of the Hilbert scheme. Toward the goal of proving (9), we obtain the following result.

**Theorem 1.6.** For all $\alpha, \beta \in A^1(S) \subset A^1(X)$, we have

$$h_{\alpha \beta}(x \cdot x') = h_{\alpha \beta}(x) \cdot x' + x \cdot h_{\alpha \beta}(x')$$

for all $x, x' \in A^*(X)$.

We believe that the statement below can be proved with the machinery described in Section 4, but the combinatorics is quite involved, and we therefore do not tackle it.

**Conjecture 1.7.** For all $\alpha \in A^1(S) \subset A^1(X)$, we have

$$h_{\alpha \delta}(x \cdot x') = h_{\alpha \delta}(x) \cdot x' + x \cdot h_{\alpha \delta}(x')$$

for all $x, x' \in A^*(X)$.

An equivalent formulation of Conjecture 1.7 is that $g_{\text{NS}}$ acts on $A^*(X)$ by derivations. Since the modified Cartan subalgebra $\tilde{t}$ is contained in $g_{\text{NS}} \oplus \mathbb{Q} \cdot \tilde{h}$, Theorem 1.6 and Conjecture 1.7 (both at the level of correspondences) would imply the multiplicativity of the refined decomposition (8). On its own, Theorem 1.6 can only imply the weaker statement where the direct sum in (8) goes over the dual of a modified Cartan subalgebra of dimension 1 smaller than $t$.

1.6. **Conventions.** Throughout the present paper, Chow groups and Chow motives will be taken with $\mathbb{Q}$-coefficients. We refer to [24] for the definitions and conventions of Chow motives.

We will often switch between the languages of correspondences and operators on Chow groups, in the following sense. Every operator $f : A^*(X) \to A^*(Y)$ will arise from a correspondence $F \in A^*(X \times Y)$ by the usual construction

$$f = \pi_{2*}(F \cdot \pi_1^*)$$

and any compositions and equalities of operators implicitly entail compositions and equalities of correspondences. For example, the operator

$$\text{mult}_{\tau} : A^*(X) \to A^*(X)$$

of cup product with a fixed element $\tau \in A^*(X)$ is associated to the correspondence $\Delta_\tau(\tau) \in A^*(X \times X)$, where $\Delta : X \to X \times X$ is the diagonal embedding.

Moreover, a family of operators $f_\gamma : A^*(X) \to A^*(Y)$ labeled by $\gamma \in A^*(Z)$ will arise from a correspondence $F \in A^*(X \times Y \times Z)$, by the assignment

$$f_\gamma \text{ arises from } \pi_{12*}(F \cdot \pi_3^*(\gamma)) \in A^*(X \times Y)$$

for all $\gamma \in A^*(Z)$. We employ the language of “operators indexed by $\gamma \in A^*(Z)$” instead of cycles on $X \times Y \times Z$ because it makes manifest the fact that $\gamma$ does not play any role in taking compositions. For instance, the family of operators

$$\text{mult}_\gamma : A^*(X) \to A^*(X)$$

labeled by $\gamma \in A^*(X)$ is associated to the small diagonal $\Delta_{123} \subset X \times X \times X$. 

We will often be concerned with cycles on a variety of the form \( S^n = S \times \ldots \times S \) for a smooth algebraic variety \( S \) (most often an algebraic surface). We let

\[ \Delta_{a_1 \ldots a_k} \in A^*(S^n) \]

denote the diagonal \( \{ (x_1, \ldots, x_n) \mid x_{a_1} = \ldots = x_{a_k} \} \), for all collections of distinct indices \( a_1, \ldots, a_k \in \{ 1, \ldots, n \} \). Moreover, given a class \( \Gamma \in A^*(S^k) \), we may choose to write it as \( \Gamma_{a_1 \ldots a_k} \) in order to indicate the power of \( S \) where this class lives. Then for any collection of distinct indices \( a_1, \ldots, a_k \in \{ 1, \ldots, n \} \), we define

\[ \Gamma_{a_1 \ldots a_k} = p_{a_1 \ldots a_k}^*(\Gamma) \in A^*(S^n) \]

where we let \( p_{a_1 \ldots a_k} : S^n \to S^k \) with \( p_i : S^n \to S \) the projection to the \( i \)-th factor. Finally, if \( \bullet \) denotes any index from 1 to \( k+1 \), we write

\[ \int_\bullet : A^*(S^{k+1}) \to A^*(S^k) \]

for the push-forward map which forgets the factor labeled by \( \bullet \).

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2. Hilbert schemes

2.1. Throughout the present paper, \( S \) will denote a projective K3 surface over \( \mathbb{C} \), and \( A^*(S) \) will denote its Chow ring with coefficients in \( \mathbb{Q} \), graded by codimension. Beauville and Voisin \cite{BeauvilleVoisin} have studied the class \( c \in A^2(S) \) of any closed point on a rational curve in \( S \), and they proved the following formulas in \( A^*(S) \):

\[ c_2(Tan_S) = 24c \]
\[ \alpha \cdot \beta = \langle \alpha, \beta \rangle c \]

for all \( \alpha, \beta \in A^1(S) \) (above, we write \( \langle \cdot, \cdot \rangle : A^*(S) \otimes A^*(S) \to \mathbb{Q} \) for the intersection pairing). Moreover, we have the following identities in \( A^*(S^2) \):

\[ \Delta \cdot c_1 = \Delta \cdot c_2 = c_1 \cdot c_2 \]
\[ \Delta \cdot \alpha_1 = \Delta \cdot \alpha_2 = \alpha_1 \cdot c_2 + \alpha_2 \cdot c_1 \]

where \( \Delta \in A^*(S^2) \) is the class of the diagonal, and the following identity in \( A^*(S^3) \):

\[ \Delta_{123} = \Delta_{12} \cdot c_3 + \Delta_{13} \cdot c_2 + \Delta_{23} \cdot c_1 - c_1 \cdot c_2 - c_1 \cdot c_3 - c_2 \cdot c_3 \]

As a corollary of (10) and (12), one can prove by induction the following identity:

\[ \sum_{i=1}^k \Delta_{1 \ldots i-1,i+1 \ldots k} c_i = (k-2)\Delta_{1 \ldots k} + \sum_{i=1}^k c_1 \ldots c_{i-1} c_{i+1} \ldots c_k \]

in \( A^*(S^k) \) for all \( k \).
Proposition 2.1. For any $\Gamma_{12...k} \in A^*(S^k)$, we have the following identity:

\[
(14) \quad (\alpha_0 - \alpha_1)(\Gamma_{02...k} - \Gamma_{12...k}) = \Delta_{01} \left( \int \Gamma_{\bullet 2...k}(\alpha_0 - \Gamma_{1...k}) \right)
\]

\[
(15) \quad (\alpha_0\beta_1 - \alpha_1\beta_0)(\Gamma_{02...k} - \Gamma_{12...k}) = \Delta_{01} \int \Gamma_{\bullet 2...k}(\alpha_0\beta_0 - \alpha_0\beta_\bullet)
\]

in $A^*(S^{k+1})$, for any $\alpha, \beta \in A^1(S)$.

Proof. To prove (14), let us consider the identity (12) on $S^3$:

\[
\Delta_{01\bullet} = \Delta_0 \cdot c_1 + \Delta_1 \cdot c_0 + \Delta_0 \cdot c_\bullet - c_0 \cdot c_1 - c_0 \cdot c_\bullet - c_1 \cdot c_\bullet.
\]

Let us pull this identity back to $S^{k+2}$ with indices $\bullet, 0, ..., k$, multiply it by $\Gamma_{\bullet 2...k}$, and then push forward by forgetting the factor represented by the index $\bullet$:

\[
\Delta_{01\bullet} = c_1\Gamma_{02...k} + c_0\Gamma_{12...k} + \int \left( \Delta_{01\bullet} \Gamma_{\bullet 2...k} - c_0 c_1 \Gamma_{\bullet 2...k} - c_0 c_\bullet \Gamma_{\bullet 2...k} - c_1 c_\bullet \Gamma_{\bullet 2...k} \right).
\]

Using the identity (10) we have:

\[
\int c_0 c_\bullet \Gamma_{\bullet 2...k} = \int c_0 \Delta_{0\bullet} \Gamma_{\bullet 2...k} = c_0 \Gamma_{02...k}
\]

and similarly:

\[
\int c_0 c_1 \Gamma_{\bullet 2...k} = \Delta_{01} \int \Gamma_{\bullet 2...k}, \quad \int c_1 c_\bullet \Gamma_{\bullet 2...k} = c_1 \Gamma_{1...k}.
\]

Inserting these into (16) the formula (14) then follows from rearranging the terms.

As for (15), we start from the identity:

\[
(\alpha_0\beta_1 - \alpha_1\beta_0)(\Delta_{0\bullet} - \Delta_{1\bullet}) = \Delta_{01}(\alpha_0\beta_0 - \alpha_0\beta_\bullet)
\]

which is a straightforward application of (11). If we pull this identity back to $S^{k+2}$ with indices $\bullet, 0, ..., k$, multiply it by $\Gamma_{\bullet 2...k}$, then we obtain the following:

\[
(\alpha_0\beta_1 - \alpha_1\beta_0)(\Delta_{0\bullet} \Gamma_{02...k} - \Delta_{1\bullet} \Gamma_{12...k}) = \Delta_{01}(\alpha_0\beta_0 - \alpha_0\beta_\bullet)\Gamma_{\bullet 2...k}.
\]

If we push forward by forgetting the factor represented by $\bullet$, we obtain (15). \qed

2.2. Consider the Hilbert scheme $\text{Hilb}_n$ of $n$ points on $S$ and the Chow rings:

\[
\text{Hilb} = \bigcup_{n=0}^{\infty} \text{Hilb}_n, \quad A^*(\text{Hilb}) = \bigoplus_{n=0}^{\infty} A^*(\text{Hilb}_n)
\]

always with rational coefficients. We will consider two types of elements of the Chow rings above. The first of these are defined by considering the universal subscheme:

\[
\mathcal{Z}_n \subset \text{Hilb}_n \times S
\]

($\mathcal{Z}_n$ is flat over $\text{Hilb}_n$, and its fibers have the property that $\mathcal{Z}_n|_{\{Z\} \times S} \cong Z$ for any point $\{Z\} \in \text{Hilb}_n$ corresponding to a closed subscheme $Z \subset S$). For any $k \in \mathbb{N}$, let $\pi : \text{Hilb}_n \times S^k \to \text{Hilb}_n$ denote the usual projection, and $\mathcal{Z}^{(i)}_n \subset \text{Hilb}_n \times S^k$ denote the pull-back of $\mathcal{Z}_n \subset \text{Hilb}_n \times S$ via the $i$-th projection map $S^k \to S$. 

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Definition 2.2. A universal class is any element of $A^*(\text{Hilb}_{n})$ of the form:

\begin{equation}
\pi_* \left[ P(...) \cdot \text{ch}_j \left( O_{Z_n^{(i)}} \right), ..., \prod_{j \in \mathbb{N}} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \right]^{1 \leq i \leq k}
\end{equation}

for all $k \in \mathbb{N}$ and for all polynomials $P$ with coefficients pulled back from $A^*(S^k)$.

The following theorem holds for every smooth quasi-projective surface (see [26]), but we only prove it here in the case where $S$ is a K3 surface (the argument herein easily generalizes to any smooth projective surface using the results of [13]).

**Theorem 2.3.** Any class in $A^*(\text{Hilb}_{n})$ is universal, i.e., of the form (17).

**Proof.** Consider the product $\text{Hilb}_{n} \times S^k \times \text{Hilb}_{n}$, and we will write $\pi_1, \pi_2, \pi_3, \pi_{12}, \pi_{23}$ and $\pi_{13}$ for the various projections to its factors. As a consequence of [21] (see also [13]), the diagonal $\Delta \subset \text{Hilb}_{n} \times \text{Hilb}_{n}$ can be written as follows:

$$|\Delta| = \pi_{13*} \left[ \sum_a \pi_2^*(\gamma_a) \prod_{(i,j)} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \prod_{(i,j)} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \pi_3^* \right]$$

for suitably chosen $k \in \mathbb{N}$, where we do not care much about the specific coefficients $\gamma_a$ and indices $i,j,i,j$ which appear in the sum above (we write $Z_n$ and $\tilde{Z}_n$ for the universal subschemes in $\text{Hilb}_{n} \times S \times \text{Hilb}_{n}$ corresponding to the first and second, respectively, copies of $\text{Hilb}_{n}$). Since the diagonal corresponds to the identity operator, the equality above implies that:

$$\text{Id}_{\text{Hilb}_{n}} = \pi_{1*} \left[ \sum_a \pi_2^*(\gamma_a) \prod_{(i,j)} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \prod_{(i,j)} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \pi_3^* \right]$$

\begin{equation}
= \sum_a \pi_* \left[ \prod_{(i,j)} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \rho^* \left( \gamma_a \cdot \rho_* \left( \prod_{(i,j)} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \cdot \pi_* \right) \right) \right]
\end{equation}

where in (18), $\pi, \rho : \text{Hilb}_{n} \times S^k \to \text{Hilb}_{n}, S^k$ denote the standard projections. \qed

Formula (18) implies the existence of a surjective homomorphism:

$$\bigoplus_a A^*(S^k) \to A^*(\text{Hilb}_{n})$$

(19)

$$\sum_a \Gamma_a \to \sum_a \pi_* \left[ \prod_{(i,j)} \text{ch}_j \left( O_{Z_n^{(i)}} \right) \rho^*(\Gamma_a) \right]$$

where the sums over $a$ are in one-to-one correspondence with the sums in (18).

2.3. Let us present another important source of elements of $A^*(\text{Hilb}_{n})$, based on the following construction independently due to Grojnowski [15] and Nakajima [25] (in the present paper, we will mostly use the presentation by Nakajima). For any $n,k \in \mathbb{N}$, consider the closed subscheme:

$$\text{Hilb}_{n,n+k} = \left\{ (I \supset I') | I/I' \text{ is supported at a single } x \in S \right\} \subset \text{Hilb}_{n} \times \text{Hilb}_{n+k}$$
endowed with projection maps:

\[
\begin{array}{ccc}
\text{Hilb}_{n,n+k} & \xrightarrow{p_-} & \text{Hilb}_n \\
& S & \xleftarrow{p_+} \text{Hilb}_{n+k}
\end{array}
\]

that remember \( I, x, I' \), respectively. One may use \( \text{Hilb}_{n,n+k} \) as a correspondence:

\[
A^*(\text{Hilb}_n) \xrightarrow{q_{\pm k}} A^*(\text{Hilb}_{n\pm k} \times S)
\]

given by:

\[
(20) \quad q_{\pm k} = (\pm 1)^k \cdot (p_{\pm} \times ps)_* \circ p_{\mp}^*.
\]

Because the correspondences above are defined for all \( n \), it makes sense to set:

\[
A^*(\text{Hilb}) \xrightarrow{q_{\pm k}} A^*(\text{Hilb} \times S).
\]

We also set \( q_0 = 0 \). The main result of [25] is that the operators \( q_k \) obey the commutation relations in the Heisenberg algebra, namely:

\[
(21) \quad [q_k, q_l] = k\delta_{k+l} (\text{Id}_{\text{Hilb} \times S}^*)
\]

as correspondences \( A^*(\text{Hilb}) \to A^*(\text{Hilb} \times S^2) \). In terms of self-correspondences \( A^*(\text{Hilb}) \to A^*(\text{Hilb}) \), the identity (21) reads, for all \( \alpha, \beta \in A^*(S) \):

\[
(22) \quad [q_k(\alpha), q_l(\beta)] = k(\alpha, \beta) \text{Id}_{\text{Hilb}}.
\]

2.4. More generally, we may consider:

\[
q_{n_1} \ldots q_{n_t} : A^*(\text{Hilb}) \to A^*(\text{Hilb} \times S^t)
\]

where the convention is that the operator \( q_{n_i} \) acts in the \( i \)-th factor of \( S^t = S \times \ldots \times S \).

Then associated to any \( \Gamma \in A^*(S^t) \), one obtains an endomorphism of \( A^*(\text{Hilb}) \):

\[
q_{n_1} \ldots q_{n_t}(\Gamma) = \pi_*(\rho^*(\Gamma) \cdot q_{n_1} \ldots q_{n_t})
\]

where \( \pi, \rho : \text{Hilb} \times S^t \to \text{Hilb}, S^t \) denote the standard projections.

**Theorem 2.4** ([3]). We have a decomposition:

\[
(23) \quad A^*(\text{Hilb}) = \bigoplus_{\Gamma \in A^*(S^t), \gamma \in \gamma_{\text{sym}}} q_{n_1} \ldots q_{n_t}(\Gamma) \cdot v
\]

where “sym” refers to the part of \( A^*(S^t) \) which is symmetric with respect to those transpositions \( (ij) \in \mathfrak{S}_t \) for which \( n_i = n_j \), and \( v \) is a generator of \( A^*(\text{Hilb}_0) \cong \mathbb{Q} \).

**Proof.** Since we will need it later, we recall the precise relationship between Nakajima operators and the correspondences studied in [3]. Let \( \lambda \) be a partition of \( n \) with \( k \) parts, let \( S^\lambda = S^k \) and let \( S^\lambda \to S^{(n)} \) be the map that sends \( (x_1, \ldots, x_k) \) to the cycle \( \lambda_1 x_1 + \ldots + \lambda_k x_k \) in the \( n \)-th symmetric product of the surface \( S \). We consider the correspondence:

\[
\Gamma_\lambda = (\text{Hilb}_n \times_{S^{(n)}} S^\lambda)_{\text{red}}
\]

\[
= \{ (I, x_1, \ldots, x_k) \mid \sigma(I) = \lambda_1 x_1 + \ldots + \lambda_k x_k \}
\]
where $\sigma : S^{[n]} \to S^{(n)}$ is the Hilbert–Chow morphism sending the subscheme $I$ to its underlying support. The subscheme $\Gamma_\lambda$ is irreducible of dimension $n + k$ and the locus $\Gamma^\text{reg}_\lambda \subset \Gamma_\lambda$, where the points $x_i$ are distinct, is open and dense; see [4] Remark 2.0.1. Similarly, the Nakajima correspondence $q_{\lambda_1} \cdots q_{\lambda_k}$ is a cycle in $\text{Hilb}_n \times S^k$ of dimension $n + k$ supported on a subscheme that contains $\Gamma^\text{reg}_\lambda$ as an open subset and whose complement is of smaller dimension [25, 4(i)]. Moreover the multiplicity of the cycle on $\Gamma^\text{reg}_\lambda$ is 1. Hence we have the equality of correspondences:

$$\Gamma_\lambda = q_{\lambda_1} \cdots q_{\lambda_k} \in A^*(\text{Hilb}_n \times S^\lambda).$$

The result follows now from [4, Proposition 6.1.5], which says that:

$$\Delta_{\text{Hilb}_n} = \sum_{\lambda \vdash n} (-1)^{n - l(\lambda)} \frac{\text{Aut}(\lambda)}{\prod_i \lambda_i} (\chi^{(\lambda)})^f \circ \Gamma_\lambda$$

where $\lambda$ runs over all partitions of size $n$, and we let $l(\lambda)$ and $\lambda_i$ denote the length and the parts of $\lambda$.

**Remark 2.5.** As shown in [25], there is an explicit way to go between the descriptions (17) and (23) of $A^*(\text{Hilb})$. Concretely, for all $n_1 \geq \ldots \geq n_t$ there exists an algorithm for computing the polynomial $P_{n_1, \ldots, n_t}$ with coefficients in $\rho^*(A^*(S^t))$ such that for all $\Gamma \in A^*(S^t)$:

$$q_{n_1} \cdots q_{n_t}(\Gamma) = \pi_* \left[ P_{n_1, \ldots, n_t}(\ldots, \chi_j(O_{\mathbb{G}(i)}), \ldots)_{1 \leq i \leq t} \cdot \rho^*(\Gamma) \right]$$

where $\pi, \rho : \text{Hilb} \times S^t \to \text{Hilb}, S^t$ are the standard projections. Moreover, loc. cit. gives an algorithm for computing the polynomial $P_{n_1, \ldots, n_t}$.

2.5. We will now present another connection between universal classes and the operators $q_n$. Let us consider any class of the form:

$$\text{univ}(\Gamma) = \pi_* \left[ \prod_{i=1}^t \chi_{d_i}(O_{\mathbb{G}(i)}) \cdot \rho^*(\Gamma) \right] \in A^*(\text{Hilb}_n)$$

where $\pi, \rho : \text{Hilb}_n \times S^t \to \text{Hilb}, S^t$ are the standard projections, while the natural numbers $d_1, \ldots, d_t$ and the class $\Gamma \in A^*(S^t)$ are arbitrary. We will write:

$$\text{univ}_{d_1, \ldots, d_t}(\Gamma) = \text{univ}(\Gamma)$$

if we wish to emphasize the particular numbers $d_1, \ldots, d_t$ which appear in (26), although they will often be inconsequential. Note that the codimension of (26) is:

$$\deg \text{univ}_{d_1, \ldots, d_t}(\Gamma) = \deg \Gamma + \sum_{i=1}^t (d_i - 2).$$

**Lemma 2.6.** *The operator of multiplication by $\text{univ}(\Gamma)$ is given by:*

$$\text{mult}_{\text{univ}(\Gamma)} = \sum_{\lambda_{b_1+1} \geq \ldots \geq \lambda_{b_s} \in \mathbb{Z}, \forall s \in \{1, \ldots, t\}} \sum_{\lambda_{b_1+1} + \ldots + \lambda_{b_t} = 0, \forall s \in \{1, \ldots, t\}} c_{t, q_{\lambda_1} \cdots q_{\lambda_{b_t}}(\Delta_{b_0+1} \cdots \Delta_{b_1} \cdots \Delta_{b_{t-1}+1} \cdots \Gamma_{b_1 \ldots b_t}) \phi_{b_1} \ldots \phi_{b_t})}$$

where in each summand we write, for all $s \in \{0, \ldots, t\}$:

$$b_s = \sum_{i=1}^s (d_i - \varepsilon_s)$$
and \( \phi_{b_s} \) is either 1 or \( c_{b_s} \), depending on whether \( \varepsilon_s \) is 0 or 2. The constants “\( c_t \)” in (27) depend on the particular numbers \( d_i, \varepsilon_s \) and \( \lambda_k \) but not on \( \Gamma \).

**Proof.** In the course of this proof, let \( \pi, \rho : \text{Hilb} \times S \to \text{Hilb}, S \) denote the standard projections. Let us recall the operators of multiplication by universal classes:

\[
\mathfrak{G}_d : A^*(\text{Hilb}) \xrightarrow{\pi^*} A^*(\text{Hilb} \times S) \xrightarrow{\text{mult}_{\lambda,d}(O_Z)} A^*(\text{Hilb} \times S).
\]

The following formulas were proved in cohomology by [19] and in Chow by [22]:

(29) \[
\mathfrak{G}_d = \sum_{\lambda_1 \geq \ldots \geq \lambda_d} \sum_{\lambda_1 + \ldots + \lambda_d = 0} c_t \cdot q_{\lambda_1} \ldots q_{\lambda_d} \big|_{\Delta_1 \ldots d} + \sum_{\lambda_1 \geq \ldots \geq \lambda_{d-2}} \sum_{\lambda_1 + \ldots + \lambda_{d-2} = 0} c_t \cdot q_{\lambda_1} \ldots q_{\lambda_{d-2}} \big|_{\Delta_1 \ldots d} \cdot \rho^*(c).
\]

The constants “\( c_t \)” that appear in the formulas above are certain rational numbers that will not be important to us. The meaning of the notation \( \big|_{\Delta_1 \ldots d} \) is that after this restriction, we also multiply by the pull-back of the class \( c \) from the second factor of \( \text{Hilb} \times S \). Formula (27) simply entails composing \( t \) of the operators (29), multiplying with the pull-back of \( \Gamma \) in \( A^*(S^t) \), and pushing forward to \( \text{Hilb} \). \( \square \)

2.6. In this section, \( \Delta \) will refer to the smallest diagonal of any \( S^t \). Two interesting collections of elements of \( A^*(\text{Hilb}_n) \) can be written as universal classes: divisors and Chern classes of the tangent bundle. It is well-known that:

\[
A^1(S) \oplus \mathbb{Q} \cdot \delta \cong A^1(\text{Hilb}_n)
\]

(with the convention that \( \delta = 0 \) if \( n = 1 \)) where:

(30) \[
l \in A^1(S) \rightsquigarrow \text{univ}_2(\Delta_*(l))
\]

(31) \[
\delta \rightsquigarrow \text{univ}_3(\Delta_*(1)).
\]

Similarly, the Chern character of the tangent bundle to \( \text{Hilb}_n \) is given by the following well-known formula (let \( \pi, \rho : \text{Hilb}_n \times S \to \text{Hilb}_n, S \) denote the standard projections):

\[
\text{ch}(\text{Tan}_{\text{Hilb}_n}) = \pi_* \left[ \text{ch}(O_{Z_n}) + \text{ch}(O_{Z_n})' - \text{ch}(O_{Z_n}) \text{ch}(O_{Z_n})' \right] \rho^*(1 + 2c)
\]

where \( \cdot \)' is the operator which multiplies a codimension \( d \) class by \((-1)^d\). Therefore, the Chern character of the tangent bundle is a linear combination of the following particular universal classes:

(32) \[
\text{univ}_d(\Delta_*(\phi)) \text{ and } \text{univ}_{d,d'}(\Delta_*(\phi))
\]

where \( \phi \in \{1, c\} \), and the natural numbers \( d \) and \( d' \) are arbitrary.

3. Motivic decompositions

3.1. Let us recall the Lie algebra action \( g_{NS} \subset A^*(\text{Hilb}_n) \) from [27], which lifts the classical construction of [20, 31] in cohomology. To this end, note that the Beauville–Bogomolov form is the pairing on

\[
V = A^1(\text{Hilb}_n) \cong A^1(S) \oplus \mathbb{Q} \cdot \delta
\]

which extends the intersection form on \( A^1(S) \) and satisfies

\[
(\delta, \delta) = 2 - 2n, \quad (\delta, A^1(S)) = 0.
\]
Let $U = \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ be the hyperbolic lattice with fixed symplectic basis $e, f$. We have
\[ g_{\text{NS}} = \Lambda^2(V \oplus U_Q) \]
where the Lie bracket is defined for all $a, b, c, d \in V \oplus U_Q$ by
\[ [a \wedge b, c \wedge d] = (a, d)b \wedge c - (a, c)b \wedge d - (b, d)a \wedge c + (b, c)a \wedge d. \]

Consider for all $\alpha \in A^1(S)$ the following operators:
\[ e_\alpha = - \sum_{n>0} q_n q_{-n}(\Delta_\alpha \alpha) \]
\[ e_\delta = - \frac{1}{6} \sum_{i+j+k=0} q_i q_j q_k (\Delta_{123}) : \]
\[ f_\alpha = - \sum_{n>0} \frac{1}{n^3} q_n q_{-n}(\alpha_1 + \alpha_2) \]
\[ f_\delta = - \frac{1}{6} \sum_{i+j+k=0} q_i q_j q_k \left( \frac{1}{k^2} \Delta_{12} + \frac{1}{j^2} \Delta_{13} + \frac{1}{i^2} \Delta_{23} + \frac{2}{jk} c_1 + \frac{2}{ik} c_2 + \frac{2}{ij} c_3 \right) : . \]

Here $:\cdot:\$ is the normal ordered product defined by
\[ : q_{i_1} \cdots q_{i_k} : = q_{i_{\sigma(1)}} \cdots q_{i_{\sigma(k)}} \]
where $\sigma$ is any permutation such that $i_{\sigma(1)} \geq \cdots \geq i_{\sigma(k)}$. We define operators $e_\alpha$ and $f_\alpha$ for general $\alpha \in A^1(\text{Hilb}_n)$ by linearity in $\alpha$. By [22] we have that $e_\alpha$ is the operator of cup product with $\alpha$. If $(\alpha, \alpha) \neq 0$, the multiple $f_\alpha/(\alpha, \alpha)$ acts on cohomology as the Lefschetz dual of $e_\alpha$. In [27], it was show that the assignment
\[ \text{act} : g_{\text{NS}} \rightarrow A^\ast(\text{Hilb}_n \times \text{Hilb}_n) \]
\[ \text{act}(e \wedge \alpha) = e_\alpha, \quad \text{act}(\alpha \wedge f) = f_\alpha \]
for all $\alpha \in V$, induces a Lie algebra homomorphism. In particular, the element $e \wedge f \in g_{\text{NS}}$ acts by
\[ h = \sum_{k>0} \frac{1}{k} q_k q_{-k}(c_2 - c_1). \]

The operator $h$ specializes in cohomology to the Lefschetz grading operator, which by our normalization acts on $H^{2i}(\text{Hilb}_n)$ by multiplication by $i - n$.

From a straightforward calculation (see [27, Lemma 3.4]), one obtains the commutation relations
\[ [h, q_{\lambda_1} \cdots q_{\lambda_k}(\Phi)] = q_{\lambda_1} \cdots q_{\lambda_k}(\bar{\Phi}) \]
for all $\Phi \in A^\ast(S^k)$, where we write
\[ \bar{\Phi} = \sum_{i=1}^k (\text{Id}_{S^{i-1}} \times h \times \text{Id}_{S^{k-i}})(\Phi) \]
\[ = \sum_{i=1}^k \int_{\text{this class lies in } A^\ast(S^k \times S)} \Phi_{1 \ldots i-1, i+1 \ldots k} (c_i - c_\bullet) \]
with the last factor in $S^k \times S$ represented by the index $\bullet$. 

\[ \text{act} : g_{\text{NS}} \rightarrow A^\ast(\text{Hilb}_n \times \text{Hilb}_n) \]
\[ \text{act}(e \wedge \alpha) = e_\alpha, \quad \text{act}(\alpha \wedge f) = f_\alpha \]
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From a straightforward calculation (see [27, Lemma 3.4]), one obtains the commutation relations
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for all $\Phi \in A^\ast(S^k)$, where we write
\[ \bar{\Phi} = \sum_{i=1}^k (\text{Id}_{S^{i-1}} \times h \times \text{Id}_{S^{k-i}})(\Phi) \]
\[ = \sum_{i=1}^k \int_{\text{this class lies in } A^\ast(S^k \times S)} \Phi_{1 \ldots i-1, i+1 \ldots k} (c_i - c_\bullet) \]
with the last factor in $S^k \times S$ represented by the index $\bullet$. 

3.2. Proof of Theorem 1.3. We start with the decomposition of the diagonal into Nakajima operators:

\[
\Delta_{\text{Hilb}_n} = \sum_{\lambda \vdash n} \frac{(-1)^l(\lambda)}{j(\lambda)} q_{\lambda} q_{-\lambda}(\Delta)
\]

where \(\lambda\) runs over all partitions of \(n\),

\[
j(\lambda) = |\text{Aut}(\lambda)| \prod_i \lambda_i
\]

is a combinatorial factor, and for any \(\pi \in A^*(S \times S)\) we write

\[
q_{\lambda} q_{-\lambda}(\pi) = q_{\lambda_1} \cdots q_{\lambda_{l(\lambda)}} q_{-\lambda_1} \cdots q_{-\lambda_{l(\lambda)}} \left(\pi_{1,1}(\lambda)+1 \pi_{2,1}(\lambda)+2 \cdots \pi_{l(\lambda),2l(\lambda)}\right)
\]

\[
= \cdot q_{\lambda_1} q_{-\lambda_1}(\pi) \cdots q_{\lambda_{l(\lambda)}} q_{-\lambda_{l(\lambda)}}(\pi).
\]

The formula (38) follows directly from (24), (25), and the fact that \(q_m = (-1)^m q_{-m}\) (which is incorporated in the definition (20)).

Consider the decomposition of the diagonal of \(S\) as

\[
\Delta = \pi_0 + \pi_1 + \pi_2,
\]

where

\[
\pi_0 = \Delta - c_1 - c_2, \quad \pi_1 = c_2.
\]

It is easy to note that \(\pi_0, \pi_1, \pi_2\) are the projectors onto the \(-1, 0, +1\) eigenspaces of the action of \(h\) on \(A^*(\text{Hilb}_1) = A^*(S)\).

To define projectors corresponding to the action of \(h\) on \(A^*(\text{Hilb}_n)\), we insert the decomposition (39) into (38), and then expand and collect the terms of degree \(i\). Concretely, for every integer \(i\), we let

\[
P_i = \sum_{|\lambda| + |\mu| + |\nu| = n, \lambda, \mu, \nu \vdash n, -l(\lambda) + (\mu) + (\nu) = i} \frac{(-1)^{l(\lambda)+l(\mu)+l(\nu)}}{j(\lambda)j(\mu)j(\nu)} \cdot q_{\lambda} q_{-\lambda}(\pi_{-1}) q_{-\mu} q_{-\nu}(\pi_{0}) q_{\mu} q_{\nu}(\pi_{1}).
\]

Let us check that \(P_i\) are indeed projectors onto the eigenspaces of \(h\).

Claim 3.1. For all \(i, j \in \mathbb{Z}\) we have the following equalities in \(A^*(\text{Hilb}_n \times \text{Hilb}_n)\):

(a) \(P_i \circ P_j = P_i \delta_{ij}\)

(b) \(h \circ P_i = iP_i\).

Proof. (a) We determine \(P_i \circ P_j\) by commuting all Nakajima operators with negative indices to the right, and then using that we act on \(\text{Hilb}_n\) so all products of Nakajima operators with purely negative indices of degree \(> n\) vanish. Since every summand in \(P_i\) contains such a product of degree \(n\), we find that for a term to contribute all operators with negative indices coming from \(P_i\) have to interact with operators (with positive indices) from the second term. The interactions are described as follows. For a single term (let \(a, b > 0\) and \(r, s \in \{-1, 0, 1\}\)) we have

\[
q_a q_{-a}(\pi_r^i) q_b q_{-b}(\pi_s^i) = q_a [q_{-a}, q_b] q_{-b} (\pi_r^i)_{12} (\pi_s^i)_{34} + q_a q_b q_{-a} q_{-b} (\pi_r^i)_{13} (\pi_s^i)_{24}
\]
where by the commutation relations (21) the first term on the right is
\[ q_{\alpha} [q_{-\alpha}, q_{b}] q_{-b} \left( (\pi_{r}^t)_{12} (\pi_{s}^t)_{34} \right) = (-a) \delta_{ab} q_{a} q_{-b} \left( (\pi_{14})_{12} (\pi_{34}^t)_{34} \Delta_{23} \right) \]
\[ = (-a) \delta_{ab} q_{a} q_{-a} (\pi_{r}^t \circ \pi_{s}^t) \]
\[ = (-a) \delta_{ab} q_{a} q_{-a} \left( (\pi_{r} \circ \pi_{s})^t \right) \]
\[ = (-a) \delta_{ab} \delta_{rs} q_{a} q_{-a} (\pi_{r}^t). \]

Hence for a composition
\[ q_{\lambda} q_{\mu} q_{\nu} (\pi_{r}^t) q_{\lambda} q_{\mu} (\pi_{s}^t) q_{\mu} q_{\nu} (\pi_{r}^t) : \circ : q_{\lambda} q_{\mu} q_{\nu} (\pi_{r}^t) q_{\mu} q_{\nu} (\pi_{r}^t) q_{\mu} q_{\nu} (\pi_{r}^t) : \]
(with \( \lambda, \mu, \nu \) as in the definition of \( P_i \), and the same for the primed partitions) to act non-trivially on \( \text{Hilb}_n \) we have to have \( \lambda = \lambda', \mu = \mu', \nu = \nu' \). Moreover, if we write \( \lambda \) multiplicatively as \( (l_1 l_2 \ldots \ldots) \) where \( l_i \) is the number of parts of size \( i \), then there are precisely \( |\text{Aut}(\lambda)| = \prod_{i} l_i! \) different ways to pair the negative factors in \( q_{\lambda} q_{\mu} q_{\nu} (\pi_{r}^t) \) with the positive factors \( q_{\lambda} q_{\mu} q_{\nu} (\pi_{r}^t) \), and similarly for \( \mu, \nu \). Hence
\[ q_{\lambda} q_{\mu} q_{\nu} (\pi_{r}^t) q_{\lambda} q_{\mu} (\pi_{s}^t) q_{\mu} q_{\nu} (\pi_{r}^t) : \circ : q_{\lambda} q_{\mu} q_{\nu} (\pi_{r}^t) q_{\mu} q_{\nu} (\pi_{r}^t) q_{\mu} q_{\nu} (\pi_{r}^t) : \]
\[ = \delta_{\lambda \lambda'} \delta_{\mu \mu'} \delta_{\nu \nu'} (-1)^{(l_{\lambda} + l_{\mu} + l_{\nu})} \frac{\Delta(\lambda) \Delta(\mu) \Delta(\nu)}{\Delta(\lambda')} : q_{\lambda} q_{\mu} q_{\nu} (\pi_{r}^t) q_{\mu} q_{\nu} (\pi_{r}^t) q_{\mu} q_{\nu} (\pi_{r}^t) : \]
which implies the claim.

(b) To determine \( h \circ P_i \) we commute \( h \) into the middle, i.e., to the right of all Nakajima operators with positive indices, and to the left of all with negative ones. In the middle position \( h \) acts on the Chow ring of \( \text{Hilb}_0 \) where it vanishes. Hence again we only need to compute the commutators. For this we use (36) and that \( \pi_i \) are the projectors onto the eigenspaces of \( h \) so that
\[ (h \times \text{Id})(\pi_{r}^t) = ((\text{Id} \times h)(\pi_{r}))^t = (h \circ \pi_{r})^t = r \pi_{r}^t. \]
As desired we find
\[ h \circ P_i = (-1 \cdot l(\lambda) + 0 \cdot l(\mu) + 1 \cdot l(\nu)) P_{i} = i P_{i}. \]

Using the claim it follows that the motivic decomposition
\[ \mathfrak{h}(\text{Hilb}_n) = \bigoplus_{i=0}^{2n} \mathfrak{h}^{2i}(\text{Hilb}_n) \]
with \( \mathfrak{h}^{2i}(\text{Hilb}_n) = (\text{Hilb}_n, P_{i-n}) \) has the stated properties. The uniqueness of the decomposition follows from the uniqueness of the decomposition of \( \Delta_{\text{Hilb}_n} \) under the action of \( h \) on \( A^*(\text{Hilb}_n \times \text{Hilb}_n) \); see the proof of the refined decomposition in Section 3.3 below. \hfill \Box

By (25), an alternative way to write the projector \( P_i \) is
\[ P_i = \sum_{\lambda \vdash n} \frac{(-1)^{n-l(\lambda)}}{\Delta(\lambda)} \frac{\Gamma_{\lambda}^t \circ \tilde{P}_i \circ \Gamma_{\lambda}}{\Gamma_{\lambda}} \]
where \( \tilde{P}_i \in A^*(S^\lambda \times S^\lambda) \) is the projector
\[ \tilde{P}_i = \sum_{i_1 + \ldots + l(\lambda) = i} \pi_{i_1} \times \ldots \times \pi_{i_l}. \]
Hence the decomposition of Theorem 1.3 is precisely the Chow–Künneth decomposition constructed by Vial in [32, Section 2].
3.3. Refined decomposition. Let \( U(\mathfrak{g}_{\text{NS}}) \) be the universal enveloping algebra of \( \mathfrak{g}_{\text{NS}} \). The Lie algebra homomorphism (34) extends to an algebra homomorphism
\[
\text{act} : U(\mathfrak{g}_{\text{NS}}) \to A^*(\text{Hilb}_n \times \text{Hilb}_n).
\]

**Lemma 3.2.** The image \( W \subset A^*(\text{Hilb}_n \times \text{Hilb}_n) \) of \( \text{act} \) is finite-dimensional.

**Proof.** For every fixed \( k \geq 1 \) the subring of \( R^*(S^k) \subset A^*(S^k) \) generated by
- \( \alpha_i \) for all \( i \) and \( \alpha \in A^1(S) \)
- \( c_i \) for all \( i \)
- \( \Delta_{ij} \) for all \( i, j \)
is finite-dimensional, and preserved by the projections to the factors. Hence the space of operators \( \tilde{W} \subset A^*(\text{Hilb}_n \times \text{Hilb}_n) \) spanned by
\[
q_{\lambda_1} \cdots q_{\lambda_l} q_{-\mu_1} \cdots q_{-\mu_l}(\Gamma)
\]
for all partitions \( \lambda, \mu \) of \( n \) and all \( \Gamma \in R^*(S^l(\lambda)+l(\mu)) \) is finite-dimensional. The commutation relations (21) show that \( \tilde{W} \) is closed under compositions of correspondences. Moreover, by inspecting the expressions for the generators of \( \mathfrak{g}_{\text{NS}} \) in (33) (and using (38) to bring them into the desired form), we see that all generators of \( \mathfrak{g}_{\text{NS}} \) lie in \( \tilde{W} \). Hence \( g \in \tilde{W} \) for all \( g \in U(\mathfrak{g}_{\text{NS}}) \), i.e., \( W \subset \tilde{W} \). □

We find that \( W \) is a finite-dimensional vector space which is preserved by the action of \( U(\mathfrak{g}_{\text{NS}}) \), and hence defines a finite-dimensional representation of \( \mathfrak{g}_{\text{NS}} \). Since \( \mathfrak{g}_{\text{NS}} \) is semisimple, this representation decomposes into isotypic summands
\[
W = \bigoplus_{\psi \in \text{Irrep}(\mathfrak{g}_{\text{NS}})} W_{\psi}.
\]

Let us look at the image of \( \Delta_{\text{Hilb}_n} \in W \) under this decomposition
\[
\Delta_{\text{Hilb}_n} = \sum_{\psi \in \text{Irrep}(\mathfrak{g}_{\text{NS}})} P_{\psi}
\]
where \( P_{\psi} \in W_{\psi} \).

**Claim 3.3.** The elements \( P_{\psi} \in A^*(\text{Hilb}_n \times \text{Hilb}_n) \) are orthogonal projectors.

**Proof.** Let us first show that left-multiplication by \( P_{\psi} \) maps \( W \) to \( W_{\psi} \), i.e.,
\[
P_{\psi} \circ W \subset W_{\psi}.
\]
Indeed, for all \( a \in W \), right multiplication by \( a \) is a \( \mathfrak{g}_{\text{NS}} \)-intertwiner and thus sends \( W_{\psi} \) to \( W_{\psi} \). In other words, we have \( W_{\psi} \circ a \subset W_{\psi} \), hence \( W_{\psi} \circ W \subset W_{\psi} \), which implies (41). If we multiply any \( a \in W \) by relation (40), we obtain
\[
a = \sum_{\psi \in \text{Irrep}(\mathfrak{g}_{\text{NS}})} P_{\psi} \circ a.
\]
By (41), the summands in the right-hand side each lie in \( W_{\psi} \). If \( a = P_{\psi'} \), then by comparing summands the equality above implies
\[
P_{\psi'} \circ a = a \quad \text{and} \quad P_{\psi} \circ a = 0
\]
for all \( \psi \neq \psi' \). In particular, taking \( a = P_{\psi'} \) implies the relations \( P_{\psi} \circ P_{\psi'} = \delta_{\psi' \psi} P_{\psi} \). Moreover, this implies that the inclusion (41) is actually an identity, hence left multiplication by \( P_{\psi} \) projects \( W \) onto \( W_{\psi} \). □
From Claim 3.3 we obtain the decomposition
\[(42) \quad h(\text{Hilb}_n) = \bigoplus_{\psi \in \text{Irrep}(\text{g}_{\text{NS}})} h_\psi(\text{Hilb}_n)\]
where $h_\psi(\text{Hilb}_n) = (\text{Hilb}_n, P_\psi)$. We can now prove the main result of this section.

Proof of Theorem 1.5. It remains to show that the summands $h_\psi(\text{Hilb}_n)$ are $\psi$-isotypic and that the decomposition (42) is unique. Let $M$ be a Chow motive. The action of $\text{g}_{\text{NS}}$ on $\text{Hom}(M, h(\text{Hilb}_n))$ is defined by $g \mapsto \text{act}(g) \circ \ldots$. Hence if $f \in \text{Hom}(M, M')$ is a morphism of Chow motives, the pullback
\[f^* : \text{Hom}(M', h(\text{Hilb}_n)) \rightarrow \text{Hom}(M, h(\text{Hilb}_n))\]
is equivariant with respect to the $\text{g}_{\text{NS}}$-action. Now, for any $v \in \text{Hom}(M, h_\psi(\text{Hilb}_n))$ we have $v = P_\psi \circ w$ for some $w \in \text{Hom}(M, h(\text{Hilb}_n))$ and thus
\[U(\text{g}_{\text{NS}})v = U(\text{g}_{\text{NS}})w^*(P_\psi) = w^*(U(\text{g}_{\text{NS}}) \circ P_\psi).\]
Since $U(\text{g}_{\text{NS}}) \circ P_\psi \subset W_\psi$ this implies that $U(\text{g}_{\text{NS}})v$ is finite-dimensional and $\psi$-isotypic. Since $v$ was arbitrary we conclude that $\text{Hom}(M, h_\psi(\text{Hilb}_n))$ is $\psi$-isotypic.

The decomposition (42) is unique because (40) is unique. Indeed, suppose we had any other decomposition
\[(43) \quad \Delta_{\text{Hilb}_n} = \sum_{\psi \in \text{Irrep}(\text{g}_{\text{NS}})} P'_\psi\]
where $P'_\psi \in W_\psi$, for all $\psi$. Then we would need $P'_\psi = P_\psi \circ a_\psi$ for some $a_\psi \in W$. But multiplying (43) on the left with $P_\psi$ and using the orthogonality of the projectors would imply $P'_\psi = P_\psi \circ P_\psi \circ a_\psi = P_\psi \circ a_\psi$, hence $P'_\psi = P_\psi$. \hfill \Box

As in [23, Proof of Theorem 7.2], we could also have used Yoneda’s Lemma to conclude the existence of the decomposition (42). Our presentation above has the advantage of being constructive. It also shows that the projectors $P_\psi$ can be written in terms of the Nakajima operators applied to elements in $R^*(S^k)$.

4. Multiplicativity

4.1. Let us recall the operator (35) (in the present section, we will find it useful to use the language of operators when referring to correspondences, as explained in Section 1.6). This operator was observed in [27] to lift the (shifted) grading operator from cohomology to Chow. Let us undo this shift by considering:
\[\tilde{h} = h + n \cdot \text{Id}_{\text{Hilb}_n}.\]
The main purpose of the present section is to prove Theorem 1.4. In the language of operators, the multiplicativity of the Chow–Künneth decomposition boils down to the identity \[6.\] Alternatively, we could restate this identity as:
\[(44) \quad [h, \text{mult}_x] = \text{mult}_{x'}\]
of correspondences $A^*(\text{Hilb}_n) \rightarrow A^*(\text{Hilb}_n)$ indexed by $x \in A^*(\text{Hilb}_n)$. By applying the equality (44) to the fundamental class, we must have $x' = h(x) - x \cdot h([\text{Hilb}_n])$.

Lemma 4.1. We have $h([\text{Hilb}_n]) = -n$.

With the lemma above in mind, we conclude that $x' = \tilde{h}(x)$ in (44). Therefore, (44) is actually equivalent to (6), thus implying part (i) of Theorem 1.4.
Proof of Lemma 4.1. It is well-known that:

$$[\text{Hilb}_n] = \frac{1}{n!} q_1(1)^n [\text{Hilb}_0].$$

Because the only operator $q_k$ which fails to commute with $q_1$ is $q_{-1}$, formula (35) implies that:

$$h([\text{Hilb}_n]) = h \left( \frac{1}{n!} q_1(1)^n [\text{Hilb}_0] \right) =
$$

$$= \left[ h, \frac{1}{n!} q_1(1)^n \right] [\text{Hilb}_0] = \left[ q_1(1)q_{-1}(c) - q_1(c)q_{-1}(1), \frac{1}{n!} q_1(1)^n \right] [\text{Hilb}_0]$$

$$= \sum_{i=1}^{n} \frac{1}{n!} q_1(1)^{i-1} \cdot q_1(1)[q_{-1}(c), q_1(1)] \cdot q_{-1}(1)^{n-i} \cdot [\text{Hilb}_0] = -n[\text{Hilb}_n]$$

where the fact that $\phi_{-1}(c), q_1(1)] = -1$ is a consequence of (22). \qed

4.2. Due to the surjectivity property (19), formula (44) remains equivalent if we replace $x \in A^*(\text{Hilb}_n)$ by a class of the form (26):

$$\text{mult}(\Gamma) = \text{mult}_{\text{uni}}(\Gamma)$$

(for any $d_1, \ldots, d_t$, which will be fixed in the present section), which is indexed by $\Gamma \in A^*(S')$. Then instead of proving (44), it suffices to prove the following:

**Proposition 4.2.** We have the identity of correspondences $A^*(\text{Hilb}_n) \to A^*(\text{Hilb}_n)$

$$[h, \text{mult}_{\text{uni}}(\Gamma)] = (d_1 + \ldots + d_t - t)\text{mult}_{\text{uni}}(\Gamma) + \text{mult}_{\text{uni}}(\Gamma)$$

parametrized by $\Gamma \in A^*(S')$ (the bar notation is defined in (37)).

**Proof.** By combining (36) with (27), we have:

$$[h, \text{mult}_{\text{uni}}(\Gamma)] = \sum_{\lambda_{s-1} + 1 + \ldots + 1 = 0, s \in \{1, \ldots, t\}} \sum_{\lambda_{s-1} + 1 + \ldots + 1 = 0, s \in \{1, \ldots, t\}} \text{ct} \cdot q_{\lambda_{s-1} + 1 + \ldots + 1} \cdot [\text{Hilb}_0]$$

To compute the overlined class on the second row, we will use the following:

**Claim 4.3.** For any natural numbers $k, l$ and any $\Phi \in A^*(S')$, we have:

$$\Delta_{1-k} \Phi_{k*} = \Delta_{1-k} [ (k-1)\Phi_{k*} + \int \Phi_{\bullet*}(c_k - c_\bullet) ].$$

The notation $*$ stands for the indices $k + 1, \ldots, k + l - 1$, and it reflects the fact that the bar notation is only defined as in (37) with respect to the indices $1, \ldots, k$ only.

**Proof.** By definition, the LHS of (48) equals:

$$\text{LHS of (48)} = \sum_{i=1}^{k-1} \int \Delta_{1-i-1, i+1+\ldots+k}(c_i - c_\bullet) \Phi_{k*} + \int \Delta_{1-k-1, i}(c_k - c_\bullet) \Phi_{\bullet*}$$

$$= \Phi_{k*} \sum_{i=1}^{k} \int \Delta_{1-i-1, i+1+\ldots+k}(c_i - c_\bullet) + \Delta_{1-k-1, i}(c_k - c_\bullet)(\Phi_{\bullet*} - \Phi_{k*})$$

$$= \text{RHS of (48)}$$

where the last equality is due to (13) and (14). \qed
By applying \((48)\) a number of \(t\) times, we obtain (let \(\Delta = \Delta_{b_0+1 \ldots b_t} \ldots \Delta_{b_{t-1}+1 \ldots b_t}\)):

\[
\Delta \Gamma_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} = \Delta \left( (b_t - t) \Gamma_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} + \sum_{i=1}^{t} \int \Gamma_{b_1 \ldots b_{i-1} b_{i+1} \ldots b_t} \phi_{b_1} \ldots \phi_{b_{i-1}} \phi_{b_{i+1}} \ldots \phi_{b_t} (c_b - c_\bullet) \right).
\]

Using \((49)\) we have the following simple identities:

\[
\int \Gamma_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} (c_b - c_\bullet) = \begin{cases} 
\int \Gamma_{b_1 \ldots b_t} (c_b - c_\bullet) \phi_{b_1} \ldots \phi_{b_t} & \text{if } \phi_{b_i} = 1 \\
\int \Gamma_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} & \text{if } \phi_{b_i} = c_b \\
- \int \Gamma_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} & \text{if } \phi_{b_i} = c_\bullet
\end{cases}
\]

which we plug into \((49)\):

\[
\Delta \Gamma_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} = \Delta \left( (b_t - t + 2 \# \{ b_i | \phi_{b_i} = c_b \}) \Gamma_{b_1 \ldots b_t} + \Gamma_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} \right).
\]

Since the coefficient of \(\Gamma_{b_1 \ldots b_t}\) in the right-hand side is equal to \(d_1 + \ldots + d_t - t\), according to \((28)\), we obtain precisely formula \((46)\).

4.3. Let us now prove part (ii) of Theorem 1.4. In order to show that a certain class \(x \in A^* (\text{Hilb}_n)\) lies in the appropriate direct summand, we must show that \(h(x) = (\deg(x) - n) \cdot x\). We are interested in the situation when \(x\) is a divisor class or a Chern class of the tangent bundle, in which case we have:

\[
x = \text{univ}_{d_1, \ldots, d_t} (\Gamma)
\]

for some \(d_1, \ldots, d_t \in \mathbb{N}\) and \(\Gamma \in A^* (S^t)\). We have shown in Proposition 4.2 that:

\[
h(x) = (d_1 + \ldots + d_t - n - t) \text{univ}_{d_1, \ldots, d_t} (\Gamma) + \text{univ}_{d_1, \ldots, d_t} (\Gamma)
\]

so the class \(x\) lies in the appropriate direct summand if:

\[
(50) \quad \Gamma = \Gamma \cdot (\deg \text{univ}_{d_1, \ldots, d_t} (\Gamma) - d_1 - \ldots - d_t + t) = \Gamma \cdot (\deg \Gamma - t).
\]

Since divisors and Chern classes of the tangent bundle are linear combinations of the classes \((30), (31), (32)\), it suffices to prove \((50)\) in the particular case \(\Gamma = \Delta, (\gamma)\), where \(\Delta\) is the small diagonal and \(\gamma \in \{1, l, c\}_{l \in A^1 (S)}\). In this case, we have:

\[
\Gamma \overset{48}{=} \Delta_0 \left( (t - 1) \gamma + \int \gamma_\star (c - c_\bullet) \right) = (t - 2 + \deg \gamma) \cdot \Gamma
\]

where the last equality is a simple case-by-case study for all \(\gamma \in \{1, l, c\}_{l \in A^1 (S)}\). Since \(\deg \Gamma = 2t - 2 + \deg \gamma\), this implies formula \((50)\).

4.4. Let \(\alpha, \beta \in A^1 (S) \rightarrow A^1 (\text{Hilb}_n)\). As a straightforward application of formulas \((33)\), the element \(\alpha \wedge \beta \in g_{NS}\) acts on \(A^* (\text{Hilb}_n)\) by the correspondence:

\[
h_{\alpha, \beta} = \sum_{k=1}^{\infty} \frac{1}{k} q_k q_{-k} (\alpha_2 \beta_1 - \alpha_1 \beta_2).
\]

In the present section, we will prove Theorem 1.6 that is to show that \(h_{\alpha, \beta}\) acts by derivations. As in the setup of Proposition 4.2, it boils down to statement \((51)\) below, for all \(\alpha, \beta \in A^1 (S)\) and all universal classes of the form \((45)\).
Proposition 4.4. We have the identity of correspondences $A^*(\text{Hilb}_n) \to A^*(\text{Hilb}_n)$
\begin{equation}
[h_{\alpha \beta}, \text{mult}_{\text{univ}}(\Gamma)] = \text{mult}_{\text{univ}}(\Gamma)
\end{equation}
parametrized by $\Gamma \in A^*(S^t)$, where for any $k$ and any $\Phi \in A^*(S^k)$, we write:
\begin{equation}
\bar{\Phi} = \sum_{i=1}^{k} \int \Phi_{1 \ldots i-1 \bullet i+1 \ldots k}(\alpha_i \beta_i - \alpha_i^s \beta_i) \quad \text{this class lies in } A^*(S^k \times S)
\end{equation}
where the last factor in $S^k \times S$ is the one represented by the index $\bullet$.

Proof. The proof follows that of Proposition 4.2 very closely, so we will only point out the differences. We start from the following identity:
$$[h_{\alpha \beta}, q_\lambda_1 \ldots q_\lambda_k(\Phi)] = q_\lambda_1 \ldots q_\lambda_k(\bar{\Phi})$$
which is a straightforward analogue of (36). Therefore, formulas (27) and (47) for $\text{mult}_{\text{univ}}(\Gamma)$ and $[h_{\alpha \beta}, \text{mult}_{\text{univ}}(\Gamma)]$ continue to hold, but with the definition (52) for the double bar operation instead of the single bar operation in (37). We leave the following analogue of (48) as an exercise to the interested reader:
$$\overline{\Delta_{1 \ldots k}\Phi_{k+1}} = \Delta_{1 \ldots k} \int \Phi_{\bullet k}(\alpha_k \beta_k - \alpha_k^s \beta_k).$$

By iterating the formula above $t$ times, we obtain (let $\Delta = \Delta_{b_0 + 1 \ldots 1 \ldots b_{t-1} + 1 \ldots b_t}$):
$$\overline{\Delta_{b_1 \ldots b_t}} = \Delta_{b_1 \ldots b_{t-1} \bullet b_{t+1} \ldots b_t} \phi_{b_1} \ldots \phi_{b_t} \phi_{b_{t+1}} \ldots \phi_{b_t} (\alpha_b \beta_b - \alpha_b^s \beta_b).$$

As a consequence of the following simple formulas:
\begin{align*}
\int \Phi_{b_1 \ldots b_t} & (\alpha_b \beta_b - \alpha_b^s \beta_b) \\
= \int \Phi_{b_1 \ldots b_t} & (\alpha_b \beta_b - \alpha_b^s \beta_b) \\
= \int \Phi_{b_1 \ldots b_t} (\alpha_b \beta_b - \alpha_b^s \beta_b) = 0 & \quad \text{if } \phi_{b_i} = c_{b_i}
\end{align*}
we have $\overline{\Delta_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t}} = \overline{\Delta_{b_1 \ldots b_t} \phi_{b_1} \ldots \phi_{b_t}}$. This concludes the proof of (51). □

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