On O’Grady’s Generalized Franchetta Conjecture

Nebojša Pavic, Junliang Shen*, and Qizheng Yin

Departement Mathematik, ETH Zürich, Rämistrasse 101, Zürich 8092, Switzerland

*Correspondence to be sent to: e-mail: junliang.shen@math.ethz.ch

We study relative zero cycles on the universal polarized $K3$ surface $X \to \mathcal{F}_g$ of degree $2g-2$. It was asked by O’Grady if the restriction of any class in $\text{CH}^2(X)$ to a closed fiber $X_s$ is a multiple of the Beauville–Voisin canonical class $c_{X_s} \in \text{CH}_0(X_s)$. Using Mukai models, we give an affirmative answer to this question for $g \leq 10$ and $g = 12, 13, 16, 18, 20$.

1 Introduction

Throughout, we work over the complex numbers. Let $S$ be a projective $K3$ surface. In [2], Beauville and Voisin studied the Chow ring $\text{CH}^*(S)$ of $S$. They showed that there is a canonical class $c_S \in \text{CH}_0(S)$ represented by a point on a rational curve in $S$, which satisfies the following properties:

(i) The intersection of two divisor classes on $S$ always lies in $\mathbb{Z}c_S \subseteq \text{CH}_0(S)$.

(ii) The second Chern class $c_2(T_S)$ equals $24c_S \in \text{CH}_0(S)$.

This result is rather surprising since the Chow group $\text{CH}_0(S)$ is infinite-dimensional by Mumford’s theorem [11].
Let \( \mathcal{F}_g \) denote the moduli space of (primitively) polarized \( K3 \) surfaces of degree \( 2g - 2 \). For \( g \geq 3 \), let \( \mathcal{F}_g^0 \subset \mathcal{F}_g \) be the open dense subset parametrizing polarized \( K3 \) surfaces with trivial automorphism groups, which carries a universal family \( X \to \mathcal{F}_g^0 \). Motivated by Franchetta’s conjecture on the moduli spaces of curves (see [1]), O’Grady asked the following question in [12], referred to as the generalized Franchetta conjecture.

**Question 1.1** (Generalized Franchetta conjecture). Given a class \( \alpha \in \text{CH}^2(X) \) and a closed point \( s \in \mathcal{F}_g^0 \), is it true that \( \alpha|_{X_s} \in \mathbb{Z}c_{X_s} \)?

The goal of this article is to give an affirmative answer to Question 1.1 for a list of small values of \( g \). By the work of Mukai [7–10], for these \( g \) a general polarized \( K3 \) surface can be realized in a variety with “small” Chow groups as a complete intersection with respect to a vector bundle.

**Theorem 1.2.** The generalized Franchetta conjecture holds for \( g \leq 10 \) and \( g = 12, 13, 16, 18, 20 \).

This article is organized as follows. In Section 2 we review Mukai’s constructions and make some comments about Question 1.1. In Section 3 we prove Theorem 1.2 for all cases except \( g = 13, 16 \). Two independent proofs are presented, one using Voisin’s result [17], the other via a direct calculation. The cases \( g = 13, 16 \) have a different flavor and are treated in Section 4.

This work is inspired by a recent preprint of Pedrini [13]. However, contrary to what was claimed there, it does not suffice to show that \( \text{CH}^2(X)_{\mathbb{Q}} \) is finite-dimensional. Our proof relies deeply on the result of Beauville–Voisin [2].

## 2 Mukai Models and the Basic Setting

In this section we review Mukai’s work [7–10] on the projective models of general polarized \( K3 \) surfaces of small degrees. Using Mukai’s models, we set up the framework for the proof of Theorem 1.2.

The following table summarizes the ambient varieties \( G_g \) and vector bundles \( \mathcal{U}_g \) involved in the constructions. It is also accompanied by a glossary.
<table>
<thead>
<tr>
<th>$g$</th>
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<tr>
<td>2</td>
<td>$\mathbb{P}(1, 1, 1, 2)$</td>
<td>$O(6)$</td>
<td>9</td>
<td>$G(3, 6)$</td>
<td>$O(1)^{\oplus 4} \oplus \wedge^2 Q$</td>
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<td>$\mathbb{P}^3$</td>
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<td>$G(2, 7)$</td>
<td>$O(1)^{\oplus 3} \oplus \wedge^4 Q$</td>
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<td>4</td>
<td>$\mathbb{P}^4$</td>
<td>$O(2) \oplus O(3)$</td>
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<td>5</td>
<td>$\mathbb{P}^5$</td>
<td>$O(2)^{\oplus 3}$</td>
<td>13</td>
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<td>$(\wedge^2 \mathcal{E}^\vee)^{\oplus 2} \oplus \wedge^3 Q$</td>
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<tr>
<td>6</td>
<td>$G(2, 5)$</td>
<td>$O(1)^{\oplus 3} \oplus O(2)$</td>
<td>16</td>
<td>$G(2, 3, 4)$</td>
<td>$\mathcal{V}<em>{16}^{\oplus 2} \oplus \tilde{\mathcal{V}}</em>{16}^{\oplus 3}$</td>
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<tr>
<td>7</td>
<td>$O G(5, 10)$</td>
<td>$\mathcal{V}_7^{\oplus 8}$</td>
<td>18</td>
<td>$O G(3, 9)$</td>
<td>$\mathcal{V}_{18}^{\oplus 5}$</td>
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<td>8</td>
<td>$G(2, 6)$</td>
<td>$O(1)^{\oplus 6}$</td>
<td>20</td>
<td>$G(4, 9)$</td>
<td>$(\wedge^2 \mathcal{E}^\vee)^{\oplus 3}$</td>
</tr>
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$\mathbb{P}(1, 1, 1, 2)$ 3-dimensional weighted projective space with weights $(1, 1, 1, 2)$

$G(r, n)$ Grassmannian parametrizing $r$-dimensional subspaces of a fixed $n$-dimensional vector space

$O(i)$ Line bundle on $G(r, n)$ with respect to the Plücker embedding

$O G(r, n)$ Orthogonal Grassmannian parametrizing $r$-dimensional isotropic subspaces of a fixed $n$-dimensional vector space equipped with a nondegenerate symmetric 2-form

$\mathcal{V}_7$ Line bundle on $O G(5, 10)$ corresponding to a spin representation

$Q$ Universal quotient bundle on $G(r, n)$

$\mathcal{E}$ Universal subbundle on $G(r, n)$

$G(2, 3, 4)$ Ellingsrud–Piene–Strømme moduli space of twisted cubic curves, constructed as the GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ by the action of $\text{GL}_2 \times \text{GL}_3$ (see [3])

$\mathcal{V}_{16}$ Rank 3 tautological vector bundle on $G(2, 3, 4)$

$\tilde{\mathcal{V}}_{16}$ Rank 2 tautological vector bundle on $G(2, 3, 4)$

$\mathcal{V}_{18}$ Rank 2 vector bundle on $O G(3, 9)$ corresponding to a spin representation

For all $g$ listed above, Mukai showed that a general $K3$ surface over $\mathcal{F}_g$ is given as the zero locus of a general global section of $U_g$ (the cases $g \leq 5$ are classical).

Let

$$\mathbb{P}_g = \mathbb{P} H^0 (G_g, U_g)$$
be the projectivization of the space of global sections of \( \mathcal{U}_g \), and let

\[ Y = \{(s, x) \in \mathbb{P}_g \times \mathbb{G}_g \mid s(x) = 0\} \]

be the incidence scheme. We have a diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{i} & \mathbb{G}_g \\
\downarrow & & \downarrow \\
\mathbb{P}_g & & \\
\end{array}
\]

where \( \pi, \iota \) are the two projections.

The discussion above shows that a general fiber of \( \pi : Y \to \mathbb{P}_g \) is a polarized K3 surface of degree \( 2g - 2 \) and that \( \mathbb{P}_g \) rationally dominates the moduli space \( \mathcal{F}_g \). Moreover, since \( \mathcal{U}_g \) is globally generated, we know that \( \iota : Y \to \mathbb{G}_g \) is a projective bundle. Its fiber over a point \( x \in \mathbb{G}_g \) is given by

\[ \mathbb{P}H^0(\mathbb{G}_g, \mathcal{U}_g \otimes I_x), \]

where \( I_x \) is the ideal sheaf of \( x \). We have the following lemma regarding the Chow group \( \text{CH}^2(Y) \) and its restriction to a general fiber of \( \pi \).

**Lemma 2.1.** Given a closed point \( s \in \mathbb{P}_g \) with K3 fiber \( Y_s \), let \( \phi_s : Y_s \hookrightarrow Y \) and \( \iota_s : Y_s \hookrightarrow \mathbb{G}_g \) be the natural embeddings. Then we have

\[ \text{Im}(\phi_s^* : \text{CH}^2(Y)_\mathbb{Q} \to \text{CH}_0(Y_s)_\mathbb{Q}) = \text{Im}(\iota_s^* : \text{CH}^2(\mathbb{G}_g)_\mathbb{Q} \to \text{CH}_0(Y_s)_\mathbb{Q}). \]  

**Proof.** Let \( \xi \in \text{CH}^1(Y) \) be the relative hyperplane class of \( \iota : Y \to \mathbb{G}_g \). By the projective bundle formula, we have for \( k \geq 0 \),

\[ \text{CH}^k(Y) = \xi^k \cdot \iota^* \text{CH}^0(\mathbb{G}_g) \oplus \xi^{k-1} \cdot \iota^* \text{CH}^1(\mathbb{G}_g) \oplus \cdots \oplus \iota^* \text{CH}^k(\mathbb{G}_g). \]  

(2.1)

Let \( h \in \text{CH}^1(\mathbb{P}_g) \) be the hyperplane class. Then we have

\[ \pi^* h = a \cdot \xi + \iota^* \beta \]

for some \( a \in \mathbb{Z} \) and \( \beta \in \text{CH}^1(\mathbb{G}_g) \). We claim that \( a \neq 0 \), otherwise

\[ \pi^*(h^{\text{dim} \mathbb{P}_g}) = \iota^*(\beta^{\text{dim} \mathbb{P}_g}). \]
Since \( \dim \mathcal{F}_g > \dim \mathcal{G}_g \), the right-hand side vanishes, but the left-hand side is the pullback of a point class and is nonzero. Contradiction. Hence

\[
\xi = \frac{1}{a} (\pi^* h - \iota^* \beta) \in \text{CH}^1(\mathcal{Y})_\mathbb{Q}.
\]

The lemma then follows from (2.1) for \( k = 2 \) and the fact that \( \phi_\ast^\pi \pi^* h = 0 \). ■

We end this section by a few remarks on the generalized Franchetta conjecture.

(i) By a standard “spreading out” argument (see [16, Chapter 1]), it is equivalent to answer Question 1.1 for general (in fact, very general) fibers \( X_s \) over \( \mathcal{F}_0 \). Moreover, classes in \( \text{CH}^2(X) \) supported over a proper closed subset of \( \mathcal{F}_g \) vanish when restricted to a fiber \( X_s \).

Hence one may work with a family \( Y \to B \) such that a general fiber \( Y_s \) is a polarized K3 surface of degree \( 2g - 2 \) and that \( B \) rationally dominates \( \mathcal{F}_g \) via the natural rational map \( B \to \mathcal{F}_g \). It then suffices to answer (the analog of) Question 1.1 for classes in \( \text{CH}^2(Y) \) and K3 fibers \( Y_s \). See Section 4 for an even more precise statement.

One may also formulate Question 1.1 in terms of the Chow group \( \text{CH}_0(X_\eta) \) of the generic fiber \( X_\eta \), but we omit this point of view.

(ii) By Roitman’s theorem [14], the Chow group \( \text{CH}_0(S) \) of a complex K3 surface \( S \) is torsion-free. Hence in Question 1.1 it is equivalent to work with \( \mathbb{Q} \)-coefficients. This also means that under Lemma 2.1, we have

\[
\text{Im}(\phi_\ast^\pi : \text{CH}^2(Y) \to \text{CH}_0(Y_s)) \subset \mathbb{Z}c_{Y_s}
\]

if and only if

\[
\text{Im}(\iota_\ast^\pi : \text{CH}^2(\mathcal{G}_g) \to \text{CH}_0(Y_s)) \subset \mathbb{Z}c_{Y_s}.
\]

(iii) Instead of restricting to \( \mathcal{F}_g^0 \), one may work with the moduli stack and the universal family over it. Question 1.1 can then be formulated using the Chow groups of smooth Deligne–Mumford stacks with \( \mathbb{Q} \)-coefficients. This notably covers the case \( g = 2 \), where a general K3 surface over \( \mathcal{F}_2 \) carries an involution. Our proof in Section 3 works in this case without change.
3 Polarized $K3$ Surfaces as Unique Complete Intersections

In this section we deal with the cases $g \leq 10$ and $g = 12, 18, 20$. For these $g$, the Mukai model embeds a general polarized $K3$ surface of degree $2g - 2$ in $G_g$ as a complete intersection with respect to $U_g$, and the embedding is unique up to automorphisms of $G_g$ and $U_g$. Moreover, the variety $G_g$ is a Grassmannian or an orthogonal Grassmannian.

Since $P_g$ rationally dominates the moduli space $F_g$, to prove Theorem 1.2 it suffices to show that the restriction of any class in $\text{CH}^2(Y)$ to a $K3$ fiber $Y_s$ lies in $\mathbb{Z}c_{Y_s}$. By Lemma 2.1, it is equivalent to show that

$$\text{Im}(i_s^* : \text{CH}^2(G_g) \rightarrow \text{CH}^2(Y_s)) \subset \mathbb{Z}c_{Y_s}.$$  

This allows us to work with a single $K3$ surface $S$ with an embedding

$$i : S \hookrightarrow G_g.$$  

If $g \leq 5$, the variety $G_g$ is a projective space and its Chow ring is generated by the hyperplane class. Thus Theorem 1.2 follows from property (i) of $c_S$ in Section 1.

Now assume that $G_g$ is not a projective space. It is well known that the Chow group $\text{CH}^2(Gr(r, n))$ of the Grassmannian is generated by the Chern classes $c_1(Q)^2$ and $c_2(Q)$, where $Q$ is the universal quotient bundle. For the orthogonal Grassmannians, we have instead

$$\text{CH}^2(OG(5,10)) = \mathbb{Z}\left(\frac{1}{2}c_2(Q)\right) \oplus \mathbb{Z}\left(\frac{1}{4}c_1(Q)^2\right)$$

and

$$\text{CH}^2(OG(3,9)) = \mathbb{Z}\left(\frac{1}{2}c_2(Q)\right) \oplus \mathbb{Z}c_1(Q)^2,$$

where $Q$ is the corresponding universal quotient bundle (see [15]). Hence in all cases, a class $\alpha \in \text{CH}^2(G_g)$ can be uniquely expressed as

$$\alpha = a \cdot c_2(Q) + b \cdot c_1(Q)^2,$$

with $a \in \mathbb{Z}$ if $G_g$ is a Grassmannian, or $a \in \frac{1}{2}\mathbb{Z}$ if $G_g$ is an orthogonal Grassmannian. For convenience we define the index $I(\alpha)$ of $\alpha \in \text{CH}^2(G_g)$ to be the coefficient $a$.

By property (i) of $c_S$ in Section 1, we have $i^*(c_1(Q)^2) \in \mathbb{Z}c_S$. Hence the following proposition implies Theorem 1.2 for $g = 6, 7, 8, 9, 10, 12, 18, 20$. 

Proposition 3.1. With the notation as above, we have $i^*c_2(Q) \in \mathbb{Z}c_S$. □

We give two independent proofs of the proposition.

First proof. Mukai showed in [7, 8] that the restriction of either $Q$ or $\mathcal{E}^\vee$ to a general $S$ is simple and rigid, where $\mathcal{E}$ is the universal subbundle. In fact, the rigidity ensures that the embedding of $S$ in $G_g$ is unique. The proposition follows from a strong result of Voisin [17, Corollary 1.10] that the second Chern class of any simple rigid vector bundle on a $K3$ surface $S$ lies in $\mathbb{Z}c_S$, which was conjectured by Huybrechts earlier in [5]. ■

Since part of the original motivation of the generalized Franchetta conjecture was to make Huybrechts’ conjecture as its consequence (see [12, Section 5]), we give a direct proof of Proposition 3.1 without using Voisin’s result.

Second proof. We first consider the cases where $G_g$ is a Grassmannian. The standard exact sequence of normal bundles

$$0 \to T_S \to i^*T_{G_g} \to i^*\mathcal{U}_g \to 0$$

yields the following relation in $\text{CH}_0(S)$:

$$i^*c_2(T_{G_g}) = c_2(T_S) + i^*c_2(\mathcal{U}_g). \tag{3.1}$$

Here $T_{G_g}$ and $T_S$ are the corresponding tangent bundles. Using the index of classes in $\text{CH}^2(G_g)$, the relation (3.1) can be written as

$$\left(I(c_2(T_{G_g})) - I(c_2(\mathcal{U}_g))\right) \cdot i^*c_2(Q) = c_2(T_S) + \gamma, \tag{3.2}$$

where $\gamma$ can be expressed in terms of divisor classes on $S$. By properties (i) and (ii) of $c_S$ in Section 1, both $c_2(T_S)$ and $\gamma$ lie in $\mathbb{Z}c_S$. Hence it suffices to verify that

$$I(c_2(T_{G_g})) - I(c_2(\mathcal{U}_g)) \neq 0. \tag{3.3}$$

The tangent bundle $T_{G(r,n)}$ of the Grassmannian is $\mathcal{H}om(\mathcal{E}, Q)$, where $\mathcal{E}$ is the universal subbundle. By computing the Chern character

$$\text{ch}(\mathcal{H}om(\mathcal{E}, Q)) = \text{ch}(\mathcal{E}^\vee \otimes Q) = \text{ch}(\mathcal{E}^\vee) \cdot \text{ch}(Q)$$
and the standard relation \( c(E) \cdot c(Q) = 1 \) between the total Chern classes, we have
\[
I(\overline{c}_2(T_{G(r,n)}) = 2r - n.\]

The following is a case-by-case study:

\( g = 6, 8 \) Here \( U_g \) is a direct sum of line bundles. Hence \( I(\overline{c}_2(U_g)) = 0 \) and
\[
I(\overline{c}_2(T_{G_g})) - I(\overline{c}_2(U_g)) = 2r - n \neq 0.
\]

\( g = 9 \) We have \( I(\overline{c}_2(T_{G_9})) = 0 \) and \( I(\overline{c}_2(U_9)) = I(\overline{c}_2(\wedge^2 Q)) = 1 \). Hence
\[
I(\overline{c}_2(T_{G_9})) - I(\overline{c}_2(U_9)) = -1 \neq 0.
\]

\( g = 10 \) We have \( I(\overline{c}_2(T_{G_{10}})) = -3 \) and \( I(\overline{c}_2(U_{10})) = I(\overline{c}_2(\wedge^4 Q)) = 1 \). Hence
\[
I(\overline{c}_2(T_{G_{10}})) - I(\overline{c}_2(U_{10})) = -4 \neq 0.
\]

\( g = 12, 20 \) We have \( I(\overline{c}_2(T_{G_g})) = -1 \) and \( I(\overline{c}_2(U_g)) = 3I(\overline{c}_2(\wedge^2 E^\vee)) \). Hence
\[
I(\overline{c}_2(T_{G_g})) - I(\overline{c}_2(U_g)) = -1 - 3I(\overline{c}_2(\wedge^2 E^\vee)) \neq 0.
\]

The orthogonal Grassmannian cases \((g = 7, 18)\) are similar. The relation (3.2) still holds, and it suffices to show (3.3). Here the left-hand side of (3.3) may be a half integer.

The natural embedding \( j : \mathcal{O}_G(r,n) \rightarrow G(r,n) \) can be realized as the zero locus of a smooth section of the vector bundle \( W \), which is given by the cohomology group \( H^0(\mathbb{P}^{r-1}, \mathcal{O}(2)) \) over every closed point
\[
[\mathbb{P}^{r-1} \subset \mathbb{P}^{n-1}] \in G(r,n).
\]

Hence we have
\[
I(\overline{c}_2(T_{OG(r,n)}) = I(j^*c_2(T_{G(r,n)})) - I(j^*c_2(W)).
\]

The term \( I(j^*c_2(T_{G(r,n)}) \) was already calculated, and the term \( I(j^*c_2(W)) \) can be determined by the following Grothendieck–Riemann–Roch calculation.

We consider \( p : \mathbb{P}(E) \rightarrow G(r,n) \) the projective bundle on \( G(r,n) \) associated to the universal subbundle \( E \). Let \( L = \mathcal{O}_{\mathbb{P}(E)}(1) \) and let \( \xi \) be the relative hyperplane class \( c_1(L) \).
We have $R^k p_\ast L = 0$ for $k > 0$. Hence by the Grothendieck–Riemann–Roch theorem, we have

$$\text{ch}(\mathcal{W}) = \text{ch}(R p_\ast L^\otimes 2) = p_\ast(\exp(2\xi) \cdot \text{td}(T_p)).$$

Together with the exact sequence

$$0 \to \mathcal{O}_{\mathcal{W}} \to p^\ast \mathcal{E} \otimes L \to T_p \to 0,$$

we obtain for $r = 3, 5$,

$$I(c_2(\mathcal{W})) = -(r + 2).$$

We finish the proof of Proposition 3.1:

$g = 7$  Here $\mathcal{U}_7$ is a direct sum of line bundles. Hence $I(c_2(\mathcal{U}_7)) = 0$ and

$$I(c_2(T_{G7})) - I(c_2(\mathcal{U}_7)) = 0 - (-7) = 7 \neq 0.$$

$g = 18$  We have $I(c_2(T_{G18})) = -3 - (-5) = 2$ and $I(c_2(\mathcal{U}_{18})) = 5I(c_2(\mathcal{V}_{18}))$. Hence

$$I(c_2(T_{G18})) - I(c_2(\mathcal{U}_{18})) = 2 - 5I(c_2(\mathcal{V}_{18})) \neq 0. \quad \blacksquare$$

4  Polarized $K3$ Surfaces as Nonunique Complete Intersections

In this section we treat the remaining cases $g = 13, 16$. In both cases, the embedding of a polarized $K3$ surface $S$ of degree $2g - 2$ in $\mathbb{G}_g$ is not unique and the restriction of the tautological bundles to $S$ may not be rigid. Hence the methods in Section 3 break down.

We keep the notation of Section 2 and write $\Phi: \mathbb{P}_g \dashrightarrow \mathcal{F}_g$ for the dominant rational map. Let $t \in \mathcal{F}_g^0$ be a closed point outside the indeterminacy locus of $\Phi$ in $\mathcal{F}_g$. Given two closed points $s_1, s_2 \in \mathbb{P}_g$ with $\Phi(s_1) = \Phi(s_2) = t$, there are canonical isomorphisms

$$Y_{s_1} \cong Y_{s_2} \cong X_t.$$

We identify $CH_0(Y_{s_1}), CH_0(Y_{s_2})$ with $CH_0(X_t)$, and define

$$CH^2(Y)_{\text{inv}} = \{ \alpha \in CH^2(Y) | \phi_{s_1}^\ast \alpha = \phi_{s_2}^\ast \alpha \text{ for all } s_1, s_2 \in \mathbb{P}_g \text{ above} \}.$$

Recall that $\phi_s: Y_s \hookrightarrow Y$ is the natural embedding for $s \in \mathbb{P}_g$. 


Again by the “spreading out” argument and the fact that classes supported over a proper closed subset of $\mathcal{F}_g^0$ do not contribute, to prove Theorem 1.2 it suffices show that

$$\text{Im}(\phi_s^*: \text{CH}^2(Y)_{\text{inv}} \to \text{CH}_0(Y_s)) \subset \mathbb{Z}c_{Y_s}$$

for all (or general, or very general) $K3$ fibers $Y_s$.

First we consider the case $g = 13$. As described by the Mukai model, let

$$i : S \hookrightarrow \mathbb{G}(3, 7)$$

be the embedding of a $K3$ surface $S$ in $\mathbb{G}(3, 7)$. The restriction of $E^\vee$ (dual of the universal subbundle) to $S$ is semi-rigid, which carries a 2-dimensional deformation. Let $M_S$ be the moduli space of stable vector bundles on $S$ with Mukai vector $(3, H, 4)$, where $H$ is the polarization class. A general point of $M_S$ is represented by $i^*E^\vee$ for some $i$; see [9] for details. Note that $M_S$ is also a polarized $K3$ surface with $g = 13$.

Let $s \in \mathbb{P}_{13}$ be a closed point with $K3$ fiber $Y_s$, and let $i_s : Y_s \hookrightarrow \mathbb{G}(3, 7)$ be as in Section 2. By Lemma 2.1, the restriction $\phi_s^*\alpha$ of a class $\alpha \in \text{CH}^2(Y)_{\text{inv}}$ can be expressed as

$$\phi_s^*\alpha = a \cdot i_s^*c_2(Q) + b \cdot i_s^*(c_1(Q)^2), \quad (4.1)$$

where $Q$ is the universal quotient bundle and $a, b \in \mathbb{Q}$ are constants independent of $s \in \mathbb{P}_{13}$. By property (i) of $c_{Y_s}$ in Section 1, we have $i_s^*(c_1(Q)^2) \in \mathbb{Z}c_{Y_s}$.

Theorem 1.2 for $g = 13$ is a direct consequence of the following lemma.

**Lemma 4.1.** In the expression (4.1), the coefficient $a$ is zero. \qed

**Proof.** We choose closed points $s_1, s_2 \in \mathbb{P}_{13}$ with $\Phi(s_1) = \Phi(s_2) = t \in \mathcal{F}_g^0$, such that the vector bundles $i_{s_1}^*E^\vee, i_{s_2}^*E^\vee$ represent different point classes in $\text{CH}_0(M_{X_t})$. This is possible by [9, Theorem 2], which shows that $\mathbb{P}_{13}$ rationally dominates the moduli space of triples $(S, H, E)$, where $S$ is a $K3$ surface, $H$ is a polarization with $H^2 = 24$, and $E$ is a stable vector bundle with Mukai vector $(3, H, 4)$.

Since $\alpha \in \text{CH}^2(Y)_{\text{inv}}$, we have by definition $\phi_{s_1}^*\alpha = \phi_{s_2}^*\alpha$ and hence

$$a \cdot i_{s_1}^*c_2(Q) = a \cdot i_{s_2}^*c_2(Q), \quad (4.2)$$

viewed as an equality in $\text{CH}_0(X_t)_Q$. 

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On the other hand, let $\mathbb{F}$ be a universal sheaf over $M_{X_t} \times X_t$ (which exists by the numerics of the Mukai vector; see [6, Corollary 4.6.7]). The correspondence

$$\text{ch}(\mathbb{F}) \cdot \sqrt{\text{td}(T_{M_{X_t} \times X_t})} \in \text{CH}^*(M_{X_t} \times X_t)_\mathbb{Q}$$

induces an isomorphism of (ungraded) Chow groups

$$\theta : \text{CH}^*(M_{X_t}) \rightarrow \text{CH}^*(X_t).$$

Here for $[E] \in M_{X_t}$, we have

$$\theta([E]) = 3[X_t] + H + 15c_{X_t} - c_2(E) \in \text{CH}^*(X_t).$$

According to our choice of $s_1, s_2 \in \mathbb{P}_{13}$, the vector bundles $t^*_1 \mathcal{E}^\vee, t^*_2 \mathcal{E}^\vee$ represent different classes in $\text{CH}_0(M_{X_t})$. By applying $\theta$, we find

$$t^*_1 c_2(\mathcal{E}^\vee) \neq t^*_2 c_2(\mathcal{E}^\vee)$$

in $\text{CH}_0(X_t)$, and together with (4.2) we obtain $a = 0$.

Finally we consider the case $g = 16$. The variety $G_{16} = G(2, 3, 4)$ is realized as a GIT quotient of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ by the obvious action of $\text{GL}_2 \times \text{GL}_3$ on the first two factors. As described in [3] (see also [10]), there are two tautological vector bundles $\mathcal{V}_{16}$ and $\tilde{\mathcal{V}}_{16}$ of ranks 3 and 2, respectively, as well as a morphism

$$\mathcal{V}_{16} \otimes (\mathbb{C}^4)^\vee \rightarrow \tilde{\mathcal{V}}_{16}.$$

Further, it was shown in [3, Proposition 2] that the Chow ring $\text{CH}^*(G(2, 3, 4))$ is generated by the Chern classes of $\mathcal{V}_{16}, \tilde{\mathcal{V}}_{16}$. To prove Theorem 1.2 we have to take care of the second Chern classes of both tautological bundles.

Let $i : S \hookrightarrow G(2, 3, 4)$ be the embedding of a K3 surface $S$ in $G(2, 3, 4)$ as in the Mukai model. By the same reasoning as in Section 3, we have the following relation in $\text{CH}_0(S)$:

$$i^* c_2(T_{G(2,3,4)}) = c_2(T_S) + i^* c_2(\mathcal{U}_{16}). \quad (4.3)$$

Here $\mathcal{U}_{16} = \mathcal{V}^{\otimes 2}_{16} \oplus \tilde{\mathcal{V}}^{\otimes 2}_{16}$. Using the exact sequence (see [4, (4-4)])

$$0 \rightarrow \mathcal{O}_{G(2,3,4)} \rightarrow \mathcal{E}nd(\mathcal{V}_{16}) \oplus \mathcal{E}nd(\tilde{\mathcal{V}}_{16}) \rightarrow \mathcal{H}om(\mathcal{V}_{16} \otimes (\mathbb{C}^4)^\vee, \tilde{\mathcal{V}}_{16}) \rightarrow T_{G(2,3,4)} \rightarrow 0,$$

...
the relation (4.3) can be written as

$$6c_2(i^*\widetilde{V}_{16}) = c_2(T_S) + \gamma,$$

where $\gamma$ can be expressed in terms of divisor classes on $S$. By properties (i) and (ii) of $c_S$ in Section 1, this verifies that $i^*c_2(\widetilde{V}_{16}) \in \mathbb{Z}c_S$.

Alternatively, by Mukai’s results [10, Propositions 1.3 and 2.2], for a general $S$ the vector bundle $i^*\widetilde{V}_{16}$ is simple and rigid. The statement $i^*c_2(\widetilde{V}_{16}) \in \mathbb{Z}c_S$ also follows from Voisin’s result [17, Corollary 1.10].

Let $s \in \mathbb{P}_{16}$ be a closed point with K3 fiber $Y_s$, and let $\iota_s : Y_s \hookrightarrow G(2,3,4)$ be as before. Again by Lemma 2.1 and property (i) of $c_{Y_s}$ in Section 1, the restriction $\phi_s^*\alpha$ of a class $\alpha \in CH^2(Y)_{\text{inv}}$ can be expressed as

$$\phi_s^*\alpha = a \cdot \iota_s^*c_2(V_{16}) + b \cdot c_{Y_s},$$

where $a, b \in \mathbb{Q}$ are constants independent of $s \in \mathbb{P}_{16}$. Moreover, the fact that $\iota_s^*c_2(\widetilde{V}_{16}) \in \mathbb{Z}c_{Y_s}$ implies

$$\phi_s^*\alpha = a \cdot \iota_s^*c_2(V_{16}) + b' \cdot c_{Y_s} \quad (4.4)$$

for some $a, b' \in \mathbb{Q}$ independent of $s \in \mathbb{P}_{16}$. Since $\iota_s^*V_{16}$ is semi-rigid with Mukai vector $(3, H, 5)$ by [10, Proposition 2.2], an identical argument as in the proof of Lemma 4.1 yields $a = 0$ in the expression (4.4).

This finishes the proof of Theorem 1.2 for $g = 16$.

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