Commutator subgroups, nilpotent subgroups, p-groups

<u>Definition</u> For $x,y \in G$, define $[x,y] := x^{-1}y^{-1}xy$, the <u>commutator of x and y</u> (交换子)

($\underline{\text{Note}}$: $xy = yx \Leftrightarrow [x,y] = e$, $g[x,y]g^{-1} = [gxg^{-1}, gyg^{-1}]$.)

$G^{der} = G' = \langle [x,y] \,;\, x,y \in G \rangle$ is called the <u>commutator subgroup</u> of G (交换子群)

or the <u>derived subgroup</u> of G (导出子群)

(<u>Caveat</u>: Not true that every element of G' is a commutator itself.)

This G' is a normal subgroup of G

and G/G' is abelian b/c $xG' \cdot yG' \overset{?}{=} yG' \cdot xG'$

$\Leftrightarrow x^{-1}y^{-1}xy\,G' = G'$ yes.

* G/G' is the "maximal abelian quotient" of G in the following sense.

<u>Lemma</u> If A is an abelian group and $\varphi: G \longrightarrow A$ a homomorphism

then $G' \subseteq \ker\varphi$ (b/c $\varphi(x^{-1}y^{-1}xy) = 1$.)

Thus, φ factors as $G \twoheadrightarrow G/G' \xrightarrow{\bar{\varphi}} A$

* There is a bijection: for A an abelian group,

$$\mathrm{Hom}_{gp}(G,A) = \mathrm{Hom}_{gp}(G/G', A)$$

$$\varphi \longmapsto (\bar{\varphi} : gG' \mapsto \varphi(g))$$

In other words, if we want to Hom a group to an abelian group, it is enough to Hom out from G/G'

<u>Example</u>: $G = D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ srs^{-1} = r^{-1} \rangle$

Then G' contains $srs^{-1}r^{-1} = r^{-2}$.

* If n is odd, $\langle r \rangle = \langle r^{-2} \rangle \subseteq G'$ ← expect an equality

On the other hand, we have $\psi: G \longrightarrow \{\pm 1\}$

rigorous way to prove that $G'=\langle r\rangle$

$$\psi(r)=1,\quad \psi(s)=-1$$

(Check: $\psi(r)^n=\psi(s)^n=1$ and $\psi(s)\psi(r)\psi(s)^{-1}=\psi(r)^{-1}$)

$$\overset{\text{Lemma}}{\Longrightarrow} G'\leq \ker\psi=\langle r\rangle$$

So. $G'=\langle r\rangle$ and $G/G'\simeq\{\pm1\}$

* If n is even, $\langle r^{-2}\rangle=\langle r^2\rangle\leq G'$

We define $\psi: G\longrightarrow\{\pm1\}\times\{\pm1\}$

$$\psi(r)=(-1,1),\quad \psi(s)=(1,-1)$$

(Check: $\psi(r)^n=\psi(s)^2=1$, $\psi(s)\psi(r)\psi(s)^{-1}=\psi(r)^{-1}$.)

$\Rightarrow G'\leq\ker\psi=\langle r^2\rangle$ So, $G'=\ker\psi$ and $G/G'\simeq\{\pm1\}\times\{\pm1\}$

To find all $\mathrm{Hom}(G,\mathbb{C}^\times)\simeq\mathrm{Hom}(\{\pm1\}\times\{\pm1\},\mathbb{C}^\times)$

$\psi: G\longrightarrow\mathbb{C}^\times$ $\rightsquigarrow$ $\bar\psi(-1,1)=\lambda\in\{\pm1\}$

$\psi(r)=\lambda,\ \psi(s)=\mu.$ $\quad\bar\psi(1,-1)=\mu\in\{\pm1\}$

• Solvable groups

Recall: A group G is called a solvable group, if

$$\exists\ 1=G_0\trianglelefteq G_1\trianglelefteq\cdots\trianglelefteq G_r=G \quad \text{s.t. } G_i/G_{i-1} \text{ is abelian.}$$

(When G is finite, this is equivalent to existing such a series with $G_i/G_{i-1}\simeq\mathbb{Z}_{p_i}$ with p_i prime)

· In particular, abelian groups are solvable.

A good way to test solvable groups is:

Definition For any group G, define the following sequence of subgroups inductively,

$$G^{(0)}=G,\quad G^{(1)}=[G,G],\qquad G^{(i+1)}=[G^{(i)},G^{(i)}]\quad\forall i\in\mathbb{N}$$

This is called the derived or commutator series of G (导出序列)

Example: $G=\left\{\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}\right\} \supseteq G^{(1)}=\left\{\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}\right\} \supseteq G^{(2)}=\left\{\begin{pmatrix} 1 & 0 & * & * \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right\} \supseteq G^{(3)}=\left\{\begin{pmatrix} 1 & 0 & 0 & * \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right\} \supseteq G^{(4)}=\{1\}$

Proposition A group is solvable if and only if $G^{(n)}=\{1\}$ for some finite $n\in\mathbb{N}$.

Proof: "$\Leftarrow$" Note each $G^{(i+1)}$ is a normal subgroup of $G^{(i)}$

So $\{1\}=G^{(n)} \trianglelefteq G^{(n-1)} \trianglelefteq \cdots \trianglelefteq G^{(0)}=G$ satisfies $G^{(i)}/G^{(i+1)}$ is abelian.

"$\Rightarrow$" $\exists \{1\}=H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_r=G$ s.t. H_i/H_{i-1} is abelian

$\Rightarrow [H_i,H_i]\subseteq H_{i-1}$.

From this, we see $G^{(1)}=[G,G]\subseteq H_{r-1}$

$G^{(2)}=[G^{(1)},G^{(1)}]\subseteq[H_{r-1},H_{r-1}]\subseteq H_{r-2}$, ...

$G^{(i)}\subseteq H_{r+1-i} \Rightarrow G^{(r+1)}\subseteq H_0=\{1\}$ □

Remark: The derived series is the "fastest-decreasing" series so that the subquotients are abelian

(The smallest $n\in\mathbb{Z}_{\geq 0}$ for which $G^{(n)}=\{1\}$ is called the *solvable length* of G.)

Lemma. All $G^{(i)}$ are normal subgroups of G

Moreover, they are charaiteristic subgroups of G.

Proof: $G^{(1)}=[G,G]=\langle x^{-1}y^{-1}xy \,;\, x,y\in G\rangle$

If $\phi: G \xrightarrow{\sim} G$ is an automorphism, then

$$\phi(G^{(1)})=\langle \phi(x^{-1}y^{-1}xy)\,;\, x,y\in G\rangle=\langle \phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1}\,;\, x,y\in G\rangle=G^{(1)}$$

Inductively, we prove $\phi(G^{(i)})=\phi[G^{(i-1)},G^{(i-1)}]$

$$=[\phi(G^{(i-1)}),\phi(G^{(i-1)})]=[G^{(i-1)},G^{(i-1)}]=G^{(i)}$$

Basic properties without proof

① If $H \leq G \Rightarrow H^{(i)} \leq G^{(i)}$. So if G is solvable $\Rightarrow$ H is solvable

② $G \overset{\varphi}{\twoheadrightarrow} K$ a surjective homomorphism $\Rightarrow \varphi(G^{(i)}) = K^{(i)}$

So G solvable $\Rightarrow$ K solvable

③ If $N \trianglelefteq G$ and both N & G/N are solvable, then G is solvable

- {cyclic groups} ⊆ {abelian groups} ⊆ {nilpotent groups} ⊆ {solvable groups} ⊆ {all groups}

<u>Definition</u> For a group G, define the following subgroups:

$$G^0 = G, \quad G^1 := [G,G], \quad G^{i+1} := [G, G^i] \text{ for } i$$

$\rightsquigarrow G^0 \trianglerighteq G^1 \trianglerighteq G^2 \trianglerighteq \cdots$ This is called <u>lower central series</u> of G

(Similar to above, each G^i is normal inside G and $G^i \geq G^{(i)}$.)

The group is called <u>nilpotent</u> if $G^c = \{1\}$ for some $c \in \mathbb{N}$

<u>Corollary</u>. G nilpotent $\Rightarrow$ solvable

<u>Proof</u>: $G^c = \{1\}$ & $G^c \geq G^{(c)} \Rightarrow G^{(c)} = \{1\}$. □

<u>Example</u>: $N = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ is nilpotent, $B = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\}$ is solvable but not nilpotent.

"<u>Dual picture</u>"

<u>Definition</u> For any group G, define the following subgroups inductively:

$$Z_0(G) = \{1\}, \quad Z_1(G) = Z(G)$$

Consider $G \overset{\pi_1}{\twoheadrightarrow} G/Z(G) =: \bar{G}$

$$\pi^{-1}(Z(\bar{G})) \longrightarrow Z(G/Z(G))$$

(with $\pi^{-1}(Z(\bar{G}))$ normal in G and $Z(G/Z(G))$ normal in $\bar{G}$)

Put $Z_2(G) := \pi_1^{-1}(Z(G/Z(G)))$

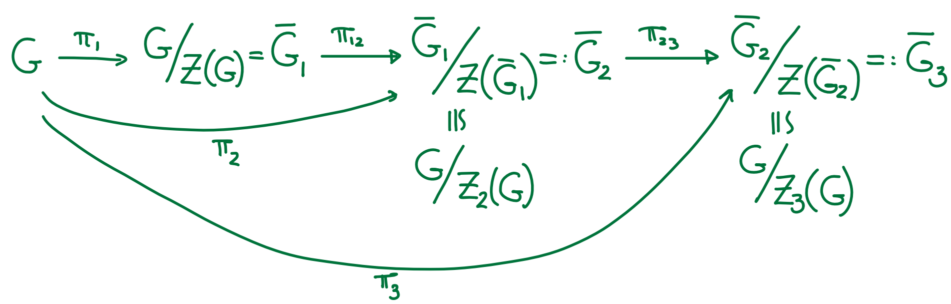
$$\text{Inductively, } \quad \begin{array}{ccc} G & \xrightarrow{\pi_i} & G/Z_i(G) \\ \cup\!\!\mid \text{ normal} & & \cup\!\!\mid \\ \pi_i^{-1}\left(Z\left(G/Z_i(G)\right)\right) & \longrightarrow & Z\left(G/Z_i(G)\right) \end{array}$$

Put $Z_{i+1}(G) := \pi_i^{-1}\left(Z\left(G/Z_i(G)\right)\right)$

The sequence : $Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq \cdots$ is called <u>upper central sequence</u>

<u>Notation</u> : subscript indexing = increasing filtration, superscript indexing = decreasing filtration.)

· Alternative explanation:



So, $Z_i(G) = \ker \pi_i = \pi_i^{-1}(\text{“}1\text{”})$

<u>Note</u> : $Z_i(G) = \pi_1^{-1}\pi_{1i}^{-1}(\text{“}1\text{”}) = \pi_1^{-1}\left(Z_{i-1}\left(G/Z(G)\right)\right)$

<u>Theorem</u> A group G is nilpotent if and only if $Z_c(G) = G$ for some $c \in \mathbb{N}$

More precisely, for $c \in \mathbb{N}$, $G^c = \{1\} \Leftrightarrow Z_c(G) = G$.

In this case, we have $\quad G^{c-i} \leq Z_i(G)$ for $i = 0, 1, \cdots, c$.

(There is an error in Dummit-Foote P194 Thm 8, not true for $Z_i(G) \leq G^{c-i-1}$)

Proof: Use induction on c. Consider

$$\begin{array}{cccc} \pi : & G & \twoheadrightarrow & G/Z(G) =: \bar{G} \\ & \cup\!\!\mid & & \cup\!\!\mid \\ & H & \longrightarrow & \pi(H) = \bar{H} \end{array}$$

Will prove

$$\begin{array}{ccc} G^c = \{1\} & & Z_c(G) = G \\ \Updownarrow (1) & & \Updownarrow (2) \\ (\bar{G})^{c-1} = \{1\} & \xLeftrightarrow{\text{inductive hypo}} & Z_{c-1}(\bar{G}) = \bar{G} \end{array}$$

(2) As explained above, we have $Z_{i-1}(\bar{G}) = \overline{Z_i(G)}$

(1) $(\bar{G})^{c-1} = \{1\} \iff G^{c-1} \subseteq Z(G) \iff [G, G^{c-1}] = \{1\} \iff G^c = \{1\}$

Applying inductive hypothesis to $\bar{G}$:

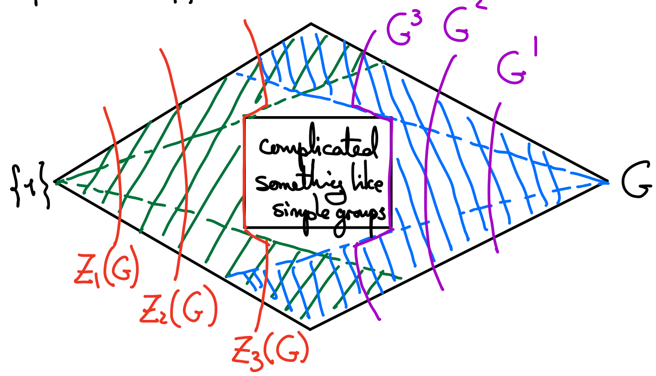
$$\bar{G}^{c-1-i} \leqslant Z_i(\bar{G})$$

Taking π^{-1} gives $G^{c-1-i} \leqslant \pi^{-1}(\bar{G}^{c-1-i}) \leqslant \pi^{-1}(Z_i(\bar{G})) = Z_{i+1}(G)$

<u>Philosophical understanding</u>:

- Abelian groups are easier to understand.
- If H is a nonabelian simple finite group, then $[H,H] = H$, $Z(H) = \{1\}$

  (b/c $[H,H] \neq 1$ & is normal in H.)
- Visualization of lower / upper central series:



<u>Example</u>: All p-groups are nilpotent.

<u>Theorem</u>: Let P be a p-group.

(1) $Z(P) \neq \{1\}$ (proved earlier)

(2) If $1 \neq H \trianglelefteq P$ is normal, then $H \cap Z(P) \neq \{1\}$

(Proof: Consider $P \overset{Ad}{\curvearrowright} H$ acting on H by conjugation

$\Rightarrow H = \bigsqcup P/\ldots$

$\rightarrow H = \coprod_i \cdot / \mathrm{Stab}_P(a_i)$ for $a_1, \dots, a_r$ representatives of orbits

- $\mathrm{Stab}(a_i) = P \iff \forall x \in P,\ x a_i x^{-1} = a_i \iff a_i \in Z(P) \cap H$

So $\#H = \sum_i \#P / \#\mathrm{Stab}_P(a_i) \equiv \#(Z(P) \cap H) \bmod p$

$\#H \equiv 0 \pmod p$ $\quad \Rightarrow Z(P) \cap H \neq \{1\}$.)

(3) If $H \lneqq P$, then $H \lneqq N_P(H)$

Cor. If $H < P$ has index $p \Rightarrow H$ is normal.

(Proof: Induction on $\#P$

Case 1: If $Z(P) \nsubseteq H$, then $Z(P) \subseteq N_P(H)$ so $H \lneqq N_P(H)$

Case 2: If $Z(P) \subseteq H$, consider $\bar{H} := H/Z(P) \subseteq \bar{P} := P/Z(P)$

By inductive hypothesis, $\bar{H} \lneqq N_{\bar{P}}(\bar{H}) \Rightarrow H \lneqq N_P(H)$ □)

Theorem (Structure theorem for nilpotent)

G finite group of order $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $P_i \in \mathrm{Syl}_{p_i}(G)$. TFAE

(1) G is nilpotent

(2) if $H \lneqq G$, then $H \lneqq N_G(H)$

(3) all Sylow subgroups P_i are normal.

(4) $G \cong P_1 \times \cdots \times P_r$.

Proof: (3) ⇒ (4) by criterion of direct product:

$$P_1P_2 \cong P_1 \times P_2,\ P_1P_2P_3 \cong P_1P_2 \times P_3 \cong P_1 \times P_2 \times P_3,\ \dots$$

(4) ⇒ (1) as each P_i is nilpotent

(2) ⇒ (3) Recall that, for each P_i, $N_G(N_G(P_i)) = N_G(P_i)$

So (2) implies that $N_G(P_i) = G \Rightarrow$ each P_i is normal.

(1) ⇒ (2) Same as Thm (3) above, noting that G nilpotent $\Rightarrow G/Z(G)$ nilpotent.