Stabilizers and orbits of group actions, class equation, outer automorphisms

Today: More advanced topics on group actions

Definition. Let G be a group acting on a set X. ← sometimes, this is called a G-set.

For each $x \in X$, write $\mathrm{Stab}_G(x) := \{ g \in G \mid g \cdot x = x \}$

called the stabilizer subgroup of x (稳定子群)

For each $x \in X$, write $\mathrm{Orb}_G(x) := G \cdot x := \{ g \cdot x \mid g \in G \}$

called the orbit of x (轨道)

Properties:

(1) $\mathrm{Stab}_G(x)$ is a subgroup of G

(NTS $\forall g, h \in \mathrm{Stab}_G(x) \Rightarrow gh^{-1} \in \mathrm{Stab}_G(x)$ i.e. $gh^{-1}x = x$

need to show

$\Downarrow$

$gx = x$ So $gh^{-1}x = gx = x$.

and $hx = x \Rightarrow \underbrace{h^{-1} \cdot hx}_{= x} = h^{-1}x$)

(2) For $x, y \in X$, either $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$ or $\mathrm{Orb}_G(x) \cap \mathrm{Orb}_G(y) = \phi$

So, we have $X = \coprod_{\text{orbits } \mathcal{O}} \mathcal{O}$

(NTS If $z \in \mathrm{Orb}_G(x) \cap \mathrm{Orb}_G(y)$, then $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$

$\Downarrow$

$z = gx = hy$ for some $g, h \in G$

Then for $w \in \mathrm{Orb}_G(x) \rightsquigarrow w = k \cdot x$ for some $k \in G$

$\Rightarrow w = kx = kg^{-1}z = kg^{-1}hy \in \mathrm{Orb}_G(y)$

So $\mathrm{Orb}_G(x) \subseteq \mathrm{Orb}_G(y)$. The other inclusion can be proved similarly.)

(3) For $y \in \mathrm{Orb}_G(x)$, say $y = g \cdot x$, then $\mathrm{Stab}_G(y) = g\,\mathrm{Stab}_G(x)\,g^{-1}$

Namely, the stabilizers at different elements of one orbit are conjugate to each other.

($h \in Stab_G(y) \Leftrightarrow hy = y \Leftrightarrow hgx = gx \Leftrightarrow g^{-1}hgx = x$

$\Leftrightarrow g^{-1}hg \in Stab_G(x) \Leftrightarrow h \in g\, Stab_G(x)\, g^{-1}$.)

<u>A particular case</u>: Conjugation action of G on itself

<u>Definition</u> Two elements $a, b \in G$ are <u>conjugate</u> (共轭) if $a = gbg^{-1}$ for some $g \in G$

The orbits of G under the conjugation action are called <u>conjugacy classes</u> (共轭类)

E.g. ① If G is abelian, the conjugacy class of a is $\{a\}$

② $GL_n(\mathbb{C})$: every matrix is conjugated into a Jordan block

So conjugacy classes $\longleftrightarrow$ Jordan forms (with nonzero eigenvalues)

③ S_n, conjugacy classes $\longleftrightarrow$ partitions of n into sums of positive integers

Proof: Recall for $\sigma \in S_n$, $\sigma (a_1 a_2 \cdots a_r)(b_1 b_2 \cdots b_s) \cdots \sigma^{-1}$

$= (\sigma(a_1)\sigma(a_2) \cdots \sigma(a_r))(\sigma(b_1)\sigma(b_2) \cdots \sigma(b_s)) \cdots$

So conjugation does not change the type of cycle decomposition

Moreover, if τ_1, τ_2 have the same cycle type $\Rightarrow \exists\, \sigma \tau_1 \sigma^{-1} = \tau_2$ $\square$

<u>Definition</u> Let $H < G$ be a subgroup, $S \subseteq G$ a subset

(1) $C_G(S) := \{ g \in G \mid \text{for every } s \in S,\ gsg^{-1} = s \}$ <u>centralizer</u> of S (中心化子)

For the conjugation action, $Stab_G(s) = C_G(s)$

$C_G(S) = \bigcap_{s \in S} Stab_G(s)$ (In particular, it is a subgroup)

(2) $Z(G) := \{ g \in G \mid \forall h \in G,\ ghg^{-1} = h \} = C_G(G)$ <u>center</u> of G (G的中心)

• Alternative point of view: the conjugation action induces $Ad : G \to S_G$

Then $Z(G) = \ker(Ad)$

(3) $N_G(H) := \{ g \in G \mid gHg^{-1} = H \}$ <u>normalizer</u> of H (正规化子)

Note: $H \trianglelefteq G \iff N_G(H) = G$

- If we consider $G \overset{\mathrm{Ad}}{\curvearrowright} \{\text{all subgroups of } G\}$

$$g \star H := gHg^{-1}$$

then $N_G(H) = \mathrm{Stab}_G(H)$.

<u>Definition</u> Let G be a group acting on both sets X and Y

We say a map $\phi : X \longrightarrow Y$ is <u>G-equivariant</u> (G-等变映射) if

$$\forall g \in G,\ x \in X, \quad \phi(g \cdot x) = g \cdot \phi(x)$$

<u>Remark</u>:

Algebraic structure on a set	vector spaces	groups	sets with group actions
$\rightsquigarrow$	$\rightsquigarrow$	$\rightsquigarrow$	$\rightsquigarrow$
maps between sets with alg. structure	linear maps	homomorphisms	G-equivariant maps.

<u>Definition</u> $G \curvearrowright X$. We say the action is <u>transitive</u>

if $\forall x, y \in X$, $\exists g \in G$, s.t. $y = gx$

In this case, for every $x \in X$, denote $H := \mathrm{Stab}_G(x)$.

Then $\varphi : G/H \xrightarrow{\sim} X$ is a G-equivariant bijection

$$gH \longmapsto gx$$

(Indeed, φ is well-defined: if $g_1 H = g_2 H \Rightarrow g_1 = g_2 h \Rightarrow g_1 x = g_2 h x = g_2 x$.

φ is surjective: b/c G-action is transitive.

φ is injective: if $\varphi(g_1 H) = \varphi(g_2 H)$

$$\Rightarrow g_1 x = g_2 x \Rightarrow g_2^{-1} g_1 x = x \Rightarrow g_2^{-1} g_1 \in \mathrm{Stab}_G(x) = H$$

$$\Rightarrow g_1 H = g_2 H$$

φ is G-equivariant b/c $g' \cdot \varphi(gH) = g'gx = \varphi(g'gH)$ $\square$

- In the general case, $G = \coprod_{\text{orbits } \mathcal{O}} \mathcal{O}$

  $\forall x \in X$, G acts transitively on $\mathrm{Orb}_G(x)$

  $$\Rightarrow \mathrm{Orb}_G(x) \simeq G/\mathrm{Stab}_G(x)$$

  & $$X \simeq \coprod_{\substack{G\text{-orbits} \\ G\cdot x}} G/\mathrm{Stab}_G(x)$$

<u>Theorem</u> Let G be a finite group (acting on itself by conjugation)

(1) For each $g \in G$, the number of elements in its conjugacy class is

$$\#\left(\mathrm{Ad}_G(g)\right) = \#G/\#C_G(g) = [G : C_G(g)]$$

b/c $\mathrm{Ad}_G(g) \simeq G/C_G(g)$

(2) <u>Class equation</u>. If $g_1, \dots, g_r$ are representatives of conjugacy classes of G,

then $$\#G = \sum_{i=1}^{r} [G : C_G(g_i)]$$

Proof: Consider the conjugation action of G on itself

$$\Rightarrow \#G = \sum \#\text{Orbits } \mathrm{Ad}_G(g_i) = \sum_{i=1}^{r} [G : C_G(g_i)]$$

(3) (A more useful version) In the above formula,

$$\text{Orbit } \mathrm{Ad}_G(g_i) \text{ is a singleton} \Leftrightarrow \forall h \in G,\ \underbrace{\mathrm{Ad}_h(g_i) = g_i}_{\Updownarrow \atop hg_i = g_ih} \Leftrightarrow g_i \in Z(G)$$

So $$\#G = \#Z(G) + \sum_{\substack{\text{nontriv} \\ \text{orbits}}} [G : C_G(g_i)]$$

<u>Example</u>: $G = S_5$, the class equation is

$$120 = 5! = 1 + \frac{\#S_5}{\#S_2 \cdot \#S_3} + \frac{\#S_5}{\#Z_3 \cdot \#S_2} + \frac{\#S_5}{\#(Z_2^2 \rtimes Z_2)} + \frac{\#S_5}{\#Z_4} + \frac{\#S_5}{\#(Z_2 \times Z_3)} + \frac{\#S_5}{\#Z_5}$$

$$= 1 + 10 + 20 + 15 + 30 + 20 + 24$$

Partition type	1+1+1+1+1	1+1+1+2	1+1+3	1+2+2	1+4	2+3	5
Typical element	1	(12)	(123)	(12)(34)	(1234)	(12)(345)	(12345)
Stabilizer		$S_2 \times S_3$	$Z_3 \times S_2$	$Z_2^2 \rtimes Z_2$	Z_4	$Z_2 \times Z_3$	Z_5

<u>Application</u>. Let p be a prime number.

A finite group G is called a <u>p-group</u> if $\#G$ is a power of p.

<u>Theorem</u>. For a nontrivial p-group G, $Z(G)$ is non-trivial.

Proof: Use class equation:
$$\underbrace{\#G}_{p\text{-power}} = \#Z(G) + \sum_{\text{nontriv. conj class}} \underbrace{[G : C_G(g_i)]}_{\text{nontrivial } p\text{-power}}$$

$$\Rightarrow p \mid \#Z(G) \qquad \square$$

<u>Automorphism group revisit</u>:

Let G be a group. $\mathrm{Aut}(G) = \{\phi : G \xrightarrow{\simeq} G \text{ isomorphism}\}$

Recall that conjugation gives a homomorphism
$$\mathrm{Ad}: G \longrightarrow \mathrm{Aut}(G)$$
$$g \longmapsto (\mathrm{Ad}_g : h \longmapsto ghg^{-1})$$

Have seen: $\ker(\mathrm{Ad}) = Z(G)$

$\mathrm{Im}(G) =: \mathrm{Inn}(G)$ is called the group of <u>inner automorphisms</u> (内自同构群)

<u>Proposition</u> $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$

Proof: Need to show: if $\sigma : G \xrightarrow{\sim} G$ is an automorphism

then $\sigma \, \mathrm{Inn}(G) \, \sigma^{-1} = \mathrm{Inn}(G)$

(suffices to show $\subseteq$, and then $\supseteq$ follows from "$\subseteq$ for σ^{-1}")

Take $g \in G \rightsquigarrow \mathrm{Ad}_g \in \mathrm{Inn}(G)$

Claim: $\sigma \circ \mathrm{Ad}_g \circ \sigma^{-1} : G \to G$ as an automorphism of G is equal to $\mathrm{Ad}_{\sigma(g)}$,

so belongs to $\mathrm{Inn}(G)$

Indeed, $\sigma \circ \mathrm{Ad}_g \circ \sigma^{-1}(h) = \sigma\left(\mathrm{Ad}_g(\sigma^{-1}(h))\right) = \sigma(g\sigma^{-1}(h)g^{-1})$

$= \sigma(g)\,\sigma(\sigma^{-1}(h))\,\sigma(g^{-1}) = \sigma(g)\,h\,\sigma(g)^{-1} = \mathrm{Ad}_{\sigma(g)}(h)$. $\square$

Definition The quotient $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the group of outer automorphisms of G (外自同构群)

Interesting examples:

① $G = GL_n(\mathbb{Q})$, $\mathrm{Ad} : GL_n(\mathbb{Q}) \to \mathrm{Aut}(G)$

$\ker(\mathrm{Ad}) = Z(GL_n(\mathbb{Q})) = \{a \cdot I_n \mid a \in \mathbb{Q}^\times\} \simeq \mathbb{Q}^\times$

Thus, $\mathrm{Inn}(G) = GL_n(\mathbb{Q})/\mathbb{Q}^\times =: PGL_n(\mathbb{Q})$

projective general linear group (射影线性群)

Automorphisms that are not inner?

- $\psi : A \longmapsto {}^tA^{-1}$ note: $\psi(AB) = \psi(A)\psi(B)$

$\rightsquigarrow PGL_n(\mathbb{Q}) \rtimes \{1, \psi\} \curvearrowright GL_n(\mathbb{Q})$ as automorphisms

Fact: $\mathrm{Aut}(SL_n(\mathbb{Q})) \cong \mathrm{Aut}(PGL_n(\mathbb{Q})) \simeq \begin{cases} PGL_n(\mathbb{Q}) \rtimes \{1,\psi\} & \text{when } n \geq 3 \\ PGL_2(\mathbb{Q}) & \text{when } n = 2. \end{cases}$

But for $GL_n(\mathbb{Q})$, there are automorphisms coming from $\mathbb{Q}$

In general, if K is a field, $\mathrm{char}(K) \neq 2$, $n \geq 3$,

$$Aut(SL_n(K)) \cong Aut(PGL_n(K)) \simeq (PGL_n(K) \rtimes Aut_{fld}(K)) \rtimes \{1, \psi\}$$

② $G = S_n$ $\quad S_n \xrightarrow{Ad} Aut(S_n)$ is injective

Interesting fact: If $n \neq 6$, $Ad : S_n \xrightarrow{\simeq} Aut(S_n)$ is an isomorphism

i.e. all automorphisms of S_n are "inner"

But when $n = 6$, $\exists \psi : S_6 \xrightarrow{\sim} S_6$ that is not inner

$$(12) \longmapsto (12)(34)(56)$$

Fact: $Aut(S_6) \cong S_6 \rtimes \{1, \psi\}$.

Definition. Let G be a group. We say a subgroup H is characteristic, denoted as H char G if for any automorphism σ of G, $\sigma(H) = H$

Properties and examples:

(1) If $H \leq G$ is the unique subgroup of that order $\Rightarrow H$ is characteristic

e.g. in $\mathbb{Z}_n$, $\forall d | n$, $\langle d \rangle \subseteq \mathbb{Z}_n$ is characteristic

(2) Characteristic subgroups are normal.

If H char G and $g \in G$, $Ad_g : G \to G$ is an automorphism of G

$$\Rightarrow Ad_g(H) = H \quad \Rightarrow H \trianglelefteq G$$

(3) If K char H and $H \trianglelefteq G$, then $K \trianglelefteq G$

and K char H and H char $G \Rightarrow K$ char G (characteristic subgroups is transitive)

Prove this: $\forall g \in G$, as $H \trianglelefteq G \Rightarrow gHg^{-1} = H$

So $Ad_g : H \to H$ is an automorphism $\Rightarrow Ad_g(K) = K$..