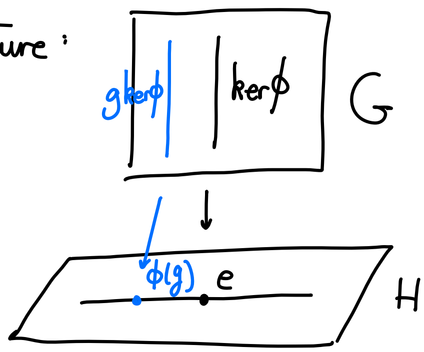
Lecture 3 Isomorphism theorems, composition series, Hölder program

The First Isomorphism Theorem

proved last time

If $\phi: G \to H$ is a homomorphism of groups, then $\ker\phi \trianglelefteq G$ and $G/\ker\phi \cong \phi(G)$

Picture:



(Imagine that we are working with linear spaces)

Proof: Define a map $\psi: G/\ker\phi \longrightarrow \phi(G)$

$$g\ker\phi \longmapsto \phi(g)$$

(1) ψ is well-defined:

If $g_1\ker\phi = g_2\ker\phi \Rightarrow g_2^{-1}g_1 \in \ker\phi$

then $\phi(g_1) = \phi(g_2\cdot g_2^{-1}g_1) = \phi(g_2)\cdot\underbrace{\phi(g_2^{-1}g_1)}_{= e_H} = \phi(g_2)$

(2) ψ is surjective. b/c every element of $\phi(G)$ takes the form of $f(g)$ for some $g \in G$

it is the image of $g\ker\phi$.

(3) ψ is a homomorphism $\psi(g_1\ker\phi \cdot g_2\ker\phi) = \psi(g_1g_2\ker\phi) = \phi(g_1g_2)$

$\|?$ $\qquad\qquad\qquad\qquad \|$

$\psi(g_1\ker\phi)\psi(g_2\ker\phi) =\!=\!=\!= \phi(g_1)\phi(g_2)$

(4) ψ is injective. Enough to check $\ker\psi = \{\ker\phi\}$

This is b/c if $\underbrace{\psi(g\ker\phi)}_{\phi(g)} = e_H \Rightarrow g \in \ker\phi \Rightarrow g\ker\phi = \ker\phi$. $\square$

The Second Isomorphism Theorem

(Slightly weaker than the version from the book)

Let G be a group, and let $A \leq G$, $B \trianglelefteq G$ be subgroups — only need A normalizes B

Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and (i.e. $\forall a \in A$, $aBa^{-1} = B$)

$$AB/B \cong A/(A \cap B)$$

<u>Rmk</u>: This is analogous to the statement in linear alg $W_1, W_2 \subset V$

$$\Longrightarrow W_1 + W_2/W_2 \cong W_1/(W_1 \cap W_2)$$

<u>Picture</u>:

$$\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & AB \arrow[dl, dash] \arrow[dr, dash, "AB/B"] & \\
A \arrow[dr, dash, "A/A\cap B"'] & & B \arrow[dl, dash] \\
 & A \cap B \arrow[d, dash] & \\
 & \{1\} &
\end{tikzcd}$$

<u>Proof</u>: Have proved that $AB \leq G$

First show $B \trianglelefteq AB$: given $a \in A$, $b \in B$, $abB(ab)^{-1} = abBb^{-1}a^{-1} = aBa^{-1} = B$

So the quotient group AB/B makes sense.

Define a homomorphism $\phi : A \longrightarrow AB \twoheadrightarrow AB/B$

$$a \longmapsto a \longmapsto aB$$

* ϕ is clearly surjective, b/c any $abB = aB = \phi(a)$.
* $\ker\phi = \{a \in A, \underset{\substack{\Updownarrow \\ a \in B}}{aB = B}\} = A \cap B$ (In particular, it's normal in A)

By the first isomorphism theorem, $A/A \cap B \xrightarrow{\simeq} AB/B$.

<u>Remark</u> A common way to prove $G/H \simeq G'/H'$ is to

* first construct a homomorphism $G \to G'$ (and then compose to $G \to G'/H'$)
* then show surjectivity + compute kernel. (Finally, apply First Isomorphism Theorem)

The Third Isomorphism Theorem

$$\begin{array}{c} G \\ | \\ K \\ | \\ H \end{array} \qquad G/K \;\cong\; (G/H)/(K/H)$$

Let G be a group and H and K be normal subgroups of G with $H \leq K$.

Then $K/H \trianglelefteq G/H$, and $(G/H)/(K/H) \cong G/K$

(If we denote the quotient by H by a bar, this says $\bar{G}/\bar{K} \cong G/K$.)

Proof: Consider $\phi : G/H \longrightarrow G/K$

$$gH \longmapsto gK$$

* ϕ is well-defined : if $g_1H = g_2H$, then $g_1K = g_1HK = g_2HK = g_2K$
* ϕ is a homomorphism : $\phi(g_1H \cdot g_2H) = \phi(g_1g_2H) = g_1g_2K$

$$\overset{?}{\|} \qquad\qquad\qquad \|$$

$$\phi(g_1H)\cdot\phi(g_2H) = g_1K \cdot g_2K = g_1g_2K$$

* ϕ is surjective : clear
* $\ker\phi = \{gH \text{ s.t. } \underbrace{gK = K}_{\Updownarrow \atop g \in K}\} = \{gH \,;\, g \in K\} = K/H.$

In particular, $K/H \trianglelefteq G/H$.

The first isomorphism theorem $\Rightarrow (G/H)/(K/H) \cong G/K$.

The Fourth Isomorphism Theorem (Lattice Isomorphism Theorem)

Let G be a group & $N \trianglelefteq G$. Then there's a bijection

$$\{\text{subgroups of } G \text{ containing } N\} \longleftrightarrow \{\text{subgroups of } G/N\}$$

$$A \longmapsto A/N$$

$$\pi^{-1}(\bar{A}) \longleftarrow\!\shortmid \bar{A}$$

where $\pi : G \to G/N$ is the natural projection, that preserves

* inclusion
* index of subgroups
* intersections
* normality, ...

Visually, the lattice of subgroups of G containing N $\leftrightarrow$ the lattice of subgroups of G/N.

* A useful point of view on homomorphisms from a quotient group

Let $\phi : G \to H$ be a group homomorphism

& let $N \trianglelefteq G$ be a normal subgroup

Hope to define $\Phi : G/N \longrightarrow H$ by
$$gN \longmapsto \phi(g)$$

Such Φ is well-defined if and only if $N \leq \ker\phi$.

($\underline{\text{note}}$: if $g_1N = g_2N$, then $g_1 = g_2 n$ for some $n \in N$

$$\rightsquigarrow \phi(g_1) = \phi(g_2 n) = \phi(g_2)\phi(n) \overset{?}{=} \phi(g_2)$$

if and only if $\phi(n) = e_H \Leftrightarrow n \in \ker\phi$)

When $N \leq \ker\phi$, we say $\phi : G \to H$ <u>factors through</u> G/N

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, dashed, "\Phi"'] & \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, mapsto] \arrow[d, mapsto] & \phi(g) \\ gN \arrow[ur, dashed, mapsto] & \end{tikzcd}$$

We say that Φ <u>makes the diagram commute</u>

<u>Remark</u>: The philosophical meaning of this construction is :

when we consider $\phi : G \to H$, we may first "group together" the information differed by N, and then map to H.

<u>Example</u>: All homomorphisms $\phi : \mathbb{Z} \to \mathbb{C}^\times$ is determined by $\lambda_\phi := \phi(1) \in \mathbb{C}^\times$

How to get a homomorphism $Z_n = \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{C}^\times$?

Need $n\mathbb{Z} \subseteq \ker \phi$

$\Leftrightarrow \phi(n\mathbb{Z}) = 1 \Leftrightarrow \phi(n) = 1 \Leftrightarrow \lambda_\phi$ is an n^{th} root of unity

<u>Ultimate goal of group theorists</u>: Classify all finite groups

Observation: If $N \lhd G \rightsquigarrow$ "$G = N + G/N$"

<u>Definition</u> A (finite or infinite) group G is called <u>simple</u> (單群) if

$\# G > 1$ and the only normal subgroups of G are $\{e\}$ and G

<u>Examples</u>. Z_p for p a prime number (these are all abelian simple groups)

A_n for $n \geq 5$ (to be introduced later)

<u>Note</u>: There are infinite simple groups; not so easy to define.

<u>Hölder's program</u>: (1) Classify all finite simple groups

(2) Find all ways of "putting simple groups together" to form larger groups

<u>BIG THEOREM</u> (單群分類定理)

Every finite simple group is isomorphic to one in

* 18 (infinite) families of simple groups, or

* 26 sporadic simple groups

E.g. Z_p, A_n $(n \geq 5)$, or $SL_n(\mathbb{F})/Z(SL_n(\mathbb{F}))$ when $n \geq 2$ and $\mathbb{F}$ a finite group

(with exception of $SL_2(\mathbb{F}_2)$, $SL_2(\mathbb{F}_3)$, ...)

Feit-Thompson Theorem If G is a simple group of odd order, then $G \simeq \mathbb{Z}_p$

(235 pages of hard math)

Definition In a group G, a sequence of subgroups

$$\{e\} = N_0 < N_1 < \cdots < N_k = G$$

is called a **composition series** (合成序列) if $N_{i-1} \lhd N_i$ and N_i/N_{i-1} is a simple group for $1 \le i \le k$.

In this case, N_i/N_{i-1} are called **composition factors** or **Jordan-Hölder factors** of G

E.g. $1 \lhd \langle s \rangle \lhd \langle s, r^2 \rangle \lhd D_8$

$1 \lhd \langle r^2 \rangle \lhd \langle r \rangle \lhd D_8$

Theorem (Jordan-Hölder) Let G be a finite group. $G \neq \{e\}$

Then (1) G has a composition series.

(b/c if G is simple, then take $\{e\} \lhd G$ and done!

if G has a nontrivial normal subgroup N,

we may reduce this theorem to N and G/N as follows

$\{e\} = A_0 \lhd A_1 \lhd \cdots \lhd A_r = N$, $\{e\} = B_0 \lhd B_1 \lhd \cdots \lhd B_s = G/N$; $\pi : G \twoheadrightarrow G/N$

$\rightsquigarrow \{e\} = A_0 \lhd A_1 \lhd \cdots \lhd A_r = N = \pi^{-1}(B_0) \underbrace{\lhd}_{\text{quotient isom to } B_1/B_0,\ \ldots} \pi^{-1}(B_1) \lhd \cdots \lhd \pi^{-1}(B_s) = G$)

(2) The composition factors are unique, i.e. if we have two composition series

$$\{e\} = N_0 \lhd N_1 \lhd \cdots \lhd N_r = G \text{ and } \{e\} = M_0 \lhd M_1 \lhd \cdots \lhd M_s = G$$

Then $r = s$, and there's a bijection $\sigma : \{1, \ldots, r\} \xrightarrow{\sim} \{1, \ldots, s = r\}$ s.t.

$$N_i/N_{i-1} \simeq M_{\sigma(i)}/M_{\sigma(i)-1}$$

(We prove (2) in next lecture.)

Definition. A group G is solvable (可解群) if there is a chain of subgroups

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_s = G$$

s.t. G_i / G_{i-1} is abelian for $i = 1, \cdots, s$

Corollary. For G a finite group,

G is solvable $\iff$ all of its composition factors are of the form $\mathbb{Z}_p$ with p prime

Example. $B = G = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in GL_3(\mathbb{C}) \right\}$ upper triangular invertible matrices

$$\begin{array}{rl}
(\mathbb{C}^\times)^3 & \quad \triangledown\!\!/ \\
 & N = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \\
\mathbb{C}^2 & \quad \triangledown\!\!/ \\
 & N' = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = \mathbb{C}
\end{array}$$