2021 秋: 代数学一 (实验班) 期末考试

时间：120 分钟　　满分：110 分，最高得分不超过 100 分

所有的环都有乘法单位元, 且与其加法单位元不相等; 所有环同态把 1 映到 1.

All rings contains 1_R and $1_R \neq 0_R$; all ring homomorphism takes 1 to 1.

判断题 请在答卷纸上整齐编号书写 T 或 F (10 分)

1. 群 G 忠实作用在集合 X 上. 若 $g_1, g_2 \in G$ 和 $x \in X$ 满足 $g_1 \cdot x = g_2 \cdot x$，则 $g_1 = g_2$.

A group G acts faithfully on a set X. If $g_1, g_2 \in G$ and $x \in X$, then $g_1 \cdot x = g_2 \cdot x$ implies $g_1 = g_2$.

2. 设 H 是群 G 的正规子群. 假设 H 中元素的最大阶 $\leq m$, G/H 中元素的最大阶 $\leq n$. 则 G 中元素的最大阶 $\leq mn$.

Let H be a normal subgroup of a group G. Suppose that the maximal order of elements in H is $\leq m$, and the maximal order of elements in G/H is $\leq n$. Then the maximal order of elements in G is $\leq mn$.

3. 设 $\varphi : R \to R'$ 为一交换环之间的满同态. 若 R 为整环, 则 $\varphi(R) = R'$ 也为整环.

Let $\varphi : R \to R'$ be a surjective homomorphism of commutative rings. If R is an integral domain, then $\varphi(R) = R'$ is an integral domain.

4. 给定环 R 和 R-左模 M, N. 则 $\mathrm{Hom}_R(M, N)$ 也有自然的 R-左模结构.

Let R be a ring and M and N be left R-modules. Then $\mathrm{Hom}_R(M, N)$ is a left R-module.

5. 在一个唯一分解整环中, 每个非零元都可以唯一的写成素元的乘积.

In a UFD, every nonzero element can be uniquely written as products of prime elements.

6. 设域 L 是域 F 的扩张, K_1 和 K_2 为中间域. 若 K_1/F 是正规扩张, 则 K_1K_2/K_2 也是正规扩张.

Let L be a field extension of a field F with intermediate fields K_1 and K_2. Suppose that K_1/F is normal, then K_1K_2 is normal over K_2.

7. 任一指数为 n 的循环扩张 K/F 一定形如 $K = F(\sqrt[n]{a})$ $(a \in F)$.

Every cyclic extension K over F of degree n is of the form $K = F(\sqrt[n]{a})$ for some $a \in F$.

8. 设域 L 是域 F 的扩张, K_1 和 K_2 为中间域. 若 K_1 和 K_2 为 F 上的 Galois 扩张, 则 $[K_1K_2 : F] = [K_1 : F] \cdot [K_2 : F]$.

Let L be a field extension of a field F with intermediate fields K_1 and K_2. Suppose that K_1 and K_2 are Galois over F. Then $[K_1K_2 : F] = [K_1 : F] \cdot [K_2 : F]$.

9. 对任一有限域 F 和正整数 n, (在同构意义下) F 恰有一个次数为 n 的循环扩张.

For any finite field F and any positive integer n, there exists a unique (up to isomorphism) cyclic extension of F of degree n.

10. 若 $f(x) \in F[x]$ 是一个不可约多项式且在一个 F 的扩域中存在单根 α, 则 $f(x)$ 在 F 上的正规闭包在 F 上是 Galois 的.

If $f(x) \in F[x]$ is an irreducible polynomial and there exists a simple zero α of $f(x)$ in some field extension of F, then the normal closure of $f(x)$ over F is Galois over F.

解答题一 (12 分) 令 R 是一个交换环. 若所有 R 自由模的子模都是自由的, 则 R 是一个主理想整环.

Let R be a commutative ring. If all submodules of finitely generated free modules over R are free over R, then R is a PID.

解答题二 (12 分) 是否存在一个有限群 G 使得 $G/Z(G)$ 恰有 143 个元素? (这里 $Z(G)$ 是 G 的中心.)

Is there a finite group G such that $G/Z(G)$ has 143 elements? ($Z(G)$ is the center of G.)

解答题三 (13 分) 设 k 为一有 q 个元素的有限域.

(1) $k[x]$ 中有多少个首一的不可约多项式次数为 $d = 2, 3, 4, 5, 6$?

(2) 一个 5 次 (不一定不可约) k 上多项式分裂域的 Galois 群可能是什么? 为什么?

Let k be a finite field with q elements.

(1) How many monic irreducible polynomials are there in $k[x]$ of each degree $d = 2, 3, 4, 5, 6$?

(2) What are the possible Galois groups of the splitting field of a (not necessarily irreducible) polynomial of degree 5 over k? Why?

解答题四 (13 分) 记 $\alpha = \sqrt{i+2}$ ($i = \sqrt{-1}$).

(1) 计算 α 在 $\mathbb{Q}$ 上的极小多项式 $f(x)$. (需间接或者直接的证明 $f(x)$ 的不可约性.)

(2) 记 F 为 $f(x)$ 在 $\mathbb{Q}$ 上的分裂域. 确定 F 在 $\mathbb{Q}$ 上的 Galois 群.

(3) 写出所有 $\mathrm{Gal}(F/\mathbb{Q})$ 子群和所有 $F/\mathbb{Q}$ 中间域的一一对应图. (此问无需解释过程, 但子群需要用元素或者生成元标注.)

Let $\alpha = \sqrt{i+2}$ where $i = \sqrt{-1}$.

(1) Compute the minimal polynomial $f(x)$ of α over $\mathbb{Q}$. (Need to show the irreducibility of $f(x)$, directly or indirectly.)

(2) Let F be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine the Galois group of F over $\mathbb{Q}$.

(3) Draw the corresponding diagram representing the field extensions of $\mathbb{Q}$ and subgroups of $\mathrm{Gal}(F/\mathbb{Q})$. (No reasoning is needed for (3), but express the subgroups by elements or generators.)

解答题五 (10 分) 多项式 $f(x)=\prod_{i=1}^{n}(x-r_i)$ 的判别式为 $\prod_{i<j}(r_i-r_j)^2$. 设 $f(x)\in\mathbb{Q}[x]$ 为一 4 次的首一不可约多项式, 其根为 $\alpha,\beta,\gamma,\delta$.

(1) 证明 $\alpha\beta+\gamma\delta$, $\alpha\gamma+\beta\delta$, $\alpha\delta+\beta\gamma$ 是一个首一三次多项式 $g(x)\in\mathbb{Q}[x]$ 的根, 且它的判别式和 $f(x)$ 的判别式相同.

(2) 简短说明为什么 f 在 $\mathbb{Q}$ 上的 Galois 群是五个群 S_4, A_4, Z_4, D_8, $Z_2\times Z_2$ 之一. (提示: 解答涉及到 S_4 的子群分类, 你需要说明是用什么条件选出的这五个群, 无需证明满足所列条件的群恰好这五个群; 但要说明 $Z_2\times Z_2$ 是具体 S_4 的哪个子群.)

(3) 在哪种上述情况下, 多项式 $g(x)$ 是不可约的?

The *discriminant* of a polynomial $f(x)=\prod_{i=1}^{n}(x-r_i)$ is $\prod_{i<j}(r_i-r_j)^2$. Let $f(x)\in\mathbb{Q}[x]$ be a monic irreducible polynomial of degree 4 with roots $\alpha,\beta,\gamma,\delta$.

(1) Prove that $\alpha\beta+\gamma\delta$, $\alpha\gamma+\beta\delta$, and $\alpha\delta+\beta\gamma$ are roots of a monic cubic polynomial $g(x)\in\mathbb{Q}[x]$ whose discriminant is the same as the discriminant of f.

(2) Give a short explanation of why the Galois group of f over $\mathbb{Q}$ is one of the five groups S_4, A_4, Z_4, D_8, or $Z_2\times Z_2$. (Hint: The solution would involve the classification of subgroups of S_4; you need only to specify the condition that allows you to pin down these groups, but do not need to verify that the subgroups exactly satisfying your conditions are these five groups. However, do explain how $Z_2\times Z_2$ is realized as a subgroup of S_4.)

(3) In which of the above case, is the polynomial g irreducible?

解答题六 (10 分) 令 p 为一素数. 假设域 F 的每一个有限扩张的次数都被 p 整除. 证明所有 F 的有限扩张的次数都是 p 的幂次.

Let p be a prime integer. Suppose that the degree of every finite extension of a field F is divisible by p. Prove that the degree of every finite extension of F is a power of p.

解答题七 (10 分) 设 R 为一唯一分解整环, 其中所有非零素理想皆为极大理想. 证明 R 是一个主理想整环.

Suppose that R is a unique factorization domain (UFD) for which every nonzero prime ideal is maximal. Show that R is a principal ideal domain (PID).

解答题八 (10 分) 设群 G 由两个元素生成.

(1) 证明 G 至多有 17 个指数为 3 的子群. (提示: 考虑从 G 到 S_3 的同态.)

(2) 证明 G 至多有 13 个指数为 3 的子群.

注: 可以直接证明 (2). 事实上, 存在这样的群 G 恰有 13 个指数为 3 的子群, 但无需证明此结论.

Let G be a group which is generated by two elements.

(1) Prove that G has at most 17 subgroups of index 3. (Hint: think about homomorphisms from G to S_3.)

(2) Prove that G has at most 13 subgroups of index 3.

Remark: Clearly, you can choose to prove (2) directly. In fact, there exists a such group G with exactly 13 subgroups of index 3; but you do not need to prove that.

解答题九 (5 分) 令 k 为一特征为 $p > 0$ 的完美域. 设 $F = k(t)$ 为 k 上单变元的函数域. 证明 F 的任一有限扩张 E 都是单扩张, 即存在 $\alpha \in E$ 使得 $E = F(\alpha)$.

Let k be a perfect field of characteristic $p > 0$. Let $F = k(t)$ be the field of rational functions in one variable over k. Show that every finite extension E of F can be generated by one element, that is, there exists $\alpha \in E$ such that $E = F(\alpha)$.

解答题十 (5 分) 设域 F 满足 $\mathbb{Q} \subseteq F \subset \mathbb{C}$, 是一个 $\mathbb{Q}$ 上的有限交换 Galois 扩张. 设 $\alpha \in F$ 的极小多项式为 $f(x) \in \mathbb{Q}[x]$, 且满足 $|\alpha| = 1$.

(1) 证明 F 在复共轭下保持稳定.

(2) 证明 $f(x)$ 的任一复根 β 满足 $|\beta| = 1$.

(3) 记 $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, 证明对所有 $0 \le i < n$, $|a_i| \le 2^n$.

(4) 证明 F 只包含有限多个绝对值为 1 的代数整数 (即满足极小多项式的系数为整数的 F 中的元素).

(5) 证明 (4) 中的这些代数整数都是单位根.

Let F be a field with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$, where $F/\mathbb{Q}$ is a finite *abelian* Galois extension. Let $\alpha \in F$ and let $f(x) \in \mathbb{Q}[x]$ be its minimal monic polynomial. Assume that $|\alpha| = 1$.

(1) Show that F is closed under complex conjugation.

(2) Prove that $|\beta| = 1$ for every complex root β of $f(x)$.

(3) Writing $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, show that $|a_i| \le 2^n$ for all i with $0 \le i < n$.

(4) Prove that F contains only finitely many algebraic integers (i.e. elements in F whose minimal polynomial over $\mathbb{Q}$ have coefficients in $\mathbb{Z}$) having absolute value 1.

(5) Deduce that each of the algebraic integers in (4) is a root of unity.