What is a Coarse Index, physically[1]?

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[1]Collaborators: M. Ludewig (Regensburg), Y. Kubota (Shinshu)

Atiyah–Singer:

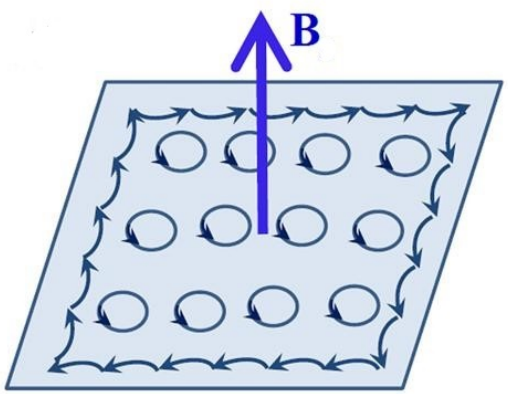
$$\#\text{Dirac zero modes} = \text{Topological } \#X, A.$$

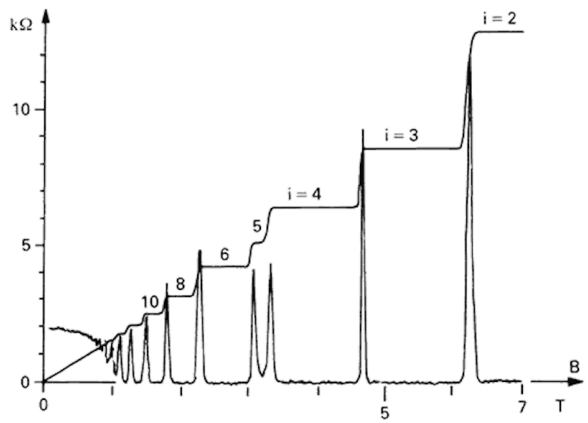
$X =$ compact (spin) manifold
$A =$ gauge field

Zero modes cause difficulties in quantization$\rightsquigarrow$ anomalies

- Many Fields Medals, Abel Prizes...
- Applications in HEP-th, gauge theory, strings,...
- Nowadays, Topological phases $\rightsquigarrow$ (Coarse?) Index Theory in the lab!

30s: Landau solved spectral problem for electron on Euclidean plane subject to uniform magnetic field ($H_{\text{Landau}} = d_A^* d_A$).
80s: Quantum Hall Effect: "Topology"?! (Nobel prizes!)
90s: Bellissard, Connes, Kellendonk, Prodan, Schulz-Baldes,... put NCG of "momentum space" to great use!



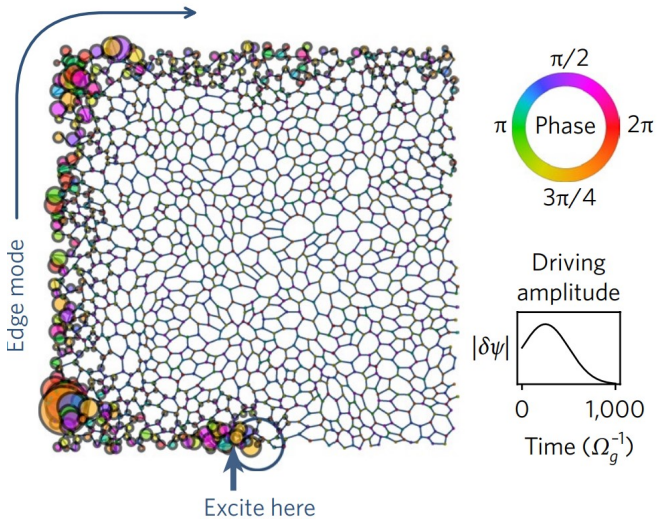


90s: Coarse index theory developed disjointly from physics[2]...

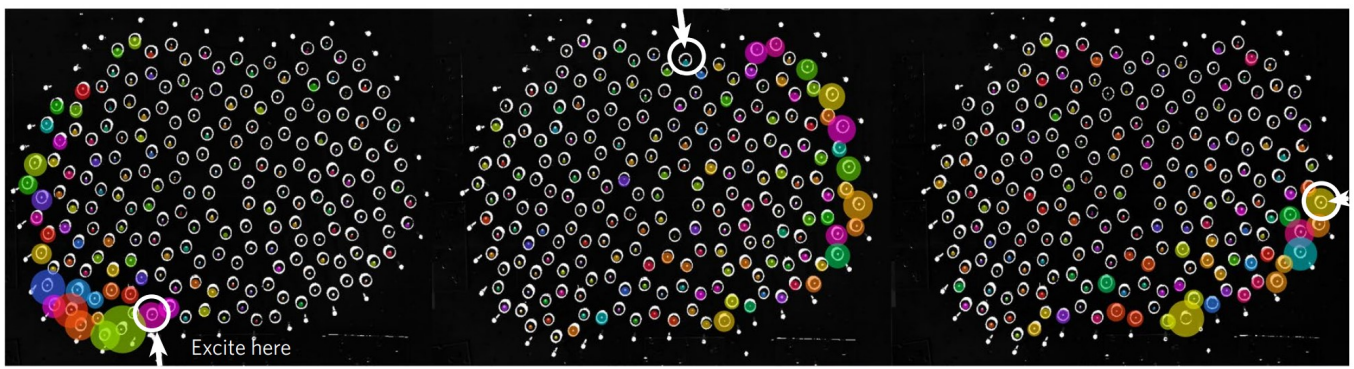
[2] Kubota '16, Ewert–Meyer '19: coarse geometry in lattice models.

Homogeneous (old) vs Amorphous (new)

Applications/experiments go beyond Euclidean geometry.
"Large-scale geometry" matters, "small-scale topology" doesn't!



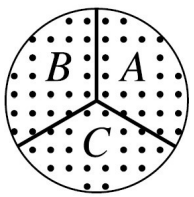
N. Mitchell et al, Nature Phys. (2018)



Coarse geometry/homology was anticipated (A. Kitaev)

Anyons in an exactly solved model and beyond

Alexei Kitaev *

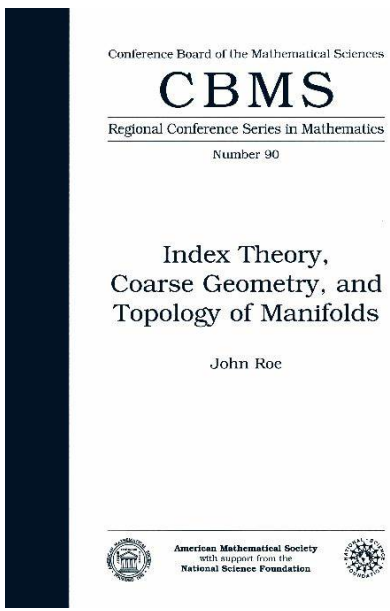


$$\nu(P) = h(A,B,C) \stackrel{\text{def}}{=} \sum_{j\in A}\sum_{k\in B}\sum_{l\in C} h_{jkl}.$$

In general, a *quasidiagonal matrix* is a lattice-indexed matrix $A = (A_{jk})$ with sufficiently rapidly decaying off-diagonal elements. Technically, one requires that

$$|A_{jk}| \leqslant c|j-k|^{-\alpha}, \quad \alpha > d,$$

where c and α are some constants, and d is the dimension of the space. Note that "lattice" is simply a way to impose coarse $\mathbb{R}^d$ geometry at large distances. We may think about the problem in these terms: matrices are operators acting in some Hilbert space, and lattice points are basis vectors. But the choice of the basis need not be fixed. One may safely replace the basis vector corresponding to a given lattice point by a linear combination of nearby points. One may also use some kind of coarse-graining, replacing the basis by a decomposition into orthogonal subspaces corresponding to groups of points, or regions in $\mathbb{R}^d$.



Periodic table for topological insulators and superconductors

Alexei Kitaev

Theorem: *Any gapped local free-fermion Hamiltonian in $\mathbb{R}^d$ is equivalent to a texture.*

(That is the key technical result, but I cannot explain it in any detail in such a short note.) Discrete systems on a compact metric space L are classified by the K-homology group $K_q^{\mathbb{R}}(L)$.

30. N. Higson, and J. Roe, *Analytic K-homology*, Oxford University Press, New York, 2000.
31. A. Connes, *Noncommutative geometry*, Academic Press, San Diego, 1994.

Roe algebra

For a Riemannian manifold X, one considers finite propagation and locally compact operators on $L^2(X)$, completed to Roe C^*-algebra.

Amazingly, $C^*_{\mathrm{Roe}}(X)$ can have non-trivial K-theory,
"Assembled geometrically from K-homology data".

"Dirac operator has finite propagation speed,..., thus Roe-coarse index,..., obstructs psc metrics,..."

As a beginner, some basic questions puzzled me:

- What is a concrete generator of

$$K_1(C^*_{\mathrm{Roe}}|\mathbb{R}|) \cong \mathbb{Z} \cong K_0(C^*_{\mathrm{Roe}}|\mathbb{R}^2|)?$$

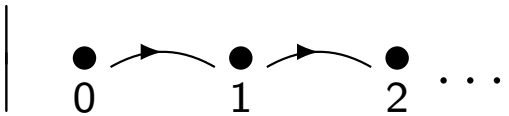
- Index theory meaning of K-theory of $C^*_{\mathrm{Roe}}(X_{\mathrm{discrete}})$?

I. One-dimensional story

The very first index theorem

Basic example of Fredholm operator is unilateral shift on $\ell^2(\mathbb{N})$,

$$\check{S} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$



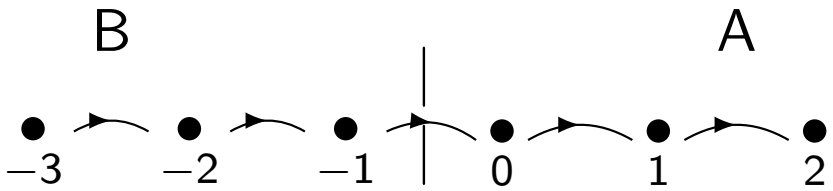
Truncation of bilateral shift S on $\ell^2(\mathbb{Z})$, whose symbol function is

$$\{\sigma_S : z \mapsto z\} \in C(\mathbb{T}).$$

Theorem [F. Noether, 1921]:

$$\mathrm{Index}(\check{S}) := \underbrace{\dim\ker(\check{S})}_{0} - \underbrace{\dim\ker(\check{S}^*)}_{1} = -\mathrm{Wind}(\sigma_S) = -1.$$

Quantized flow of unitaries



$$[A, S] \text{ is finite-rank,}$$
$$\mathrm{Tr}(S^*[A, S]) = \mathrm{Tr}(S^*AS - A).$$

RHS unchanged by truncation to finite-subspace near partition $\rightsquigarrow$ difference of traces of two finite projections, thus an integer.

Idea generalizes to finite-propagation, locally trace-class $U - 1 \ldots$

Lemma 4.1: *Suppose that E and F are projections in $\mathfrak{L}^\infty$, and that $E - F$ is a trace class operator. Then trace $(E - F) \in \mathbb{Z}$.*

E. Effros, Math. Intell. '89, Avron–Seiler–Simon, JFA, '94,
A. Kitaev, Ann. Phys. '06.

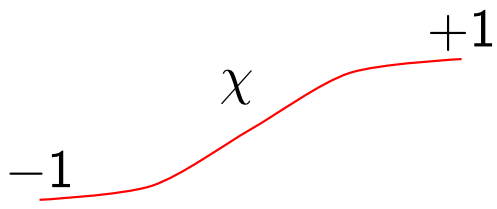
Roe's odd index

On $L^2(\mathbb{R})$, translations are generated by momentum

$$S_t = e^{-itD}, \quad D = -i\frac{d}{dx} = \text{1D Dirac}.$$

Regularize D by applying χ.

- D has full spectrum in $\mathbb{R} \Leftrightarrow$ $\exp(\pi i \chi(D))$ has full spectrum in $\mathrm{U}(1)$.
- Odd coarse index [Roe '89] is



$$[\exp(\pi i \chi(D))] \in K_1(C^*_{\mathrm{Roe}}|\mathbb{R}|).$$

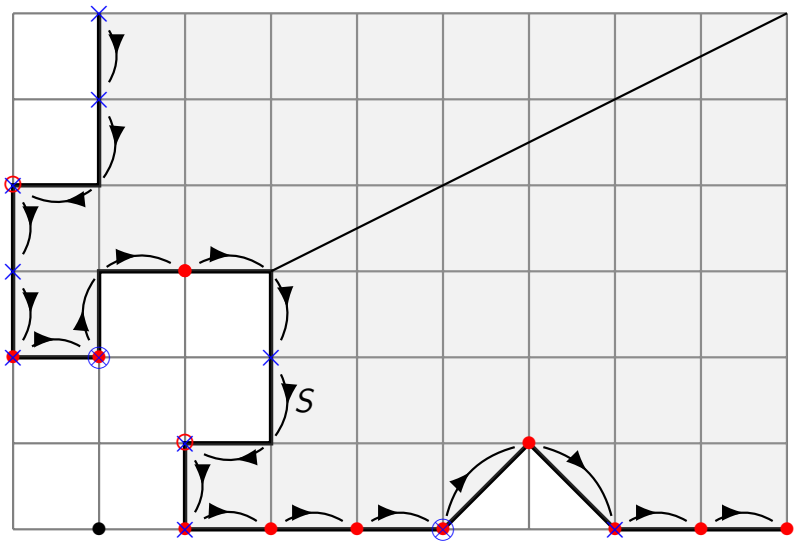
- Roe constructed cyclic 1-cocycle using partition, same formula as before. "Flow of regularized unitary".
- S embeds as generator of $K_1(C^*_{\mathrm{Roe}}|\mathbb{R}|) \cong \mathbb{Z}$.

Odd index and spectral gaplessness

In physics jargon, 1D Dirac has a "chiral anomaly".
$\rightsquigarrow$ $D + V$ has full spectrum $\mathbb{R}$, for *any* bounded V.

But $D \oplus -D$ has no coarse index/anomaly:

$$\begin{pmatrix} D & m \\ m & -D \end{pmatrix} \longrightarrow \mathrm{Spec} = (-\infty, -m] \underbrace{\cup}_{\text{gap}} [m, \infty).$$



Actually "anomalous D" appears at the boundary of a 2D topological insulator system.

Edge-following topological states [T, JGP '20]

II. Two-dimensional story

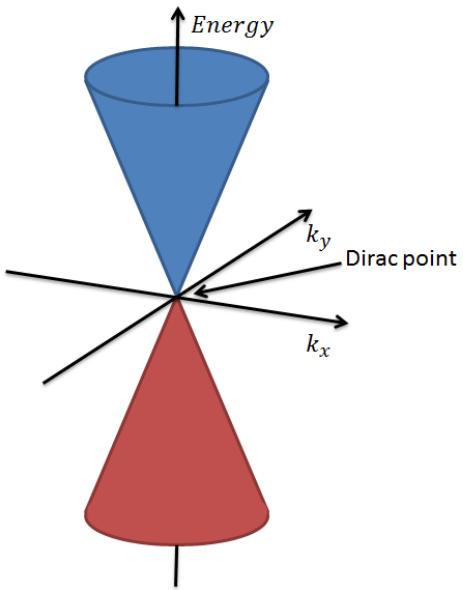
On Euclidean plane, Dirac Hamiltonian is the odd operator,

$$D = \begin{pmatrix} 0 & -i\partial \\ -i\bar{\partial} & 0 \end{pmatrix}, \qquad \partial = \partial_x - i\partial_y.$$

Abstract Roe-coarse index: generator of $K_0(C^*_{\mathrm{Roe}}|\mathbb{R}^2|) \cong \mathbb{Z}$.
But what does this count?!

Fourier transform of D is

$$D(z) = \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix}, \; z \in \hat{\mathbb{R}}^2 \cong \mathbb{C}.$$



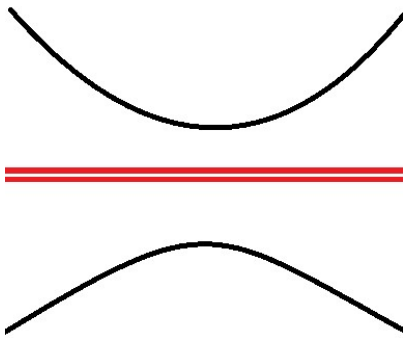
- Invertible self-adjoint family, except at $z = 0$.
  Pointwise index $= 0 \ldots$ can Dirac point be killed?

Correct zero mode counting: Families index

$$\tilde{D}(z) = \left(\begin{array}{cc|cc} 0 & 0 & \bar{z} & 0 \\ 0 & 0 & t & 0 \\ \hline z & t & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{has kernel} \quad \begin{cases} [1 : -z/t], & L, \\ [0 : 1], & R. \end{cases}$$

- Non-zero eigenvalues of $\tilde{D}(z)$ are

$$\pm\sqrt{t^2 + |z|^2}.$$



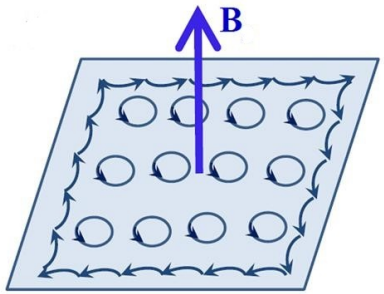
- Perturbed $\tilde{D}$ acquires gaps above/below 0.

Index bundle is $\underbrace{\mathcal{O}(1) \ominus \underline{\mathbb{C}}}_{K-\text{theory}}$ $\rightsquigarrow$ equivariant coarse index.

$\mathbb{Z}^2$-equivariant version: Baum–Connes assembly map (T-duality).

Couple D to constant curvature gauge field

$$D_b = \begin{pmatrix} 0 & -i\partial + ibx \\ -i\bar{\partial} - ibx & 0 \end{pmatrix}.$$



$$D_b^2 = \begin{pmatrix} H_{\text{Landau}} - b & 0 \\ 0 & H_{\text{Landau}} + b \end{pmatrix} \overset{\text{SUSY}}{\Rightarrow} \text{Spec}(H_{\text{Landau}}) = \underbrace{(2\mathbb{N}+1)b}_{\text{quantized}}.$$

Landau levels $\quad b \quad 3b \quad 5b \quad 7b \quad \ldots$

- **Pure geometry:** Lichnerowicz identity

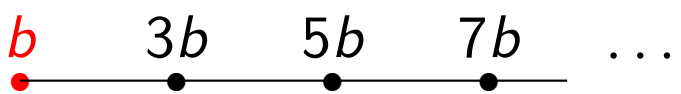
$$\text{Dirac}^2 = \text{Laplace} + \text{curvature}$$

- Lowest Landau level $= \infty$-dimensional $\underbrace{\text{twisted Dirac kernel}}_{\text{coarse index}}$.

Identification of Landau eigenspace as Dirac coarse index is new [Ludewig+T, CMP '21, ATMP '22].

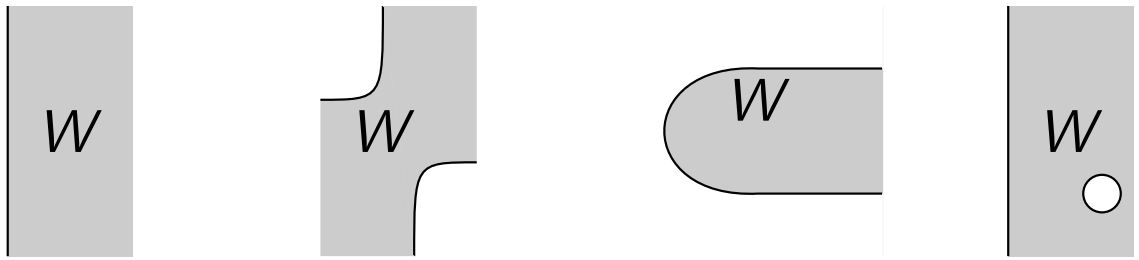
Gap-filling at boundary:
When H_{Landau} is restricted from X to a coarse half-space W, the spectrum becomes unbroken,

$$\sigma(H_{\mathrm{Landau},W}) \subset [b, \infty).$$



Conceptually, this is "$\partial D_W = D_{\partial W}$", or coarse homology boundary map.
Recall: $D_{\partial W}$ is *gapless* 1D Dirac, associated with "chiral current"!

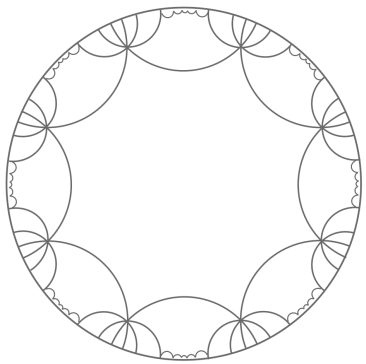
Calculations are coarse-geometry-invariant $\rightsquigarrow$ boundary gaplessness is extremely robust!

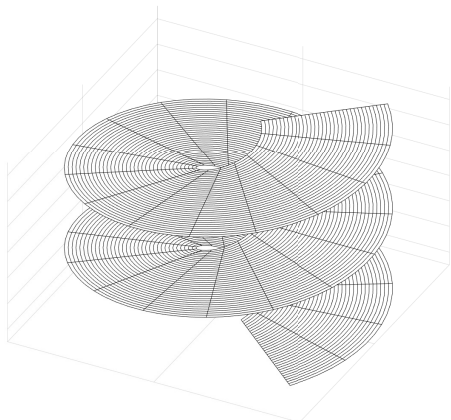


Using a more intricate supersymmetry argument + coarse index, we proved similar result for X the hyperbolic plane,

This phenomenon survives a wide class of perturbations.

Spectrum of Landau operator on complete X can be estimated from scalar curvature R and gauge curvature Θ.





If curvatures vary too much, the Landau levels broaden and spectral gaps close.

Introduce "*delocalized* coarse index", prove existence of gapless helical states along axis of a helical surface[3].

$\sim$ mod out Roe-subalgebra localized near submanifolds.

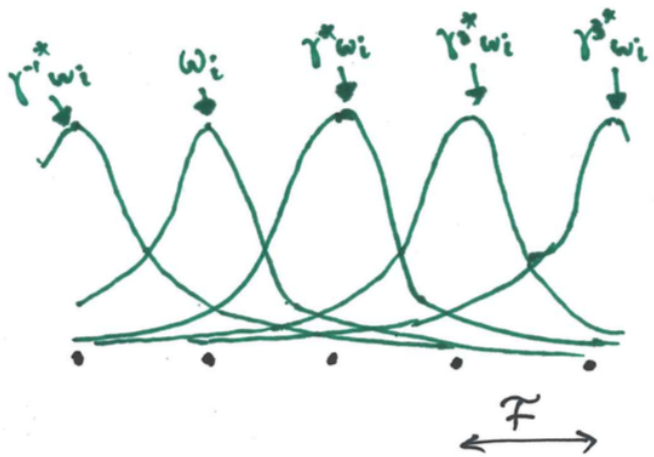
[3] [Kubota+L+T, 2201.05416]

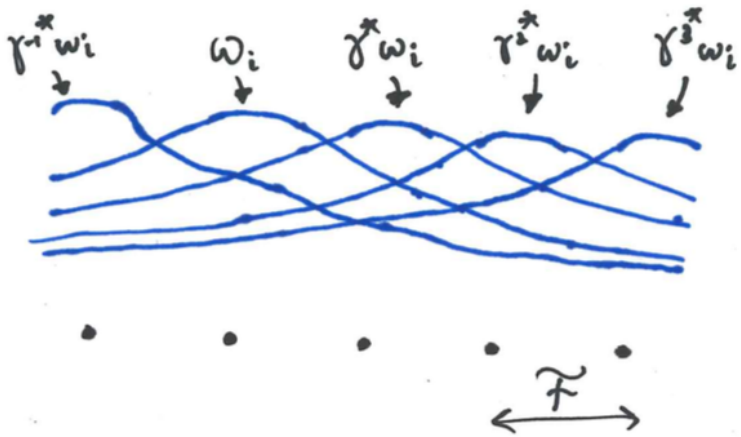
III. Roe algebras & anomalous discretization of space.

Twisted Dirac kernel $\rightsquigarrow$ Nontrivial projection in $C^*_{\mathrm{Roe}}(X)$.
How to "see/measure" non-triviality of a general projection?

Answer: Range of non-trivial projection is "highly delocalized".

Fun Fact: Inside $\ell^2_{\mathrm{reg}}(\mathbb{Z}^2) \oplus \ell^2_{\mathrm{reg}}(\mathbb{Z}^2)$, there exists a copy of $\ell^2_{\mathrm{reg}}(\mathbb{Z}^2)$ whose basis functions are "delocalized". [L+T, JMP '20]





Wannier bases

On homogeneous Euclidean space $\mathbb{R}^2$, this idea is known to physicists as "Wannier non-localizability[4]" .

- Atomic cores $\sim$ "classical lattice" $\Gamma \cong \mathbb{Z}^2$ in $\mathbb{R}^2$.
- Wannier basis' wavefunctions, $\gamma^* w \in L^2(X), \gamma \in \Gamma$, are orthonormal "electron orbitals", spanning some relevant spectral subspace $\mathcal{H} \subset L^2(\mathbb{R}^2)$.
- Effective "tight-binding" Hilbert space $\mathcal{H} \cong \ell^2_{\mathrm{reg}}(\mathbb{Z}^2)$.
- Classical discretization $\mathbb{Z}^2 \subset \mathbb{R}^2$ is "lifted" to a quantum one, $\ell^2_{\mathrm{reg}}(\mathbb{Z}^2) \subset L^2(\mathbb{R}^2)$.

Usually want w to be well-localized.
Otherwise, above process is bad "lattice approximation".

[4] [Brouder, Panati et al, PRL '07]

General problem: Given a Hilbert subspace $\mathcal{H} \subset L^2(X)$, does there exist a uniformly discrete $\Gamma \subset X$, and a *uniformly localized* orthonormal basis $\{w_\gamma\}_{\gamma\in\Gamma}$ for $\mathcal{H}$?

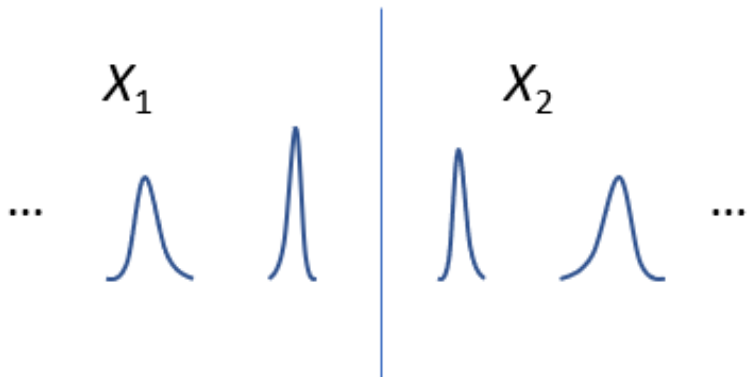
$$|w_\gamma(x)| \leq C_\mu(1 + d(x,\gamma))^{-\mu}, \qquad \forall \gamma \in \Gamma, x \in X.$$

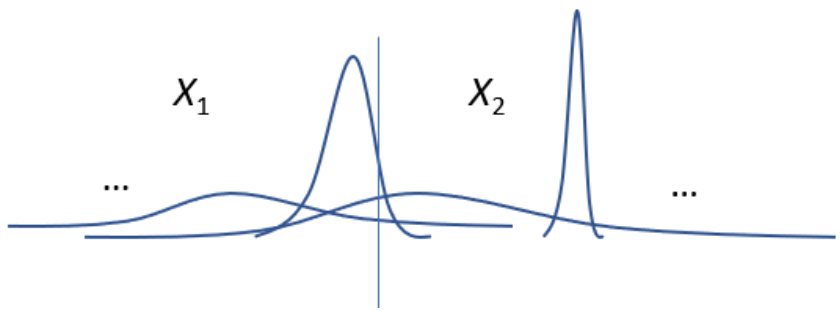
Motivation: We're looking for a "point-like" basis to describe $\mathcal{H}$.

Theorem [L+T, 2204.12895].
If $p_\mathcal{H} \in C^*_{\mathrm{Roe}}(X)$ is K-theoretically non-trivial, then there exists no uniformly localized ONB for $\mathcal{H}$, for any choice of uniformly discrete $\Gamma \subset X$ whatsoever[5].

[5] Assume $X = X_1 \cup X_2$ with $K_0(C^*_{\mathrm{Roe}}(X_i)) = 0$, mild geometric regularity.

If $\mathcal{H}$ is spannable by disjointly-compactly-supported w_γ, then classical splitting of $X = X_1 \cup X_2$ gives quantum splitting $p_{\mathcal{H}} = p_{\mathcal{H}_1} \oplus p_{\mathcal{H}_2}$ into trivial projections.





If $\mathcal{H}$ has uniformly localized basis, construct von Neumann equivalence in $C^*_{\mathrm{Roe}}(X)$ to previous case.

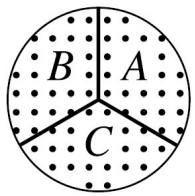
If $\mathcal{H}$ has non-trivial $p_{\mathcal{H}}$, it can never be built from "pointlike basis". There are irremovable long-range quantum overlaps!

Dirac and Landau operators have spectral subspaces $\mathcal{H} = pL^2(X)$,

$$0 \neq [p] \in K_0(C^*_{\mathrm{Roe}}(X)).$$

Q: How to quantify the "large-scale quantum overlaps" in $\mathcal{H}$?

Kitaev (on Euclidean lattice): "2-current",



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In progress:

- NCG: cyclic cocycles on suitable Roe-subalgebras, pairings with K-theory, physical meaning...
- Relation to coarse homology/Mayer–Vietoris [Higson-Roe-Yu]; Quantitative K-theory [Oyono-Oyono–Yu]
- Coarse KO-homology (torsion)