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Proposition 2.1: $F : V \to \overline{\mathbb{R}}$ is convex iff its epigraph is convex.

Proof: Suppose F is convex. Take (u,a) & (v,b) in $\operatorname{epi} F$. Then obviously $F(u) \le a < \infty$ and $F(v) \le b < \infty$ and $\forall \lambda \in [0,1]$, we have

$$F(\lambda u + (1-\lambda) v) \le \lambda F(u) + (1-\lambda) F(v) \le \lambda a + (1-\lambda) b.$$

This means $\lambda(u,a) + (1-\lambda)(v,b) \in \operatorname{epi} F$.

Conversely, let $\operatorname{epi} F$ be convex. Then its projection

$$\operatorname{dom} F = \{ u \in V \mid F(u) < \infty \}$$

is convex. Let $u, v \in \operatorname{dom} F$, $a \ge F(u)$, $b \ge F(v)$. By hypothesis,

$$\lambda(u,a) + (1-\lambda)(v,b) \in \operatorname{epi} F \quad \forall \lambda \in [0,1]$$

so that $F(\lambda u + (1-\lambda) v) \le \lambda a + (1-\lambda) b$.

If $F(u)$ and $F(v)$ are finite, we can take $a = F(u)$ and $b = F(v)$. If either $F(u)$ or $F(v)$ is equal to $-\infty$, we can simply let a or b tend to $-\infty$.

Either way, we have $F(\lambda u + (1-\lambda) v) \le \lambda F(u) + (1-\lambda) F(v)$.

Proposition 2.3: F is l.s.c. $\Leftrightarrow$ $\operatorname{epi} F$ is closed.

Proof: Let $\phi(u,a) = F(u) - a$. Then ϕ l.s.c. $\Leftrightarrow$ F l.s.c. Now, for every $r \in \mathbb{R}$, $\{(u,a) \mid \phi(u,a) \le r\}$ is closed $\Leftrightarrow$ ϕ l.s.c.

Note that this set is just a translate of $\operatorname{epi} F$ by $(r,0)$.

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<u>Existence and Uniqueness of Convex Variational Problems.</u>

Proof: Let u_n be a minimizing sequence of $\inf_{v\in\mathcal{C}} F(v)$, i.e. $F(u_n) \to \inf_{v\in\mathcal{C}} F(v) = \alpha$, $u_n \in \mathcal{C}$.

Note that $\alpha \in [-\infty, \infty)$. The sequence $\{u_n\}$ is bounded when F is coercive. Thus, $\exists \{u_{n_k}\}$ s.t. $u_{n_k} \rightharpoonup u \in \mathcal{C}$.

Since F is convex and l.s.c., then it's weakly l.s.c. on $\mathcal{C}$ and hence $F(u) \le \lim_{n_k\to\infty} F(u_{n_k}) = \alpha$.

This shows that u is a solution and $\alpha \neq -\infty$.

If $\exists u_1 \& u_2 \in \mathcal{C}$ are two solutions. When F is strictly convex, we have

$$F\left(\frac{u_1+u_2}{2}\right) < \frac{1}{2}F(u_1) + \frac{1}{2}F(u_2) = \alpha.$$

Thus, we must have $u_1 = u_2$ □.

<u>Note on weak topology and weak l.s.c. :</u>

Definition of weak convergence: Let V be a normed vector space and V^* its dual space associated with the bilinear form $\langle \cdot, \cdot \rangle$. We say that u_k converges to u weakly if $\langle u^*, u_k \rangle \to \langle u^*, u \rangle$ $\forall u^* \in V^*$. Denote it as $u_k \rightharpoonup u$.

Definition of weak l.s.c.: $\forall \bar{u} \in V$ and $u_k \rightharpoonup \bar{u}$, we have $\varliminf_{u_k \rightharpoonup \bar{u}} F(u_k) \ge F(\bar{u}) \iff \operatorname{epi} F$ is weakly closed.

Fermat's Theorem:

"$\Rightarrow$" If $F(u) = \min_{v \in V} F(v)$, then $F(u) \leq F(v)$, $\forall v \in V$.

This means $\langle v-u, 0 \rangle + F(u) \leq F(v) \Rightarrow 0 \in \partial F(u)$.

"$\Leftarrow$" If $0 \in \partial F(u)$, then $\langle v-u, 0 \rangle + F(u) \leq F(v)$, $\forall v \in V$.

Thus $F(u) \leq F(v)$ $\forall v \in V$.

Gâteaux derivative of $\frac{1}{2}\int_\Omega |\nabla u|^2 dx$.

Let $F(u) : W^{1,2}(\Omega) \to \mathbb{R}$ and $F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx$

$$\frac{F(u+\lambda v) - F(u)}{\lambda} = \frac{d}{d\lambda} F(u+\lambda v)\Big|_{\lambda=0}$$

$$= \frac{1}{2}\frac{d}{d\lambda} \int_\Omega (\nabla u + \lambda \nabla v)\cdot(\nabla u + \lambda \nabla v)\, dx$$

$$= \frac{1}{2}\frac{d}{d\lambda}\left(\int_\Omega |\nabla u|^2 + 2\lambda \int_\Omega \nabla u \cdot \nabla v + \lambda^2 \int_\Omega |\nabla v|^2\right)$$

$$= \int \nabla u \cdot \nabla v$$

(Green's formula) $= -\int_\Omega \Delta u\, v + \int_{\partial\Omega} \frac{\partial u}{\partial N} v\, ds = F'(u;v)$

Euler-Lagrange Equation

$$F'(u;v) = 0 \quad \forall v \iff -\Delta u = 0 \text{ and } \frac{\partial u}{\partial N} = 0$$