ON PROPER MODULI SPACE OF SMOOTHABLE KÄHLER-EINSTEIN FANO VARIETIES

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ABSTRACT. In this paper, we investigate the geometry of the orbit space of the closure of the subscheme parametrising smooth Fano Kähler-Einstein manifolds inside an appropriate Hilbert scheme. In particular, we establish the uniqueness for the Gromov-Hausdorff limit for a punctured flat family of Fano Kähler-Einstein manifolds, which corresponds to a minimal orbit in a limiting orbit. We then construct a proper scheme parameterizing all smoothable K-polystable \(\mathbb{Q}\)-Fano varieties, and establish various properties which make it a good moduli space.

CONTENTS

1. Introduction 1
1.1. Main results 2
2. Preliminaries 5
3. Linear action of reductive groups on projective spaces 6
4. Gromov-Hausdorff continuity of conical Kähler-Einstein metric on smooth Fano pair 9
4.1. Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds 9
4.2. Gromov-Hausdorff continuity of conical Kähler-Einstein metric on smooth Fano family 10
5. Strong uniqueness for \(0 < \beta \ll 1\) 13
6. Continuity method 15
7. K-semistability of the nearby fibers 22
7.1. Orbit of K-semistable points 22
7.2. Zariski Openness of K-semistable varieties 23
7.3. Proof of Theorem 1.1 and 1.2 25
8. Local geometry near a smoothable K-polystable \(\mathbb{Q}\)-Fano variety 26
9. Appendix 33
9.1. Constructibility of \(\text{kst}\) 33
9.2. Stabilizer Preserving Property 35
References 38

1. INTRODUCTION

Constructing moduli spaces for higher dimensional algebraic varieties is a fundamental problem in algebraic geometry. For dimension one case, the moduli space parametrizing Deligne-Mumford stable curves was constructed via various kind of methods, e.g. geometric invariant theory (GIT), Teichmüller space quotient by mapping class group, etc. For higher dimensional case, one of the natural classes to consider is all canonically polarised manifolds, for which GIT machinery is quite successful (see [Aub78, Yau78, Vie95, Don01]). However, to construct a geometrically natural compactification for these moduli spaces, the GIT method in its classical form fails to produce that(cf. [WX14]), thus people have to develop substitutes. In fact, it has been quite a while for people to realize what kind of varieties should be included in order to form a proper moduli (cf. [KSBS88]). Thanks to the recent breakthrough coming from the theory of minimal model program (see [BCHM10] etc.), one is able to obtain a rather satisfactory theory on proper projective moduli spaces parameterizing KSBA-stable varieties, named after Kollár-Shepher-Barron-Alexeev (see [Kol13] for a concise survey

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of this theory). We also remark that it is realized later that this compactification should coincide with the compactification from Kähler-Einstein metric/K-stability (cf. [Oda13, WX14, BG14]).

As for Fano varieties, the story is much subtler. Apart from some local properties, e.g. having only Kawamata log terminal (klt) singularities when a Fano variety is assumed to be K-semistable (cf. [Oda13]) and admitting klt Fano degenerations as long as a general fiber is a klt Fano variety in a one parameter family (cf. [LX14]), it is still not clear what kind of general Fano varieties we should parametrize in order for us to obtain a nicely behaved moduli space, especially if we aim to find a compact Hausdorff one, and how to construct it. The recent breakthrough in Kähler-Einstein problem, namely the solution to the Yau-Tian-Donaldson Conjecture (CD15a, CD15b, CD15c and [Tia15]) is a major step forward, especially for understanding those Fano manifolds with Kähler-Einstein metrics. Furthermore, it implies that the right limits of smooth Kähler-Einstein manifolds form a bounded family. In this paper, we aim to use the analytic results they established to investigate the geometry of the compact space of orbits which is the closure of the space parametrizing smooth Fano varieties.

1.1. Main results. Our first main result of this paper is the following:

**Theorem 1.1.** Let $X \to C$ be a flat family of projective varieties over a pointed smooth curve $(C, 0)$ with $0 \in C$. Suppose

1. $K_X$ is Q-Cartier and $-K_{X/C}$ is relatively ample over $C$;
2. for any $t \in C^0 := C \setminus \{0\}$, $X_t$ is smooth and $X_0$ is klt;
3. $X_0$ is K-polystable.

Then

(i) there is a Zariski open neighborhood $U$ of $0 \in C$ on which $X_t$ is K-semistable for all $t \in U$, and K-stable if we assume further $X_0$ has a discrete automorphism group;
(ii) for any other flat projective family $X' \to C$ satisfying (1)-(3) as above and $X' \times_C C^0 \cong X \times_C C_0$,

we can conclude $X'_0 \cong X_0$;
(iii) $X_0$ admits a weak Kähler-Einstein metric. If we assume further that $X_t$ is K-polystable, then $X_0$ is the Gromov-Hausdorff limit of $X_t$ endowed with the Kähler-Einstein metric for any $t \to 0$.

If both $X_0$ and $X'_0$ are assumed to be smooth Kähler-Einstein manifolds then part of Theorem 1.1 is a consequence of the work [Sze10], where the more general case for arbitrary polarization is established. When the fiber is of dimension 2, this is also implied by the work of [Tia90, OSS16] as explicit compactifications of Kähler-Einstein Del Pezzo surfaces are constructed there.

Now let us give a brief account of our approach. First we note that although part of our theorem is stated in algebro-geometric terms, the proof indeed relies heavily on known analytic results, especially the recent work in [CD15b, CD15c, Tia15] as well as [Sze10, OSS16] as explicit compactifications of Kähler-Einstein Del Pezzo surfaces are constructed there.

The first main tool for us is a continuity method very similar to the one proposed by Donaldson in [Don12a]. Indeed, by throwing in an auxiliary divisor $D \in |-mK_X|$ for some positive integer $m > 1$; we consider the following log extension of Theorem 1.1

**Theorem 1.2.** For a fixed $\beta \in [0, 1]$, let $X \to C$ be a flat family over a pointed smooth curve $(C, 0)$ with a relative codimension 1 cycle $D$ over $C$. Suppose

1. $-K_{X/C}$ is ample and $D \sim_C -mK_{X/C}$ for some positive integer $m > 1$;
2. for any $t \in C^0 := C \setminus \{0\}$, $X_t$ and $D_t$ are smooth, $(X_0, \frac{1}{m} D_0)$ is klt;
3. $(X_0, D_0)$ is $\beta$-K-polystable. (cf. Definition 2.3).

Then

(i) there is a Zariski neighborhood $U$ of $0 \in C$, on which $(X_t, D_t)$ is $\beta$-K-semistable (in fact $\beta$-K-polystable if $\beta < 1$) for all $t \in U$;
(ii) for any other flat projective family $(X', D') \to C$ with a relative codimension 1 cycle $D'$ satisfying (1)-(β) as above and

$$(X', D') \times_C C^0 \cong (X, D) \times_C C^0,$$

we can conclude $(X_0', D_0') \cong (X_0, D_0)$;

(iii) $(X_0, D_0)$ admits a conical weak Kähler-Einstein metric with cone angle $2\pi(1 - (1 - \beta)/m)$ along $D_0$, which is Gromov-Hausdorff limit of $(X_{t_i}, D_{t_i})$ endowed with the conical Kähler-Einstein metric with cone angle $2\pi(1 - (1 - \beta_i)/m)$ along $D_{t_i}$ for any sequence $t_i \to 0$ and $\beta_i \not\nearrow \beta$.

To prove Theorem 1.2, one notices that the uniqueness is well-understood when the angle is small. We give an account to this fact using a completely algebro-geometric means. To be precise, we use closure of the SL$(N)$-orbit of any point in the limiting orbit near $p$ actually contains $g \cdot p$ for some $g \in \text{SL}(N + 1)$. In particular, it guarantees that there is no nearby non-equivalent K-polystable limit which either specializes to $(X_0, D_0)$ in a test configuration or becomes the central fiber of a test configuration of $(X_0, D_0)$, violating the K-stability assumption. Similarly, this argument can also be applied to study the case when $\beta \nearrow 1$.

To finish the proof, we need to verify the assumption that all the nearby fibers $X_t$ are $K$-semistable. For this, one needs two observations. First, it follows from the work of [CDS15b, CDS15c, Tia15] that to check K-semistability of $(X_0, D_0)$ in a test configuration or becomes the central fiber of a test configuration of $(X_0, D_0)$, we can conclude that all the nearby fibers $X_t$ are $K$-semistable. For this, one needs two observations. First, it follows from the work of [CDS15b, CDS15c, Tia15] to check K-semistability of $X_t$, $t \neq 0$, it suffices to test for all one-parameter-group (1-PS) degenerations in a fixed $P^n$. Second, it follows from a straightforward GIT argument that $K$-semistable threshold (kst) (cf. Section 7.2) is a constructible function. So what remains to show is that it is also lower semi-continuous (also observed in [SY14]), which is a consequence of the upper semi-continuity of the dimension of the automorphism groups and the continuity method deployed in the proof of Theorem 1.2.

With all this knowledge in hand, we will prove that there is a well-behaved orbit space for smoothable K-semistable Fano varieties.

**Theorem 1.3.** For $N \gg 0$, let $Z^*$ be the semi-normalization of the locus inside $\text{Hilb}_1(P^n)$ parametrizing all smoothable $K$-semistable Fano varieties in $P^n$ with fixed Hilbert polynomial $\chi$ (see Section 3 for the precise definition of $Z^*$). Then the algebraic stack $(Z^*/\text{SL}(N+1))$ admits a proper semi-normal scheme $\mathcal{K}_F N$ as its good moduli space (see for the definition in [Alp13, Section 1.2]). Furthermore, for sufficiently large $N$, $\mathcal{K}_F N$ does not depend on $N$.

In particular, our quotient is a compact Hausdorff Moishezon space thanks to [Art70, Theorem 7.3]. The existence of a moduli space for Kähler-Einstein Fano manifolds is well expected after the work of [Tia90]. A local quotient picture was suggested in [Don08, Section 5.3] and [Sze10], and was explicitly conjectured in [Spo12, Section 1.3 and 1.4] and [OSS16, Conjecture 6.2]. Furthermore, the moduli space is speculated to be projective by the existence of the descending of the CM-line bundle (see e.g. [PT06] and [OSS16]). We also remark that for smooth Kähler-Einstein Fano manifolds with discrete automorphism which are known to be asymptotically Chow stable by [Don01], they admit (possibly non-proper) algebraic moduli spaces thanks to the work of [Don15] and [Oda12].

Now let us explain our approach to Theorem 1.3. Due to the lack of a global GIT interpretation of the K-stability, a new strategy is needed to prove the existence of a good quotient moduli space.
Thanks to the work of [AFS16], a good quotient can be obtained as long as the following two key properties are established:

- the stabilizer preserving condition for the local presentation of the moduli stack;
- the affineness of the quotient morphism.

Notice that both properties follow from the famous Luna’s étale slice theorem for a reductive group $G$-acting on an affine variety $Z$, that is, if $z \in Z$ and the $G$-orbit $G \cdot z \subset Z$ is closed then there is a nice slice containing $z$ satisfying the above two properties. Unfortunately, we are unable to verify the assumption of Luna’s theorem since there is no global GIT interpretation of K-stability. So instead the closedness of $G \cdot x$ in an affine variety will be a consequence of our proof which based on the existence of a nice continuous proper slice (though non-algebraic) lying over the stack. The slice is obtained via the family version of Tian’s embeddings of Kähler-Einstein Fano varieties and its properness follows from Theorem 1.1. The slice can be regarded as a replacement of the level set of moment map in the classical Kempf-Ness-Kirwan picture.

Finally we close the introduction by outlining the plan of the paper. In Section 2, we give the basic definitions. In Section 3, we review some facts on the linear action of a reductive group on a projective space. In Section 4, we list the main analytic results we need in this note. First we recall the recent results appeared in [CDS15b, CDS15c, Tia15]. Then we also state the Gromov-Hausdorff continuity for conical Kähler-Einstein metrics on a smooth family of Fano pairs (see [CDS15b, CDS15c, Tia15]). In Section 5, we prove that when the angle is small enough, the filling is always unique. In Section 6, we establish the main technical tool of our argument, which is a continuity theorem. We remark, with it we can already show Theorem 1.2 under the assumption that the nearby fibers are all $\beta$-K-polystable. In Section 7, we will prove the K-semi-stability of the nearby points by applying the continuity method. First in Section 7.1 we prove Theorem 1.2 which says that any orbit closure of a K-semistable Fano manifold contains only one isomorphic class of K-polystable $\mathbb{Q}$-Fano variety. In particular, this is an extension of the result of [CS14] for the Fano case. In Section 7.2, we show that a smoothing of a K-semistable $\mathbb{Q}$-Fano variety is always K-semistable. In Section 7.3, by putting all the results together, we finish the proof of Theorem 1.1 and 1.2. In Section 8, we apply our results and prove a Luna slice type theorem for K-stability, which is used to establish Theorem 1.3. In Section 9, we will discuss several technical results that are needed on the general theory of linear action of a reductive group on projective space.

**History.** Now we remark on some history of this paper, which was original titled as ‘Degeneration of Fano Kähler-Einstein manifolds’ (see [LWX14]), and in the first version, we first established the separateness of the moduli space. After it was posted on the arXiv, we were informed by the authors of [SSY14] who independently investigated similar questions with a circle of parallel ideas but in a more analytic fashion and obtained results which are closely related. In particular, in [SSY14], the authors obtained first the existence of weak Kähler-Einstein metrics on smoothable K-polystable $\mathbb{Q}$-Fano varieties; the analytic openness of K-stability in the case of finite automorphism group; the lower semi-continuity of the cone angle for conical Kähler-Einstein metrics. Those statements are not included in the first version of our preprint. As a consequence the uniqueness of K-stable filling with finite automorphism group was also obtained in [SSY14]. However, the approach in the first version of our paper naturally extends and give rise to a more complete picture as in the current version. We would like to thank the authors of [SSY14] for communicating their work to us. After we posted the second version of our paper on the arXiv, we were informed by Odaka [Oda15] that he had announced part of the Theorem 1.3 independently based on the results in [LWX14] and [SSY14].

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2. Preliminaries

In this section, we will fix our convention of the paper. The definitions below are recalled from [Tan97] and [Don02]. The readers may also consult the lecture notes [PS10] and [Tho06] for both analytic and algebro-geometric point of view.

Definition 2.1. Let \((X, D; L)\) be an \(n\)-dimensional projective variety polarized by an ample line bundle \(L\) together with an effective divisor \(D \subset X\). A log test configuration of \((X, D; L)\) consists of

1. A projective flat morphism \(\pi : (X, D; L) \to \mathbb{A}^1\);
2. A \(\mathbb{G}_m\)-action on \((X, D; L)\), such that \(\pi\) is \(\mathbb{G}_m\)-equivariant with respect to the standard \(\mathbb{G}_m\)-action on \(\mathbb{A}^1\) via multiplication;
3. \(L\) is relative ample and we have \(\mathbb{G}_m\)-equivariant isomorphism.

\[\mathcal{L} = \mathcal{L}|_{X^0} \cong (X \times \mathbb{G}_m, D \times \mathbb{G}_m; \pi_X : X \times \mathbb{G}_m \to X)\]

where \((X^0, D^0) = (X, D) \times_{\mathbb{A}^1} \mathbb{G}_m\) and \(\pi_X : X \times \mathbb{G}_m \to X\).

A log test configuration is called a product test configuration if \((X, D; L) \cong (X \times \mathbb{A}^1, D \times \mathbb{A}^1; \pi_X L)\) where \(\pi_X : X \times \mathbb{A}^1 \to X\), and a trivial test configuration if \(\pi : (X, D; L) \to \mathbb{A}^1\) is a product test configuration with \(\mathbb{G}_m\) acting trivially on \(X\).

In this article, we will focus on the projective pairs \((X, D)\) satisfy the following

Definition 2.2. Let \((X, D)\) be a projective pair with Kawamata log terminal (klt) singularities (see [KM98, 2.34]). Then \((X, D)\) to said to be a log Fano pair if \(-(K_X + D)\) is an ample \(\mathbb{Q}\)-Cartier divisor, and \(\mathbb{Q}\)-Fano variety if \(D = 0\).

To proceed, let \(\chi\) denote the Hilbert polynomial and we introduce \(a_i, \tilde{a}_i, b_i, \tilde{b}_i \in \mathbb{Q}\) via the following expansions.

- \(\chi(X, L^k) := \dim H^0(X, L^k) = a_0 k^n + a_1 k^{n-1} + O(k^{n-2})\);
- \(\chi(D, (L|_D)^k) := \dim H^0(D, L^k|_D) = \tilde{a}_0 k^n + O(k^{n-2})\);
- \(w(k) := \text{weight of } \mathbb{G}_m\text{-action on } \wedge^k H^0(X_0, L^k|_X) = b_0 k^{n+1} + b_1 k^n + O(k^{n-1})\);
- \(\tilde{w}(k) := \text{weight of } \mathbb{G}_m\text{-action on } \wedge^k H^0(D_0, L^k|_D) = \tilde{b}_0 k^{n} + O(k^{n-1})\).

Now we are ready to state the algebraic geometric criterion for the existence of conical Kähler-Einstein metric on a log Fano manifold \((X, D)\) with cone angle \(2\pi(1 - (1 + \beta)/m)\) along the divisor \(D \in \mid - mK_X\mid\).

Definition 2.3. For a \(\mathbb{Q}\)-Fano variety \(X\) with \(D \in \mid - mK_X\mid\) and a real number \(\beta \in [0, 1]\), we define the log generalized Futaki invariant with the angle \(\beta\) as following:

\[DF_{1-\beta}(X, D; L) \equiv DF(X; L) + (1 - \beta) \cdot CH(X; D; L)\]

with

\[DF(X; L) := \frac{a_1 b_0 - a_0 b_1}{a_0^2} \text{ and } CH(X; D; L) := \frac{1}{m} \frac{a_0 b_0 - b_0 a_0}{2a_0^2} \text{ (cf. [LS14] Definition 3.3)} \]

Then

\[DF_{1-\beta}(X, D; L^r) \equiv DF_{1-\beta}(X, D; L)\]

We say \((X, D; L)\) is called \(\beta\)-K-semistable if \(DF_{1-\beta}(X, D; L) \geq 0\) for any normal test configuration \((X, D; \mathcal{L})\), and \(\beta\)-K-polystable (resp. \(\beta\)-K-stable) if it is \(\beta\)-K-semistable with \(DF_{1-\beta}(X, D; L) = 0\) if and only if \((X, D; L)\) is a product test configuration (resp. trivial test configuration).

Thanks to the linear dependence of \(DF_{1-\beta}(X, D; L)\) on \(\beta\), we immediately obtain the following interpolation property:

Lemma 2.4. If \((X, D; L)\) is both \(\beta_1\)-K-semistable and \(\beta_2\)-K-polystable with \(\beta_1 < \beta_2\) (resp. \(\beta_2 < \beta_1\), then \((X, D; L)\) is \(\beta\)-K-polystable for any \(\beta \in (\beta_1, \beta_2]\) (resp. \(\beta \in [\beta_2, \beta_1)\)).
Remark 2.5. Notice that if for \((X, D; K_X^{(-r)})\) where \(X\) is a \(\mathbb{Q}\)-Fano variety with \(D \in \mathbb{Z} - mK_X\),
\[
\lambda : \mathbb{G}_m \to \text{SL}(N_r + 1) \quad \text{with} \quad N_r + 1 := \dim H^0(X, K_X^{(-r)})
\]
induces a test configuration \((X, D; \mathcal{L})\), then
\[
\text{CH}(X, D; \mathcal{L}) = \frac{1}{2mr^n(-K_X)^n} \cdot \left( \text{CH}(D_0) - \frac{nm}{(n+1)r} \text{CH}(X_0) \right)
\]
with \(\text{CH}(D_0)\) and \(\text{CH}(X_0)\) being precisely the \(\lambda\)-weight for the Chow points of \(D_0, X_0 \subset \mathbb{P}^{N_r}\).

3. Linear action of reductive groups on projective spaces

In this section, we prove a basic fact on a reductive group acting on \(\mathbb{P}^N\), which will be crucial for the later argument. Let \(G\) be a reductive algebraic group acting on \(\mathbb{P}^N\) via a rational representation \(\rho : G \to \text{SL}(N + 1)\) and \(z : C \to \mathbb{P}^N\) be an algebraic morphism satisfying \(z(0) = z_0 \in \mathbb{P}^N\) where \(\rho\) is a smooth pointed curve germ. Let
\[
\overline{BO} := \lim_{t \to 0} \overline{O_z(t)}
\]
with \(O_z(t) := G \cdot z(t)\) and \(\overline{O_z(t)} \subset \mathbb{P}^N\) be its closure, that is, \(\overline{BO}\) is a union of (broken) orbits that \(\overline{O_z(t)}\) specialized to.

Lemma 3.1. Suppose \(G_{z_0} < G\), the stabilizer of \(z_0 \in \mathbb{P}^N\) for the \(G\)-action on \(\mathbb{P}^N\), is reductive. Then there is a \(G\)-invariant Zariski open neighbourhood of \(z_0 \in U \subset \mathbb{P}^N\) satisfying:
\[
\overline{BO} \cap U = \bigcup_{O_p \subset \overline{BO}} O_p \cap U \quad \text{where} \quad O_p := G : p \subset \overline{BO},
\]
i.e. the closure of the \(G\)-orbit of any point in \(\overline{BO}\) near \(z_0\) contains \(g \cdot z_0\) for some (hence for all) \(g \in G\). We will call \(O_{z_0}\) a minimal orbit.

Proof. We divide the proof into two steps:

Step 1: \(G = G_{z_0}\). The representation \(\rho : G \to \text{SL}(N + 1)\) induces a \(G\)-linearization of \(\mathcal{O}_{\mathbb{P}^N}(1) \to \mathbb{P}^N\). Let \(\rho_0 : G \to \mathbb{G}_m\) be the character of the resulting \(G\)-action on \(\mathcal{O}_{\mathbb{P}^N}(1)|_{z_0}\) since \(z_0\) is fixed by \(G\). Then \(z_0\) is GIT poly-stable with respect the linearization of \(\mathcal{O}_{\mathbb{P}^N}(1)\) induced by the representation \(\rho \otimes \rho_0^{-1} : G \to \text{SL}(N + 1)\). Our claim then follows from the standard fact of GIT that semi-stable points are Zariski open, i.e., there exists a \(G\)-invariant section \(s\) such that \(s(z_0) \neq 0\) and we can take \(U\) the open set where \(s\) does not vanish. In particular, in this case \(U\) can be chosen as a \(G\)-invariant Zariski open set.

Step 2: \(G > G_{z_0}\). Since \(G_{z_0}\) is reductive, we have a decomposition of its Lie algebra
\[
\text{Lie}(G) = \mathfrak{g} = \mathfrak{g}_{z_0} \oplus \mathfrak{p}
\]
as representations of \(G_{z_0}\). The infinitesimal action of \(G\) at \(0 \neq \hat{z}_0 \in \mathbb{C}^{N+1}\), a lifting of \(z_0 \in \mathbb{P}^N\), induces a \(G_{z_0}\)-invariant decomposition \(\mathbb{C}^{N+1} = \mathbb{C} \cdot \hat{z}_0 \oplus \mathbb{C}^\perp \oplus \mathfrak{p}\). By the proof of [Don12b Proposition 1],
\[
\mathbb{P}W = \mathbb{P}(W^\perp \oplus \mathbb{C}\hat{z}_0) \subset \mathbb{P}^N
\]
satisfies the following properties:

1. \(\hat{z}_0 \in \mathbb{P}W\) and is preserved by \(G_{z_0}\);
2. \(\mathbb{P}W\) is transversal to the \(G\)-orbit of \(\hat{z}_0\) at \(\hat{z}_0\);
3. for \(w \in \mathbb{P}W\) near \(\hat{z}_0\) and \(\xi \in \mathfrak{g} := \text{Lie}(G)\), if we let \(\sigma_w : \mathfrak{g} \to T_w\mathbb{P}^N\) denote the infinitesimal action of \(G\) then
\[
\sigma_w(\xi) \in T_w\mathbb{P}W \iff \xi \in \mathfrak{g}_{z_0} := \text{Lie}(G_{z_0}).
\]
In particular, part (3) implies that there exists a Zariski open neighborhood \(U_0 \subset \mathbb{P}W\) of \(\hat{z}_0\) such that the infinitesimal action induced by \(\mathfrak{p}^\perp\) on \(\mathbb{P}W\) is transversal for all points in \(U_0\) (cf. Lemma 9.7).
Claim 3.2. Let \( S := G \cdot \text{Im} z \subset \mathbb{P}^N \) and \( H \) be the identity component of \( G_{z_0} \). Then there is a Zariski open subset \( U_W \subset U_0 \subset \mathbb{P} W \) and a finite collection of pointed arcs \( \{ z^i : (C_i, 0) \to (U_0, z_0) \} \) with \( z^i = z : C \to \mathbb{P} W \) such that

\[
\overline{S} \cap U_W = \bigcup_{i=0}^{d} \overline{O(H, z^i)} \cap U_W \quad \text{with} \quad O(H, z^i) := H \cdot \text{Im} z^i \subset \mathbb{P} W.
\]

Assume Claim 3.2 for the moment, let us define

\[
\overline{BO}^W := \lim_{t \to 0} \overline{O^W_{z^i(t)}} \subset E \quad \text{with} \quad O^W_{z^i(t)} = H \cdot z^i(t) \subset \overline{O(H, z^i)} \subset \mathbb{P} W.
\]

Next for each \( 0 \leq i \leq d \), applying Step 1 to the \( H \)-action on \( \mathbb{P} W \) and \( \overline{BO}^W \subset \mathbb{P} W \), we obtain a \( H \)-invariant Zariski open \( z_0 \in U'_i \subset \mathbb{P} W \) such that

\[
\forall p \in U'_i \cap \overline{BO}^W \implies z_0 \in G \cdot p.
\]

Then \( U = G \cdot \left( \bigcap_{i=0}^{d} U'_i \right) \) is the \( G \)-invariant Zariski open set we want.

Now let us proceed to the proof of Claim 3.2. To better illustrate the picture, let us treat the case \( \dim G_{z_0} = 0 \) first.

Case 1: \( \dim G_{z_0} = 0 \). Let us consider the variety \( S := G \cdot \text{Im} z \subset \mathbb{P}^N \) and let \( \partial S := \overline{S} \setminus S \). Then there is an open neighborhood \( U_W \subset U_0 \) such that \( \overline{S} \cap U_W \) has only finitely many irreducible components. Let us write

\[
\overline{S} \cap U_W = \bigcup_{i=0}^{d} C_i
\]

with \( C_0 = \text{Im} z(C) \) and \( C_i \) are irreducible components passing through \( z_0 \).

Since \( \partial \overline{S} \cap C_i \) is constructible, after a possible shrinking of \( C_i \) we have two possibilities:

1. \( \partial \overline{S} \cap C_i = C_i \)
2. \( \partial \overline{S} \cap C_i = \emptyset \) or \( z_0 \).

We claim that the first case does not happen and then by choosing the arc \( z^i : (C_i, 0) \to (U_0, z_0) \) we establish Claim 3.2. To prove our claim, one notices there are two kinds of possible points on \( \partial \overline{S} \):

- first kind: a boundary point of \( G \cdot z(t) \) for a fixed \( t \);
- second kind: all the remaining points on \( \partial \overline{S} \).

Notice that the set of both kinds of points form constructible sets. Any boundary point of the first kind can be indeed written as a limit of points in \( G \cdot z(t) \cap U_W \) for a fixed \( t \), but this is absurd as \( G \) acts on \( U_0 \) transversally. So we may assume all the points on \( C_i \) are of the second kind, this implies that

\[
\text{Im} z \not\subset G \cdot z(t) \text{ for a fixed } t \in C.
\]

In particular, we have \( \dim G + 1 = \dim \overline{S} \) as \( \dim G_{z_0} = 0 \). Since \( \partial \overline{S} \) is \( G \)-invariant, we have \( G \cdot C_i \subset \partial \overline{S} \). Now let us consider the \( G \)-action on \( z \in C_i \), which implies that the

\[
\dim \partial \overline{S} \geq \dim G + \dim C_i = \dim G + 1 = \dim \overline{S},
\]

a contradiction. Thus our claim is verified.

Case 2: the general case. Let us consider the variety \( S := G \cdot \text{Im} z \subset \mathbb{P}^N \) and let \( \partial S := \overline{S} \setminus S \). Then there is an \( H \)-invariant open neighborhood \( U_W \subset U_0 \) such that \( \overline{S} \cap U_W \) has only finitely many irreducible components, which are denoted by

\[
\overline{S} \cap U_W = \bigcup_{i=0}^{d} V_i
\]

with \( V_0 = \overline{O(H, z)} \) and \( z_0 \in V_i, 0 \leq i \leq d \). Moreover, \( V_i \) is \( H \)-invariant for each \( i \) since \( S \) is.

Then Claim 3.2 amounts to saying that for each \( i \), there is an arc \( z^i : C_i \to U_0 \) such that

\[
V_i = \overline{O(H, z^i)} \cap U_W.
\]

Notice that the latter case will not happen by our definition of \( \partial \overline{S} = \overline{S} \setminus S \).
To find such an arc, all we need is a \text{general} \ v \in V_i\ satisfying \ (4)\quad \dim H \cdot v + 1 \geq \dim V_i,

since that implies two situations: either \ \dim H \cdot v < \dim V_i\ for which we choose \ z^i : C_i \to V_i\ be an arc joining \ z_0\ and \ v\ so that \ \text{Im} z^i \not\subseteq H \cdot v;\ or \ \dim H \cdot v = \dim V_i\ for which we choose any nonconstant arc \ z^i : C_i \to V_i\ satisfying \ z^i(0) = z_0.\ Then \ \dim V_i = \dim (H, z^i)\ and our Claim is justified.

To find such \ v \in V_i,\ we only need it to satisfy
\[
\dim H \cdot v \geq \dim H \cdot z(t)\ for\ all\ t \in C,
\]
which again follows from the transversality. Indeed, there is a Zariski open set \ U_C\ of \ C,\ such that for any \ t_0 \in U_C,
\[
\dim H \cdot z(t_0) = \max_{t \in C} \dim H \cdot z(t).
\]

By definition of \ V_i,\ for a fixed general \ v \in V_i,\ there is a \ g_i \in G\ and \ t_0 \in U_C\ such that \ g_i \cdot z(t_0) \in B(v, \epsilon) \in \mathbb{P}^N,\ by the transversality of \ p\-action on \ U_0,\ for \ \epsilon \ll 1\ there is an \ h \in G\ close to identity such that \ h \cdot g_i \cdot z(t_0) \in V_i.\ By the genericity of \ v,\ we obtain
\[
\dim H \cdot v \geq \dim H \cdot h \cdot g_i \cdot z(t_0) = \dim H \cdot z(t_0) \geq \dim H \cdot z(t)\ for\ all\ t \in C.
\]

and hence \ \dim (H, z^i) \geq \dim (H, z)\ by our choice of \ z^i : C_i \to V_i.

Now we prove (4). Suppose (4) does not hold which is equivalent to \ \dim V_i > \dim (H, z^i),\ then we have
\[
\dim S \geq \dim G \cdot V_i
\]
\[
(\text{p}\-acting\ only\ on\ U_0) \geq \dim G/H + \dim V_i
\]
\[
> \dim G/H + \dim (H, z^i)
\]
\[
\geq \dim G/H + \dim (H, z) = \dim S,
\]
a contradiction. So the proof of the Claim is justified and hence the Lemma are completed. \qed

The necessity of the assumption that \ G_{z_0}\ is reductive can be illustrated by the following example.

\textbf{Example 3.3.} Let \ M_2(\mathbb{C}) = \{(v, w) \mid v, w \in \mathbb{C}^2\}\ be the linear space of \ 2 \times 2\ matrices, on which \ G := \text{GL}(2)\ is acting via multiplication on the left. Let \ V := M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C},\ G\ acts on \ PV\ via the representation

\text{acting via multiplication on the left. Let} \ V := M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C},\ G\ acts on \ PV\ via the representation
\[
\rho : \text{GL}(2) \to SL(V) \quad \rho(g) = \left[ \begin{array}{c} A \\ x_5 \\ x_6 \end{array} \right] = \left[ \begin{array}{c} g \cdot A \\ \det(g^{-1})x_5 \\ \det(g^{-1})x_6 \end{array} \right].
\]

Let
\[
z_0 = \left[ \begin{array}{c} 0_{2 \times 2} \\ 1 \\ 0 \end{array} \right] \quad \text{and} \quad z_0' = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right],\ z_0'' = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \in PV,
\]

then their stabilizers are \ G_{z_0} = G\ and \ G_{z_0'} = \left[ \begin{array}{c} * \\ * \end{array} \right] < \text{GL}(2).\ In particular, \ G_{z_0}\ is reductive while \ G_{z_0'}\ is not. Now let
\[
z(t) = \left[ \begin{array}{c} t \\ 0 \\ t^2 \end{array} \right] \quad \text{and} \quad z'(t) = \left[ \begin{array}{c} 0 \\ t \\ t^2 \end{array} \right] \in PV
\]
be two curves in \ PV,\ then we have
\[
\lim_{t \to 0} O_{z(t)} = \lim_{t \to 0} PV_{[1,t]} = \lim_{t \to 0} O_{z'(t)} = PV_{[1,0]}\ ,
\]

where \ V_{[1,t]} := \{tx_5 = x_6\} \subset V.\ Clearly, \ z_0 := z(0)\ satisfies (3)\ while \ z_0' := z'(0)\ does not, since
\[
z''_0 \not\in \mathbb{P}^1 \cong G_z \cdot z_0'' \subset PV_{[1,0]}\ for \ 0 < |\epsilon| \ll 1\ where \ z''_0 := \left[ \begin{array}{c} 1 \\ \epsilon \\ 0 \end{array} \right].
\]
4. Gromov-Hausdorff continuity of conical Kähler-Einstein metric on smooth Fano pair

In this section, we list the important analytic results that will be needed in our main argument.

4.1. Gromov-Hausdorff limit of Kähler-Einstein Fano manifolds. In this subsection, let us recall the main technical results obtained in the solution of Yau-Tian-Donaldson conjecture (see [CDS15b], [CDS15c], [Tia15] and [Ber16]).

**Theorem 4.1.** Let $X_i$ be a sequence of $n$-dimensional Fano manifolds with a fixed Hilbert polynomial $\chi$ and $D_i \subset X_i$ be smooth divisors in $|−mK_{X_i}|$ for a fixed $m > 0$. Let $\beta_i \in (0, 1)$ be a sequence converging to $\beta_\infty$ with $0 < \epsilon_0 \leq \beta_\infty \leq 1$. Suppose that each $X_i$ admits Kähler metric $\omega_i(\beta_i)$ solving:

$$\text{Ric}(\omega(\beta_i)) = \beta_i \omega(\beta_i) + \frac{1 - \beta_i}{m} [D] \text{ on } X_i.$$  \hspace{1cm} (5)

that is, $\omega_i(\beta_i)$ is a conical Kähler-Einstein metric will cone angle $2\pi(1-(1-\beta_i)/m)$ along the divisor $D_i \subset X_i$. Then the Gromov-Hausdorff limit of any subsequence of $\{(X_i, \omega_i(\beta_i))\}_i$ is homeomorphic to a $\mathbb{Q}$-Fano variety $Y$. Furthermore, there is a unique Weil divisor $E \subset Y$ such that

1. $(Y, \frac{1-\beta_\infty}{m} E)$ is klt;
2. $Y$ admits a weak conical Kähler-Einstein metric solving

$$\text{Ric}(\omega(\beta_\infty)) = \beta_\infty \omega(\beta_\infty) + \frac{1 - \beta_\infty}{m} [E] \text{ on } Y.$$  \hspace{1cm} \hspace{1cm} (6)

In particular, $\text{Aut}(Y, E)$ is reductive and the pair $(Y, E)$ is $\beta_\infty$-$\mathbb{K}$-polystable;

3. possibly after passing to a subsequence, there are embedings $T_i : X_i \rightarrow \mathbb{P}^N$ and $T_\infty : Y \rightarrow \mathbb{P}^N$, defined by the complete linear system $|−rK_{X_i}|$ and $|−rK_Y|$ respectively for $r = r(m, \epsilon_0, \chi)$ and $N + 1 = \chi(X_i, K_{X_i}^{\otimes r})$, such that $T_i(X_i)$ converge to $T_\infty(Y)$ as projective varieties and $T_i(D_i)$ converge to $T_\infty(E)$ as algebraic cycles.

In the following corollary, we denote by $C^{α,β}$ the space of conical Kähler metrics defined in [Don12a].

**Corollary 4.2.** Let $(X, D)$ be smooth Fano pair with $D \in |−mK_X|$. Then

1. $(X, D)$ is $β$-$\mathbb{K}$-stable if and only if it admits a conical Kähler-Einstein metric $\omega(\beta) \in C^{α,β}$ solving (5);
2. Let $γ \in (0, 1]$. Then $(X, D)$ is $γ$-$\mathbb{K}$-semistable if and only if it admits a conical Kähler-Einstein metric $\omega(\beta) \in C^{α,β}$ solving (5) for any $β \in (0, γ)$.

**Remark 4.3.** Notice that the limiting divisor $E \subset Y$ is actually $\mathbb{Q}$-Cartier. To see that, one notice that on the smooth locus of $Y$

$$E|_{Y^{\text{reg}}} \sim −mK_{Y^{\text{reg}}},$$

which implies $E|_Y \sim −mK_Y$ as $Y$ is normal. On the other hand, $Y$ being $\mathbb{Q}$-Fano implies that $K_Y$ is $\mathbb{Q}$-Cartier. This together with (6) implies that $E$ is $\mathbb{Q}$-Cartier. Also it was pointed out in [DS14] Section 4.3 and [CDS15c] Section 5 that if the sequence $\{(X_i, D_i)\} = \{(X_{i_t}, D_{i_t})\}$ is a subsequence of a projective flat family $(X^0, D^0) \rightarrow C^0$ of smooth log Fano pairs over a smooth punctured (not necessarily complete) curve $C^0 = C \setminus \{0\}$, i.e. $\{t_i\} \subset C^0$ and $t_i \xrightarrow{i \rightarrow \infty} 0$, then the Gromov-Hausdorff limit $(Y, E)$ can be realized as the central fiber of a flat degeneration

$$(X^0, D^0) \rightarrow (X, D),$$

that is, $(Y, E) = (X_0, D_0)$. This important consequence is used in [CDS15c] and [Tia15] to construct the destabilizing test configurations. In particular, the flatness of $X \rightarrow C$ is established in [DS14] Section 4.3 and the flatness of $D \rightarrow C$ can be deduced (see [Har77] Chapter III, Exercise 10.9) from the fact that $D$ is Cohen-Macaulay since we have already shown it is $\mathbb{Q}$-Cartier (see [KM98] Corollary 5.25), and the morphism $D \rightarrow C$ is equi-dimensional.
4.2. Gromov-Hausdorff continuity of conical Kähler-Einstein metric on smooth Fano family.

**Definition 4.4.** Let

\[ H^{X,N} := \text{Hilb}_X(\mathbb{P}^N) \]

denote the Hilbert scheme of closed subschemes of \( \mathbb{P}^N \) with Hilbert polynomial \( \chi \). For a closed subscheme \( X \subset \mathbb{P}^N \) with Hilbert polynomial \( \chi \) \((X, O_{\mathbb{P}^N}(k)|_\chi) = \chi(k), \) let \( \text{Hilb}(X) \in H^{X,N} \) denote its Hilbert point.

To set the scene, let

\[ (\mathcal{X}, D) \xrightarrow{i} \mathbb{P}^N \times \mathbb{P}^N \times \Delta \]

be projective flat family of Fano varieties over the disc \( \Delta = \{ |t| < 1 \} \subset \mathbb{C} \) such that:

1. \( X \) is smooth and \( D \in | - mK_X|/\Delta \) is a smooth divisor defined by a smooth section \( s_D \in \Gamma(\Delta, \omega_{\mathbb{P}^N})/\Delta \);
2. \( \pi \) is a holomorphic submersion (which is guaranteed if \( (\mathcal{X}_t, D_t) \) is smooth for all \( t \in \Delta \) and the family \( (\mathcal{X}_t, D_t) \) is flat).

To get rid of the \( U(N + 1) \)-ambiguity for the later argument, let us assume that \( \omega_X^{\otimes r} \) is relatively very ample and \( i \) be the embedding induced from the embedding \( i \) via the basis \( \{ s_i \} \). Suppose that for each \( t \in \Delta \), \( \mathcal{X}_t \) is K-semistable. Then by Lemma 2.4 \( (\mathcal{X}_t, D_t) \) is \( \beta \)-stable for any \( \beta \in (0, 1) \). So by Corollary 4.2 for any \( \beta \in (0, 1) \) there exists conical Kähler-Einstein metric \( \omega(t, \beta) \) on the pair \( (\mathcal{X}_t, \frac{1}{m}D_t) \) which satisfies

\[ \text{Ric}(\omega(t, \beta)) = \beta \omega(t, \beta) + \frac{1 - \beta}{m}[D_t]. \]

In the following, by abusing of name, sometime we will abbreviate \( \omega(t, \beta) \) as a conical Kähler-Einstein metric with cone angle \( \beta \) (instead of \( 2\pi(1 - (1 - \beta)/m) \)) along \( D \), since the integer \( m \) is fixed once for all the whole paper. Now assume \( \omega(t, \beta) = \omega_{K\mathbb{E}}(t, \beta) = \omega_{FS}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t, \beta) \) where \( r \cdot \omega_{FS}(t) \) is equal to the Fubini-Study metric induced from the embedding \( \mathcal{X}_t \to \mathbb{P}^N \) using the basis \( \{ s_i(t) \}_{i=0}^N \). Then \( \varphi(t, \beta) \) is the unique solution (c.f. [Ber15, Theorem 7.3]) to the equation

\[ (\omega_{FS}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t, \beta))^n = e^{\int_{\mathcal{X}_t}(1 - \beta \varphi(t, \beta)} \frac{\omega_{FS}(t)^m}{|s_{D_t}|^2 h_{FS}^{1/2}(\varphi)} \]

where \( f(t) \) satisfies

\[ \text{Ric}(\omega_{FS}(t)) = \omega_{FS}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t, \beta) \text{ and } \int_{\mathcal{X}_t} f(t) \cdot \omega_{FS}(t) \text{ and } \int_{\mathcal{X}_t} \omega_{FS}(t). \]

**Remark 4.5.** It’s easy to check that an equivalent form of equation (8) is:

\[ (\omega_{FS}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t, \beta))^n \cdot |s_{D_t}|^{2\beta} h_{FS}^{-\varphi}(s_{D_t} \otimes s_{D_t}) = 1. \]

We define a positive definite Hermitian matrix

\[ A_{KE}(t, \beta) = [(s_i, s_j)_{KE, \beta}(t)] \]

with

\[ (s_i, s_j)_{KE, \beta}(t) = \int_{\mathcal{X}_t} (s_i(t), s_j(t))_{h_{KE}(t, \beta)} \omega^n(t, \beta), \]

where \( h_{KE}(t, \beta) := h_{FS}(t) \cdot e^{-\varphi(t, \beta)} \). Now we introduce \( r \)-th Tian’s embedding

\[ T : (\mathcal{X}_t, D_t, \omega(t, \beta)) \rightarrow \mathbb{P}^N \]

to be the one given by the basis \( \{ g(t, \beta) \circ s_j(t) \}_{j=0}^N \) with \( g(t, \beta) = A_{KE}^{-1/2}(t, \beta). \)
Definition 4.6. We denote by
\[(\text{Hilb}(X, (1 - \beta) D_t) \in \mathbb{H}^\times : = \mathbb{H}^\times \times \mathbb{H}^\times\]
the Hilbert point of the pair \((X, D_t \subset X_t) \in \mathbb{P}^N\) using Tian’s embedding for the basis \(\{s_i\}\) with respect to Kähler form \(\omega(t, \beta)\), where \((\chi, \bar{\chi})\) are the Hilbert polynomials of \(X \subset \mathbb{P}^N\) and \(D \subset \mathbb{P}^N\) respectively. We note that when \(\beta = 1\), the second factor \(\mathbb{H}^\times\) is not trivial as we still remember \(D_t\), i.e., \(\text{Hilb}(X, 0 \cdot D)\) is not the same as \(\text{Hilb}(X)\). See Remark 4.7 below.

Remark 4.7. We make some remarks:
1. It is by definition that
\[\text{Hilb}(X, (1 - \beta) D_t) = (\text{Hilb}(X_t), \text{Hilb}(D_t); \omega(t, \beta)).\]
In the following we will always use the coefficient \((1 - \beta)\) to stress that the cycle is obtained via Tian’s embedding with respect to the metric \(\omega(t, \beta)\).
2. Tian’s embedding is well defined for any kit Q-Fano log pair with weak conical Kähler-Einstein metric \((X, (1 - \beta) D; \omega_{\text{KE}}(\beta))\). Note that for any weak conical Kähler-Einstein metric \(\omega_{\text{KE}}(\beta)\), we always assume that the local potential is bounded (see [BBE11]).
3. The advantage of fixing a basis \(\{s_i(t)\}\) for \(\mathbb{C}^{N - 1, 0}\) C-continuous \(t\) as before. Note that
\[\text{Hilb}(X, (1 - \beta) D_t) \text{ is completely determined by the isometric class of }\omega(t, \beta)\]. See Lemma 4.9.

Proposition 4.8. \(\text{Hilb}(X_t, (1 - \beta) D_t)\) varies continuously in \(\mathbb{H}^\times : = \mathbb{H}^\times \times \mathbb{H}^\times\) with respect to the pair \((\beta, t) \in (0, 1) \times \Delta\).

Proof. Using the above notations, we claim that \(\varphi_{\text{KE}}(t, \beta)\) is continuous with respect to \(t\) for any \(\beta < 1\). Assuming the claim, \(A_{\text{KE}}(t, \beta)\) is then continuous with respect to \(t\), and hence the images of Tian’s embedding given by orthonormal basis change continuously.

Now we verify the claim by applying implicit function theorem. First we notice that the complex manifold \((X_t, D_t)\) is diffeomorphic to a fixed pair \((X, D)\) endowed with the integrable complex structure \(J_t\) thanks to the assumption that \(\pi\) is a submersion. Let \(C^{\alpha, \beta}(X, D_t; J_t)\) and \(C^{\alpha, \beta}(X, D_t; J_t)\) denote the function spaces on \((X_t, D_t; J_t)\) defined in [Don12a]. For each fixed \(t \in \Delta\), we consider the map:
\[
F(t, \cdot, \cdot) : C^{\alpha, \beta}(X, D_t; J_t) \rightarrow C^{\alpha, \beta}(X, D_t; J_t)
\]
where for simplicity we write \(f(t) = f(t), \varphi(t) = \varphi(t)\) and \(h_t) = h_t\) and \(s_{D_t}\) is the defining section for \(D_t\) as before. Note that \(\varphi_{\text{KE}}(t, \beta)\) is exactly the solution to the equation \(F(t, \beta, \varphi) = 0\). We would like to apply implicit function theorem to obtain the continuity of \(\varphi_{\text{KE}}(t, \beta)\) with respect to \(t\) in order to do that, we need to work with a fixed function space, whereas the spaces \(C^{\alpha, \beta}(X_t, D_t; J_t)\) depends on the parameter \(t\). To get around this, we notice that the metrics \(\{\omega(t, J_t)\}\) change smoothly and hence \(C^{\alpha, \beta}(X, D; J_t) = C^{\alpha, \beta}(X, D; J_t)\). This key observation allows us to identify the space \(C^{\alpha, \beta}(X, D, J_t)\) and \(C^{\alpha, \beta}(X, D; J_t)\) via the following simple way. Let us fix a family of background conical Kähler metrics:
\[
\hat{\omega}_t = \omega_t + \epsilon \sqrt{-1} \partial \bar{\partial} h_t \chi|s_{D_t}|^2 h_t,
\]
with \(\gamma = 1 - \frac{1 - \beta}{\mu} \in (0, 1)\) being fixed and \(0 < \epsilon \ll 1\). Then we define a linear map:
\[
Q_{t, \beta} : C^{\alpha, \beta}(X, D; J_t) \rightarrow C^{\alpha, \beta}(X, D; J_t)
\]
\[
(\phi) \mapsto (-\Delta_{\omega_t} + 1)^{-1} \circ (-\Delta_{\omega_t} + 1) \phi.
\]
Since ker\((-\Delta_{\omega_t} + 1) = \{0\}\) by the proof of [Don12a] Proposition 8], it follows from Donaldson’s Schauder estimate in [Don12a Section 4.3] that \(Q_{t, \beta}\) is an isomorphism for \(|t| \ll 1\) and \(\beta' \in (\beta - \epsilon, \beta + \epsilon)\) with \(0 < \epsilon \ll 1\). Also using the explicit parametrix constructed in [Don12a Section 3], \(Q_t\) gives rise to a continuous local linear trivialization of the family of subspace \(C^{\alpha, \beta}(X, D; J_t) \subset C^{\alpha, \beta}(X, D; J_t)\). Denoting \(\tilde{\varphi}(t, \beta) = Q_{t, \beta}^{-1}(\varphi(t, \beta))\), we can calculate:
\[
\frac{\partial F(t, \beta, Q_{t, \beta}(\varphi))}{\partial \tilde{\varphi}} (0, \beta, \varphi_{\text{KE}}) = (\Delta_{\omega_{\text{KE}}} + \beta) \circ Q_{t, \beta} \phi = (\Delta_{\omega_{\text{KE}}} + \beta) \phi.
\]
which is invertible by \cite[Theorem 2]{Don12a} since there exists no holomorphic vector field on the pair \((X_0, D_0)\) (see \cite[Theorem 2.1]{SW12}). Now we can apply effective implicit function theorem as in \cite[Section 4.4]{Don12a} to the map \(F(t, \beta, Q(t, \beta, \cdot)) : C_{2, \alpha}^2(X, D; J_0) \rightarrow C_{2, \alpha}^2(X, D; J_0)\) to get a continuous family of solutions \(\tilde{\omega}_{\text{KE}}(t, \beta')\) to the equation \(F(t, \beta', Q(t, \beta', \cdot)) = 0\) for \(|t| < 1\) and \(\beta' \in (\beta - \epsilon, \beta + \epsilon)\) with \(0 < \epsilon < 1\). Since the argument for this last statement is standard, we will only sketch its proof. For a fixed \(\beta\) by the usual implicit function theorem we first get a family of solutions \(\varphi_{\text{KE}}^{(1)}(t, \beta)\) to the equation \(F(t, \beta, Q(t, \beta, \varphi_{\text{KE}}^{(1)})) = 0\) for \(|t| < 1\). Then we can apply Donaldson’s argument of deforming cone angles in \cite[Section 4.4]{Don12a} in a family version to further get \(\varphi_{\text{KE}}^{(1)}(t, \beta')\) for any \(|t| < 1\) and \(\beta' \in (\beta - \epsilon, \beta + \epsilon)\). More precisely, let \(\omega_{\text{KE}}(t, \beta) = \omega_{FS} + \sqrt{-1} \partial \bar{\partial} \varphi_{\text{KE}}^{(1)}(t, \beta)\) be the continuous family of \(C_{2, \alpha}^2\) conical Kähler-Einstein metric obtained earlier. For each \(\beta' \in (\beta - \epsilon, \beta + \epsilon)\) and \(t\) near 0, we define the new reference metric \(\omega(t, \beta') := \omega_{\text{KE}}(t, \beta) + \sqrt{-1} \partial \bar{\partial} \psi(t, \beta')\) where \(\psi(t, \beta') = \frac{1}{2} \log(1 + \|s_{D_1}\|^2/m - \|s_{D_2}\|^2/m)\) \(\in \mathcal{C}_{2, \alpha}^2\) is a smooth extension of Hermitian metric determined by \(h_{FS} \exp(-\varphi_{\text{KE}}^{(1)}(t, \beta))\) on \(-mK_X|_{D_i}\) (using the fact that \(\varphi_{\text{KE}}^{(1)}(t, \beta)\) is “smooth in tangential directions” by \cite[Section 4.3]{Don12a}). Then as in the proof of \cite[Proposition 7]{Don12a}, one can show that

\begin{enumerate}
  \item \(k_{\beta'} := |s_{D_1}|^2/|s_{D_2}|^2/m \frac{1}{1/m} \omega(t, \beta')^n\) satisfies \(|k_{\beta'} - 1|_{C, build} \rightarrow 0\) as \(\beta' \rightarrow \beta\).
  \item If \(\Delta_{\beta'}\) denotes the Laplace operator associated to \(\omega(t, \beta')\), then \(\Delta_{\beta'} + \beta'\) is invertible and the operator norm of its inverse is bounded by a fixed constant independent of \(\beta'\) and \(t\) near 0.
\end{enumerate}

So the effective version of implicit function theorem allows us to get a continuous family of solutions \(\varphi_{\text{KE}}^{(1)}(t, \beta')\). In fact, notice that \(\omega(t, \beta') = \omega_{FS} + \sqrt{-1} \partial \bar{\partial} \psi(t, \beta')\) where \(\psi(t, \beta') = \frac{1}{2} \log(1 + \|s_{D_1}\|^2/m - \|s_{D_2}\|^2/m)\) is an “almost solution” of the conical Kähler-Einstein equation by the item (1) above and is continuous with respect to both \(t\) and \(\beta'\). By item (2) and the effective implicit function theorem, we then know that the difference between the actual solution \(\varphi_{\text{KE}}^{(1)}(t, \beta')\) and \(\psi(t, \beta')\) approaches 0 in \(C^2, \alpha^2\) norm as \(\beta' \rightarrow \beta\). As a consequence, \(\varphi_{\text{KE}}^{(1)}(t, \beta')\) is continuous at \(\beta' = \beta\) in \(C^2, \alpha^2\)-norm with respect to both \(\beta'\) and \(t\). Noting that the argument above does not depend on the origin 0 and \(\beta\) we choose, hence \(\varphi_{\text{KE}}^{(1)}(t, \beta') = Q(t, \beta', \varphi_{\text{KE}}^{(1)}(t, \beta'))\) is continuous with respect to all \(t \in \Delta\) and \(\beta' \in (\beta - \epsilon, \beta + \epsilon)\).

By using the complex Monge-Ampère equation in \cite{Ko06} or \cite{Ti10}, we see that the family of volume forms \(\omega_{\text{KE}}(t, \beta')^n\) on the fixed smooth manifold \(X\) is continuous in \(L^p(X)\) for \(p \in [1, 1/(1 - \beta')]\) with respect to \(\beta'\) and \(t\) and since the family of matrices of \(L^2\)-inner products \(A_{\beta'}(t, \beta') := (s_{i}, s_{j})_{\varphi_{\text{KE}}^{(1)}}(t)\) is also continuous with respect to \(t\) and \(\beta'\). So Tian’s embeddings \(T(X, D; \omega(t, \beta'))\) determined by \(\{A_{\beta'}^{1/2}(t, \beta') \circ s_{\iota}(t)\}_{j=0}^{N}\) indeed produce a continuous family of Hilbert points inside \(\mathbb{H}^{X, N}\).

Let \(\{(X_i, D_i)\}\) be a sequence of smooth Fano pairs with a fixed Hilbert polynomial \(\chi\) and \(D_i \in \{-mK_X\}\). Suppose each \(X_i\) admits a unique conical Kähler-Einstein form \(\omega(i, \beta_i)\) solving

\[\text{Ric}(\omega(i, \beta_i)) = \beta_i \omega(i, \beta_i) + \frac{1 - \beta_i}{m} [D_i] \text{ on } X_i\]

with \(\inf \beta_i \geq \epsilon > 0\), we define

\[T_i : (X_i, D_i; \omega(i, \beta_i)) \rightarrow \mathbb{P}^N\]

to be the Tian’s embedding with respect to \(\omega(i, \beta_i)\) for sufficiently large \(N\) depending only on \(\epsilon\), \(m\) and the fixed Hilbert polynomial \(\chi\), and let \(\text{Hilb}(X_i, (1 - \beta_i)D_i) \in \mathbb{H}^{X_i, N} \times \mathbb{H}^{X_i, N}\) denote the Hilbert point corresponding to the Tian’s embedding of \(X_i\) with respect to \(\omega(i, \beta_i)\). Then we have

**Lemma 4.9.** Let \((X, D) \subset \mathbb{P}^N\) be a log \(\mathbb{Q}\)-Fano pair with the same Hilbert polynomial \(\chi\) and \(D \in \{-mK_X\}\). Suppose \((X, D)\) admits a weak conical Kähler-Einstein form \(\omega(\beta)\) with \(\beta = \lim_{i \rightarrow \infty} \beta_i\) solving

\[\text{Ric}(\omega(\beta)) = \beta \omega(\beta) + \frac{1 - \beta}{m} [D] \text{ on } X\]

Then

\[(X_i, D_i; \omega(i, \beta_i)) \underset{GH}{\rightarrow} (X, D; \omega(\beta)) \text{ as } i \rightarrow \infty\]
for a conical Kähler-Einstein metric $\omega'(\beta)$ is equivalent to the following statement: there is a sequence of $\{g_i\} \subset U(N+1)$ such that

$$g_i \cdot \text{Hilb}(X, (1 - \beta_i)D_i) \rightarrow \text{Hilb}(X, (1 - \beta)D) \in \mathbb{P}^N$$

as $i \to \infty$.

where $\text{Hilb}(X, (1 - \beta)D)$ denote the Hilbert point of Tian's embedding $T : (X, D; \omega(\beta)) \to \mathbb{P}^N$ for a fixed basis $\{s_i\}$.

Proof. This follows directly from Theorem 4.1 which is in turn from the works of CDS15b, CDS15c, and Ta12, Ta15. Indeed let us assume that $(X, D; \omega(i, \beta_i)) \cong \text{Hilb}(X, (1 - \beta)D)$ where we can assume the limit exists by Theorem 4.1. Then by Theorem 4.1(3) $T_i(X, D_i) \to (T'_\infty(X), T'_\infty(D))$ where $T_i$ (resp. $T'_\infty$) is given by Tian's embedding determined by an orthonormal basis of $H^0(X_i, -mK_{X_i})$ (resp. $H^0(X, -mK_X)$) with respect to $\omega(i, \beta_i)$ (resp. $\omega'(\beta)$). Assume $\omega(\beta)$ is also a conical Kähler-Einstein metric on $(X, D)$. By the uniqueness of conical Kähler-Einstein metrics proved in BBE+11, there exists a holomorphic automorphism $\sigma \in \text{Aut}(X, D)$ such that $\sigma^* \omega(\beta) = \omega'(\beta)$. Moreover because $\sigma$ lifts to $\text{Aut}(X, -mK_X)$, there is a unitary isomorphism between $(H^0(X, -mK_X), \|\cdot\|_{\omega(\beta)}^2)$ and $(H^0(X, -mK_X), \|\cdot\|_{\omega'(\beta)}^2)$ where $\|\cdot\|_{\omega(\beta)}^2$ is the inner product induced by $\omega(\beta)$ (resp. $\omega'(\beta)$). Via this isomorphism, we have $T_i(X_i, D_i) \to (T'_\infty(X), T'_\infty(D))$ where $T'_\infty$ is given by Tian's embedding determined by an orthonormal basis of $H^0(X, -mK_X)$ with respect to $\omega(\beta)$. Now the statement of the lemma holds because the orthonormal basis of a unitary vector space is defined only up to $U(N + 1)$ ambiguity. \qed

5. Strong uniqueness for $0 < \beta \ll 1$

In this section, we will give a completely algebraic proof of the fact that when the angle $\beta > 0$ is sufficiently small, then there is a unique filling.

Proposition 5.1. For a fixed finite set $I \subset [0, 1]$, there exists a number $\beta_I > 0$ such that if $(X, (1 - \beta_I)D)$ is a klt pair, $D$ is $\mathbb{R}$-Cartier and the coefficients of $\mathbb{R}$-divisor $D$ are contained in $I$, then $(X, D)$ is log canonical.

Proof. By [HMX14] Theorem 1.1, we know that for the set of all $n$-dimensional log pairs $(X, D)$ satisfying the property that $D$ is a $\mathbb{R}$-divisor and its coefficients are contained in $I$, the set of log canonical thresholds

$$\{\text{lt}(X, D) | X \text{ is } n \text{ dimensional, the coefficients of } D \text{ are in } I\}$$

satisfies the ascending chain condition (ACC). In particular, there exists a maximum $\beta_I$ among all log canonical thresholds which are strictly less than $1$.

Then we know that if $(X, (1 - \beta)D)$ is klt and $D$ is $\mathbb{Q}$-Cartier, $(X, D)$ is log canonical, since otherwise, we will have a pair whose log canonical threshold is in $(1 - \beta_I, 1)$, which is a contradiction. \qed

Let $X \to C$ be a flat family of $\mathbb{Q}$-Fano varieties over a smooth pointed curve $0 \in C$. We assume $X$ is $\mathbb{Q}$-Gorenstein. Fix $m > 1$ and $D \sim_C -mK_X$ is a divisor such that for every $t \in C$, the fiber $(X_t, \frac{1}{m}D_t)$ is klt. For instance, we can choose $m$ sufficiently divisible such that $| -mK_X|$ is relatively base point free over $C$ and $D \sim_C -mK_X$ to be a divisor in the general position. In particular, $D_t$ is smooth for $t \in C^o$.

Theorem 5.2. Let

$$\beta_0 := \min \left\{ \beta_I, \frac{1}{m + 1} \right\}$$

with $\beta_I$ being given in Proposition 5.1 for the set $I = \{ \frac{1}{m}q | q = 1, 2, ..., m \}$. For any fixed $\beta \in (0, \beta_0]$, suppose $(X', D') \to C$ is another flat family with $K_{X'} + \frac{1}{m}D'$ being $\mathbb{Q}$-Cartier and satisfies

$$(X', D') \times_C C^o \cong (X, D) \times_C C^o$$

and $(Y_0, \frac{1}{m}E_0) := (X'_0, \frac{1}{m}D'_0)$ being $\mathbb{Q}$-Fano. Then the above isomorphism can be extended to an isomorphism

$$(X', D') \cong (X, D).$$
Proof. Since \((X_t, D_t)\) is smooth for \(t \in C^0\), the coefficients of \(\frac{1}{m} E_\beta\) lies in the set \(\{ \frac{q}{m} \mid q \in \mathbb{N} \}\). By our assumption that \((Y_\beta, \frac{1}{m} E_\beta)\) is klt and \(\beta \leq \beta_0 \leq \frac{1}{m+1}\), we have

\[
\frac{1}{m+1} \leq 1 - \frac{\beta}{m} \quad \text{and} \quad 1 - \frac{\beta}{m} c_i < 1 \quad \text{hence} \quad c_i < m + 1
\]

where \(E_\beta = \sum_i c_i E_{\beta,i}\) with \(E_{\beta,i}\) being a prime divisor for each \(i\). Hence the coefficients of \(\frac{1}{m} E_\beta\) must lie in \(I = \{ \frac{q}{m} \mid q = 1, 2, \ldots, m \}\). By our assumption of \(\beta \in (0, \beta_0) \subset (0, \beta_1]\), we know that \((Y_\beta, \frac{1}{m} E_\beta)\) is log canonical by Proposition [5.1]. Furthermore, since \(Y_\beta\) is irreducible, we know that

\[
K_{X'} + \frac{1}{m} D' \sim_{\mathbb{Q}, C} 0
\]

as this holds over \(C^0\).

Let \(W\) be a common resolution

\[
\begin{tikzcd}
W \\
\mathcal{X} \arrow{rd}{p} \\
\mathcal{X}' \arrow{ru}{q}
\end{tikzcd}
\]

that is an isomorphism over \(C^0\). If the birational morphism \(\phi\) extends to a birational morphism \(\mathcal{X}_0 \sim \mathcal{X}'\), then

\[
q^* K_{X'} \sim_{\mathbb{Q}, C} p^* K_X
\]

as \(\phi\) is an isomorphism in codimension one, which implies

\[
\mathcal{X} = \text{Proj} \bigoplus_{r=0}^{\infty} \mathcal{O}_W(-rp^* K_{X'/C}) = \text{Proj} \bigoplus_{r=0}^{\infty} \mathcal{O}_W(-rq^* K_{X'/C}) = \mathcal{X}'
\]

as both \(\mathcal{X}_0\) and \(\mathcal{X}'\) are \(\mathbb{Q}\)-Fano and we are done already. So from now on we assume \(\mathcal{X}_0 \not\sim_{\mathbb{Q}} Y_\beta\).

Now let us write

\[
(16) \quad p^*(K_X + \frac{1}{m} D) + a_0 Y_\beta + \sum a_i E_i = K_W + \frac{1}{m} p^* D.
\]

Since \((\mathcal{X}_0, \frac{1}{m} D_0)\) is klt, this implies that \((\mathcal{X}, \frac{1}{m} D + \mathcal{X}_0)\) is plt near \(\mathcal{X}_0\) by inversion of adjunction [KM98 Theorem 5.50]. Hence for any divisor \(F\) whose center is contained in \(\mathcal{X}_0\) we have

\[
-1 < a(F, \mathcal{X}, \frac{1}{m} D + \mathcal{X}_0) = a(F, \mathcal{X}, \frac{1}{m} D) - v_F(\mathcal{X}_0) \leq a(F, \mathcal{X}, \frac{1}{m} D) - 1
\]

i.e. \((\mathcal{X}, \frac{1}{m} D)\) is terminal along \(\mathcal{X}_0\). Therefore, \(a_0 > 0\) and \(a_i > 0\). Similarly,

\[
(17) \quad q^* (K_{X'} + \frac{1}{m} D') + b_0 \mathcal{X}_0 + \sum b_i E_i = K_W + \frac{1}{m} q^* D',
\]

and we have \(b_0, b_i \geq 0\) because \((Y, \frac{1}{m} E)\) is log canonical thanks to Proposition [5.1] and our choice of \(\beta\). Since the right hand sides of \((16)\) and \((17)\) are equal to each other by \((15)\), \(\mathcal{X}_0 \not\sim_{\mathbb{Q}} Y_\beta\), and both \(K_{X'} + \frac{1}{m} D\) and \(K_{X'} + \frac{1}{m} D'\) are \(\mathbb{Q}\)-linearly equivalent to a relatively trivial divisor over \(C\), these imply there is a constant \(c \leq 0\) such that

\[
a_0 Y_\beta + \sum a_i E_i = b_0 \mathcal{X}_0 + \sum b_i E_i + c \cdot W_0.
\]

By comparing the coefficients of \(Y_\beta\) on both sides, we see \(c > 0\); but by comparing the coefficients of \(\mathcal{X}_0\) on both sides, we see \(c \leq 0\). This contradiction implies that \(\mathcal{X}' = \mathcal{X}\). \(\square\)

**Remark 5.3.** If \(m = 1\), the pair we get is plt instead of klt. The above argument indeed also applies to this case.

A similar uniqueness statement is observed in [Oda12, 4.3] and the above argument indeed gives a straightforward proof of it.

We also notice that the automorphism group \(\text{Aut}(X, D)\) is always finite by the following well known fact.

**Lemma 5.4.** Let \((X, D)\) be a klt pair such that \(-K_X\) is ample and \(D \sim_{\mathbb{Q}} -K_X\). Then \(\text{Aut}(X, D)\) is finite.
Proof. We can choose a sufficiently small $\epsilon > 0$ such that $(X, (1 + \epsilon)D)$ is klt and we know $K_X + (1 + \epsilon)D$ is ample. As $\text{Aut}(X, D)$ preserves $K_X + (1 + \epsilon)D$, so it gives polarized automorphisms. Therefore, to prove it is finite, we only need to show that it does not contain $\mathbb{G}_m$ or $\mathbb{G}_a$ as a subgroup. For $\mathbb{G}_m$ this follows from [HX16, Lemma 3.4]. As mentioned there, the same argument also works for $\mathbb{G}_a$ verbatim.

6. Continuity method

In this section, we will develop our continuity method which serves as the main technique of the proof of the main result. Let $(C, 0)$ be a smooth pointed curve, we define $C^\circ := C \setminus \{0\}$ as before. To begin with, let us fix $B \in (0, 1]$ and we will assume the nearby smooth fibers are all $B$-$K$-polystable for the rest of this section. We fix an $\epsilon \in (0, \beta_0)$, with $\beta_0$ being given as in Theorem 5.2. By Lemma 2.4 for any $\beta \in \epsilon, B$, $(X_t, D_t)$ is $\beta$-$K$-polystable. Applying [CDS15a, CDS15b, CDS15c, Tia12] (cf. Corollary 4.2), we conclude that $(X_t, D_t)$ conical Kähler-Einstein metric with cone angle $2\pi(1 - (1 - \beta)/m)$ along $D_t$ for all $t \in C^\circ$ near 0. This leads us to introduce the following notion.

Definition 6.1. We say

$$
\begin{align*}
\{(X, D; \mathcal{L}) &\longrightarrow (\mathcal{P}E; \mathcal{O}_{\mathcal{P}E}(1)) \\
\pi &
\end{align*}
$$

is a Kähler-Einstein degeneration of index $(r, B)$ if for any $\beta \in \epsilon, B$

1. $D \in \mathbb{R}K_X$;
2. $\mathcal{L} = K_X^{\circ-r}$ is relatively very ample and $\mathcal{E} = \pi_* \mathcal{L}$ is locally free of rank $N + 1$;
3. $\forall t \in C$, $(X_t, 1 \mathcal{L}_t)$ is klt and $(X_t, D_t)$ is a smooth Fano pair for $\forall t \in C^\circ$;
4. For $\beta < 1$ and $\forall t \in C^\circ$, $(X_t, D_t)$ admits a unique Kähler form $\omega(t, \beta) \in C^{\alpha, \beta}$ in the sense of [Don12a] solving

$$
\text{Ric}(\omega(t, \beta)) = \beta \omega(t, \beta) + \frac{1 - \beta}{m} [D_t] \text{ on } X_t.
$$

Moreover, $\omega(t, \beta)$ gives rise to $r$-th Tian’s embedding

$$
T : (X_t, D_t; \omega(t, \beta)) \longrightarrow \mathbb{P}^N.
$$

By Theorem 4.1 there is a uniform $r = r(X, D)$ being independent of $\beta \in \epsilon, B$ such that all Gromov-Hausdorff limits of subsequences of the family $\{(X_t, D_t; \omega(t, \beta))\}_{t \in C}$, $\beta \in \epsilon, B$ can be embedded in to $\mathbb{P}^N$.

Definition 6.2. Let us continue with the notation as above and define

$$
B_\epsilon(X, D) := \left\{ \beta \in \epsilon, B \mid \begin{array}{l}
\text{Ric}(\omega(\beta)) = \beta \omega(\beta) + \frac{1 - \beta}{m} [D] \text{ on } X.
\end{array} \right\}
$$

and we fix $T$ such that $\epsilon \leq T \leq \sup\{\sigma \in \epsilon, B \mid \epsilon, \sigma \in B_\epsilon(X, D)\}$.

By Theorem 4.1 the Gromov-Hausdorff limit of any subsequence of $(X_t, D_t; \omega(t, \beta))$ is a Q-Fano $Y$ together with a Q-Cartier divisor $E$ such that $(Y, \frac{1 - \beta}{m} E)$ is log Fano.

Lemma 6.3. $B_\epsilon(X, D) \supset \epsilon, \beta_0$.

Proof. After shrinking $C$ if necessary, we may choose a holomorphic basis

$$
\{s_i(t)\}_{i=0}^N \subset \Gamma(\Delta, \pi_* O_X(-TK_{X/\Delta}))
$$

for the family $X \to C$ as in Section 7.2 which gives rise to an algebraic arc

$$
\begin{align*}
z : C &\longrightarrow \mathbb{P}^{X;N} \times C \\
t &\mapsto (\text{Hilb}(X_t, D_t), t).
\end{align*}
$$
For this arc, we know that Hilb\((Y, E)\) for the Gromov-Hausdorff limit \((Y, E; \omega_Y)\) of any subsequence \(\{(X_t, D_t; \omega_{KE}(t, \beta))\}_{t \to 0}\) lies in the fiber over \(0 \in C\) of the morphism

\[
\begin{align*}
\text{SL}(N + 1) \cdot \text{Im}z & \longrightarrow \mathbb{H}^{X;N} \times C \\
\pi_C & \downarrow \\
C & \longrightarrow C.
\end{align*}
\]

By choosing an arc \(\tilde{C} : C \to \text{SL}(N + 1) \cdot \text{Im}z\) that passes through Hilb\((Y, E)\) and dominates \(C\), and comparing the universal family over \(\text{Im}z \subset \mathbb{H}^{X;N} \times C\) with the pull-back family induced by the map \(\pi_C : \tilde{C} \to C\), we conclude that \((Y, E) = (X, D)\) as long as \(\beta \leq \beta_0\) thanks to Theorem 5.2. Our proof is thus completed. \(\square\)

**Remark 6.4.** Notice that Lemma 6.3 implies that for \(\beta \in [0, \beta_0]\), \((X, D)\) is actually \(\beta\)-K-stable (see Lemma 5.4), which can also be proved by using Theorem 5.2 and a verbatim extension of the theory of special test configuration developed in [LX14] to the log setting. In fact, using the latter approach, we can indeed conclude a pair \((X_0, D_0)\) is \(\beta\)-K-stable if \(D_0 \sim -mK_{X_0}\), \((X_0, \frac{1}{m}D_0)\) is klt and \(\beta \in [0, \beta_0]\), without assuming \(X_0\) is smoothable. However, this stronger fact is not needed for the rest of the paper.

From now on, let us assume \((X_0, D_0)\) is \(\mathfrak{B}\)-K-polystable, we are going to show that \(\mathfrak{B}_\epsilon(X, D)\) is both open and closed in the set \([\epsilon, \mathfrak{B}]\), or equivalently we can choose

\[
T = \mathfrak{B} = \max_{[\epsilon, \mathfrak{B}] \subset \text{Hilb}(X, (1 - \beta)D)} \{\sigma\}.
\]

To do this, we first define a map

\[
\tau : [\epsilon, \mathfrak{B}] \times C^0 \longrightarrow \mathbb{H}^{X;N} \quad \text{cf. see Definition 4.6}
\]

Then we have

**Lemma 6.5.** \(\tau_{|[\epsilon, \mathfrak{B}] \times C^0}\) is continuous.

**Proof.** By Proposition 4.8, \(\tau(\cdot, \cdot)\) is continuous with respect to \((\beta, t)\) on \([\epsilon, \mathfrak{B}] \times C^0\). By Theorem 4.1, the Gromov-Hausdorff limit of \((X_t, D_t; \omega(t, \beta))\) for any sequence \(\beta_i \not\sim \mathfrak{B}\) is \(\mathfrak{B}\)-K-polystable and lies in \(\text{SL}(N + 1) \cdot X_t\). On the other hand, since \((X_t, D_t)\) is \(\mathfrak{B}\)-K-polystable, this implies the limit must lie in \(U(N + 1) \cdot \text{Hilb}(X_t, (1 - \mathfrak{B})D_t)\), hence the metrics \(\{h_{KE}(t, \beta)\}_{(t, \beta) \in [\epsilon, \mathfrak{B}] \times C^0}\) vary continuously for \((\beta, t) \in [\epsilon, \mathfrak{B}] \times C^0\). So \(\tau(\cdot, t)\) is also continuous at \(\beta = \mathfrak{B}\) with respect to the basis \(\{s_i\}\) in Definition 4.6. Thus the proof is completed. \(\square\)

By Lemma 6.3, we know that the continuity of \(q \circ \tau\) can be extended to \([\epsilon, \beta_0] \times \{0\}\), where \(q : \mathbb{H}^{X;N} \to \mathbb{H}^{X;N} / U(N + 1)\) is the natural quotient morphism, which is continuous with respect to the quotient topology on \(\mathbb{H}^{X;N} / U(N + 1)\). Next we will show indeed \(\beta\)-continuity of \(q \circ \tau\) can be extended to \([\epsilon, T] \times \{0\}\) (i.e. including the central fiber) as long as \(q \circ \tau\) can be continuously extended to \([\epsilon, T] \times C\) based on the fact that \((X, D)\) is a degeneration of smooth pairs \((X_t, D_t)\) admitting conical Kähler-Einstein metrics \(\omega(t, \beta)\) for any \(\beta \in [\epsilon, T]\). To do that, let us prefix a **continuous** distance function on \(\mathbb{H}^{X;N}\)

\[
\text{dist}_{\mathbb{H}^{X;N}} : \mathbb{H}^{X;N} \times \mathbb{H}^{X;N} \longrightarrow \mathbb{R}_{\geq 0}.
\]

**Lemma 6.6.** Let us continue with the above setting. In particular, \((X, D) = (X_0, D_0)\) is \(\mathfrak{B}\)-K-polystable. Then \((X, D)\) admits a conical Kähler-Einstein metric \(\omega_X(T)\) with angle \(2\pi(1 - (1 - T)/m)\) along the divisor \(D\).

Furthermore, for any sequence \(\{\beta_i\} \subset (\epsilon, T)\) satisfying \(\beta_i \not\sim T\), we have

\[
\text{dist}_{\mathbb{H}^{X;N}}(\text{Hilb}(X, (1 - \beta_i)D), U(N + 1) \cdot \text{Hilb}(X, (1 - T)D)) \longrightarrow 0,
\]

where \(\text{Hilb}(X, (1 - T)D)\) is the Hilbert point corresponding to the cycle obtained via Tian’s embedding of \((X, D; \omega_X(T))\).
Proof. By Theorem \[4.1\] and the definition of \( T \), for any \( \beta < T \), the Gromov-Hausdorff limit as \( t \to 0 \) of \( (X_t, \mathcal{D}_t; \omega(t, \beta)) \) converges to a weak conical Kähler-Einstein metric on \( (X, D; \omega(\beta)) = (X_0, \mathcal{D}_0; \omega(0, \beta)) \). This implies that for each fixed \( \beta_i < T \), there is a \( C^0 \) \( \ni t_i \to 0 \) so that
\[
\text{dist}_{\text{GH}}(\text{Hilb}(X_{t_i}, (1 - \beta_i)D_{t_i}), U(N + 1) \cdot \text{Hilb}(X, (1 - \beta_i)D)) < 1/i .
\]

It follows from Theorem \[4.1\] that any subsequence of \( \{(X_{t_i}, \mathcal{D}_{t_i}; \omega(t_i, \beta_i))\} \), there is a Gromov-Hausdorff convergent subsequence. Now suppose there is a subsequence
\[
(X_{t_{i_k}}, \mathcal{D}_{t_{i_k}}; \omega(t_{i_k}, \beta_{i_k})) \xrightarrow{\text{GH}} (Y, E; \omega_Y(T)) \quad \text{as} \quad k \to \infty,
\]
from which we obtain there are \( g_{i_k} \in U(N + 1) \) such that
\[
g_{i_k} \cdot \text{Hilb}(X_{t_{i_k}}, (1 - \beta_{i_k})D_{t_{i_k}}) \longrightarrow \text{Hilb}(Y, (1 - T)E),
\]
where \( \text{Hilb}(Y, (1 - T)E) \) is the Hilbert point corresponding to the Tian’s embedding of \( (Y, E) \) using the limiting conical Kähler-Einstein metric \( \omega_Y(T) \) of angle \( 2\pi(1 - (1 - T))/m \) along a Q-Cartier divisor \( E \). In particular, \( (Y, E) \) is \( T \)-K-polystable by \[8\text{Be12} \text{ Theorem 4.2}\]. On the other hand, by \[22\] we have
\[
\text{Hilb}(Y, (1 - T)E) \in \text{SL}(N + 1) \cdot \text{Hilb}(X, D) \subset \mathbb{H}^{X \times N},
\]
Suppose \( (Y, E) \not\cong (X, D) \), then by \[\text{Don12b} \text{ Proposition 1}\] there is a test configuration of \( (X, D) \) with central fiber \( (Y, E) \) and vanishing generalized Futaki invariant since \( (Y, E) \) is \( T \)-K-polystable. This contradicts our assumption that \( (X, D) \) is \( T \)-K-polystable. Hence we must have \( (Y, E) \cong (X, D) \). In particular, \( X \) admits a weak conical Kähler-Einstein metric with angle \( 2\pi(1 - (1 - T)) \) along \( D \).

In conclusion, we have
\[
(X_{t_{i_k}}, \mathcal{D}_{t_{i_k}}; \omega(t_{i_k}, \beta_{i_k})) \xrightarrow{\text{GH}} (X, D; \omega_X(T)),
\]
which implies
\[
\text{dist}_{\text{GH}}(\text{Hilb}(X, (1 - \beta_i)D), U(N + 1) \cdot \text{Hilb}(X, (1 - \beta_i)D)) \longrightarrow 0 .
\]
Combining with \[22\], the proof is completed. \( \Box \)

Remark 6.7. Notice that in the argument above, the existence of the conical Kähler-Einstein metric on \( X_t \) is needed only for an angle \( \beta_i < T \) instead of \( T \). So the proof remains valid by only assuming that \( X_t \) is \( T \)-K-semistable for any \( t \in C^0 \) instead of being \( T \)-K-polystable.

An immediate consequence is the following.

Corollary 6.8. \( \text{Aut}(X, D) \) is finite. If \( T = 1 \), \( \text{Aut}(X) \) is reductive.

Proof. The first part is just Lemma \[5.4\] The second part follows from \[\text{CDS15c} \text{ Theorem 6}\] thanks to the existence of weak Kähler-Einstein metric on \( X \). \( \Box \)

Let
\[
\mathcal{BO} := \lim_{t \to 0} \text{SL}(N + 1) \cdot \text{Hilb}(X_t, D_t) \subset \mathbb{H}^{X \times N},
\]
denote the limiting orbit and
\[
O_{\text{Hilb}(X, (1 - T)D)} = \text{SL}(N + 1) \cdot \text{Hilb}(X, (1 - T)D) \quad \text{and} \quad O_{\text{Hilb}(X, (1 - T)D)} \subset \mathbb{H}^{X \times N}
\]
be the \( \text{SL}(N + 1) \)-orbit of \( \text{Hilb}(X, (1 - T)D) \) and its closure. By Corollary \[6.8\] this allows us to construct an \( \text{SL}(N + 1) \)-invariant Zariski open neighborhood
\[
\text{Hilb}(X, (1 - T)D) \subset U \subset \mathbb{H}^{X \times N}
\]
satisfying the condition \[3\] in Lemma \[3.1\]. We want to remark that the open neighborhood \( U \) is independent of \( T \) (cf. part (1) of Remark \[4.7\]).

Then we have the following

Lemma 6.9. Let \( \{t_i\} \subset C \) be a sequence of points approaching \( 0 \in C \) and
\[
\{\beta_i\}, \{\beta_i^1\}, \{\beta_i^2\} \subset [\varepsilon, 1]
\]
be three sequences satisfying \( \beta_i^1 < \beta_i < \beta_i^2 \) for all \( i \).
(1) Assume $\beta_i \to T, \beta_i^* \to T$ and that there is a sequence $\{(X_{t_i}, D_{t_i}) | (X_{t_i}, D_{t_i})$ being $\beta_i$-K-polystable\} with $t_i \to 0$ such that

\[
\text{Hilb}(X_{t_i}, (1 - \beta_i^*)D_{t_i}) \xrightarrow{i \to \infty} U(N + 1) \cdot \text{Hilb}(X, (1 - T)D)
\]

and for $g_i \in U(N + 1)$

\[
g_i \cdot \text{Hilb}(X_{t_i}, (1 - \beta_i^*)D_{t_i}) \xrightarrow{i \to \infty} \text{Hilb}(Y, (1 - T)E).
\]

Then $\text{Hilb}(Y, (1 - T)E) = g \cdot \text{Hilb}(X, (1 - T)D)$ for some $g \in U(N + 1)$.

(2) Assume $\beta_i^* \not\sim T$ and that for any fixed $i$, there is a $g_i \in U(N + 1)$ such that

\[
\text{Hilb}(X_{t_i}, (1 - \beta_i^*)D_{t_i}) \xrightarrow{i \to \infty} g_i \cdot \text{Hilb}(X, (1 - \beta_i^*)D)
\]

and

\[
\text{Hilb}(X_{t_i}, (1 - \beta_i^*)D_{t_i}) \xrightarrow{i \to \infty} \text{Hilb}(Y, (1 - T)E) \in \overline{BO} \setminus O_{\text{Hilb}(X,(1-T)D)}.
\]

If $(X, D) \not\cong (Y, E)$, then there exists a sequence $\{t_i\}$ satisfying $0 < \text{dist}_C(t_i', 0) < \text{dist}_C(t_i, 0)$ such that

\[
\text{Hilb}(Y', (1 - T)E') = \lim_{i \to \infty} \text{Hilb}(X_{t_i}, (1 - \beta_i^*)D_{t_i})
\]

\[
\in \left(O_{\text{Hilb}(X,(1-T)D)} \cup (U \cap \overline{BO})\right) \setminus O_{\text{Hilb}(X,(1-T)D)} \subset \mathbb{H}^{XN}.
\]

where $\text{dist}_C : C \times C \to \mathbb{R}$ is a fixed continuous distance function on $C$.

**Proof of Lemma 6.9** To prove part 1), one first notices that (26) together with Lemma 4.9 imply that $(X, D)$ is T-K-polystable. We will show that under the above assumption and

\[
\text{Hilb}(Y, (1 - T)E) \not\in U(N + 1) \cdot \text{Hilb}(X, (1 - T)D),
\]

then one can construct a new sequence $\{\beta''_i\}$ satisfying $\beta''_i \in [\beta_i^*, \beta_i]$ such that

\[
\text{Hilb}(Y', (1 - T)E') = \lim_{i \to \infty} \text{Hilb}(X_{t_i}, (1 - \beta''_i)D_{t_i})
\]

\[
\in \left(O_{\text{Hilb}(X,(1-T)D)} \cup (U \cap \overline{BO})\right) \setminus O_{\text{Hilb}(X,(1-T)D)} \subset \mathbb{H}^{XN}.
\]

On the other hand, Lemma 4.9 implies

\[
(X_{t_i}, D_{t_i}; \omega(t_i, \beta_{t_i}^*)) \xrightarrow{\text{GH}} (Y', E'; \omega_{Y'}(T)),
\]

thus $(Y', E')$ admits weak Kähler-Einstein metric with angle $2\pi(1 - 1/m)$ along $E'$ and hence T-K-polystable. These allow one to construct either a test configuration of $(X, D)$ with central fiber $(Y', E')$ and vanishing generalized Futaki invariant or a test configuration of $(Y', E')$ with central fiber $(X, D)$ and vanishing generalized Futaki invariant, contradicting to the fact that both $(X, D)$ and $(Y', E')$ are T-K-polystable. So we must have

\[
\text{Hilb}(Y, (1 - T)E) = g \cdot \text{Hilb}(X, (1 - T)D)
\]

for some $g \in U(N + 1)$.

Now we proceed to the construction of $\{\beta''_i\}$. Let

\[
B(\text{Hilb}(X, (1 - T)D), \epsilon_1) \subset U
\]

be the radius $\epsilon_1$ open balls with respect to the distance function (21) and $U$ be given as in (25).

By shrinking the pointed curve $(0 \in C)$ if necessary, we may assume that

\[
\text{Hilb}(X_{t_i}, (1 - \beta_i^*)D_{t_i}) \in U(N + 1) \cdot B(\text{Hilb}(X, (1 - T)D), \epsilon_1)
\]

for all $i$ thanks to our assumption (26). On the other hand, by our assumption that $(X, D) \not\cong (Y, E)$, and we may assume $(Y, E)$ is not in the closure of the orbit of $(X, D)$ (otherwise, we can just let $\beta''_i = \beta_i$), then there is an $\epsilon_1 > 0$ such that

\[
\text{dist}_{XN}(\text{Hilb}(X_{t_i}, (1 - \beta_i^*)D_{t_i}), O_{\text{Hilb}(X,(1-T)D)}) > \epsilon_1 \text{ for } i \gg 1.
\]

By the $\beta$-continuity of $\tau(\cdot, t_i)$ for each fixed $i \gg 1$, for any $0 < \epsilon < \epsilon_1$ there is $XW$: correction

\[
\beta_{i,k}'' = \sup \left\{ \beta \in (\beta_i'', \beta_i) \mid \tau(t_i, t_i) \in B(O_{\text{Hilb}(X,(1-T)D)}, \epsilon/2^k) \cup U(N + 1) \cdot B(\text{Hilb}(X, (1 - T)D), \epsilon_1) \right\}
\]
where $B(O_{\text{Hilb}}(X,(1-T)D),\varepsilon/2^k)$ is the $\varepsilon/2^k$-tubular neighbourhood of $O_{\text{Hilb}}(X,(1-T)D)$, that is, $\beta''_i$ is the smallest $\beta$ such that $\tau(-,t_i)$ escapes $B(O_{\text{Hilb}}(X,(1-T)D),\varepsilon/2^k) \cup U(N+1) \cdot B(Hilb(X,(1-T)D),\epsilon_1)$.

Clearly, we have $\beta''_{i,k+1} \leq \beta''_i$. Now if
\[
\tau(\beta''_{i,0}, t_i) \in SL(N+1) \cdot B(Hilb(X,(1-T)D),\epsilon_1)
\]
we let $\beta''_i = \beta''_{i,0}$, otherwise, we let $\beta''_i = \beta_{i,k}$ where $\beta''_{i,k}$ is the first number satisfying
\[
\tau(\beta''_{i,k}, t_i) \in SL(N+1) \cdot B(Hilb(X,(1-T)D),\epsilon_1).
\]
Such $k$ exists because of (30). Now by our construction, there is a $g_i \in SL(N+1)$ such that
\[
(32) \quad \tau(\beta''_i, t_i) \in g_i \cdot B(Hilb(X,(1-T)D),\epsilon_1).
\]
We let
\[
M_i = \inf \{ \text{Tr}(g^* g) \mid g \in SL(N+1) \text{ such that (32) is satisfied} \} + 1
\]
and by passing through a subsequence we may assume $\text{Tr}(g_i^* g_i) \leq M_i$. Then we have the following dichotomy:

**Case 1.** there is a subsequence $\{M_{i_j}\}$ such that $|M_{i_j}| < M$ for some constant $M$ independent of $i$. Then we claim that
\[
\{ \tau(\beta''_{i_j}, t_{i_j}) = Hilb(X_{i_j}, (1 - \beta''_{i_j})D_{i_j}) \}
\]
is the subsequence we want, and its limit $\text{Hilb}(Y', (1 - T)E')$ lies in
\[
(U \cap \overline{\mathcal{O}}) \setminus O_{\text{Hilb}}(X,(1-T)D).
\]
To see this, one only needs to notice that it follows from our construction of $\beta''_i$ that
\[
\text{dist}_{\mathbb{H}^{X':N}}(\tau(\beta''_{i_j}, t_{i_j}), O_{\text{Hilb}}(X,(1-T)D))
\]
is uniformly bounded from below by some $\varepsilon/2^k$, since there is a $k = k(M)$ such that
\[
|z| \leq \varepsilon/2^k \quad \text{and} \quad |g| < M \quad \text{for all} \quad z \in \mathbb{H}^{X':N}, g \in SL(N+1) \cdot U.
\]

**Case 2.** $|M_i| \to \infty$. If that happens, let us replace $\varepsilon$ by $\varepsilon/2$ in (31) and repeat the above process, if for the new sequence $\{M_{i_j}\} \subset \mathbb{R}$ there is a bounded subsequence $\{M_{i_j}\}$ then we reduce to the *Case 1*, otherwise, we keep on repeating this process. Then either we stop at a finite stage or this becomes an infinite process. If we stop at a finite stage, then we obtain our subsequence as before, if the process never terminates, we claim that we are able to extract a subsequence whose limit $\text{Hilb}(Y', (1 - T)E')$ lands in the boundary
\[
\partial O_{\text{Hilb}}(X,(1-T)D) = O_{\text{Hilb}}(X,(1-T)D) \setminus O_{\text{Hilb}}(X,(1-T)D).
\]
This is because by choosing a diagonal sequence we will have
\[
\text{dist}_{\mathbb{H}^{X':N}}(\tau(\beta''_{i_j,k}, t_{i_j}), O_{\text{Hilb}}(X,(1-T)D)) < \varepsilon/2^k \to 0,
\]
so we know
\[
z := \lim_{k \to \infty} \tau(\beta''_{i_j,k}, t_{i_j}) \in \overline{O_{\text{Hilb}}(X,(1-T)D)}.
\]
On the other hand, if $z \in O_{\text{Hilb}}(X,(1-T)D)$, then
\[
z = g \cdot \text{Hilb}(X,(1-T)D)
\]
for some $g \in SL(N+1)$. In particular, $g \cdot B(Hilb(X,(1-T)D),\epsilon_1)$ contains a neighborhood of $z$. However, this violates the assumption that $|M_{i_j}| \to \infty$ as $k \to \infty$. Hence our proof is completed.

The proof of part 2) is similar. Contrast to the part 1), we will vary $t$ instead of $\beta$ in $\tau(\beta,t)$. First by our assumption (29) together with Lemma 4.9, $(Y,E)$ is $T$-K-polystable hence
\[
\text{Hilb}(Y,(1-T)E) \notin \partial O_{\text{Hilb}}(X,(1-T)D).
\]
So there is an $\epsilon_1 > 0$ such that
\[
\text{dist}_{\mathbb{H}^{X':N}}(\text{Hilb}(X_{i_j}, (1 - \beta''_{i_j})D_{i_j}), O_{\text{Hilb}}(X,(1-T)D)) > \epsilon_1 \quad \text{for} \quad i \gg 1.
\]
On the other hand, by our assumption \( \beta' \) with \( i \gg 1 \), there is a \( 0 < s_i \in \mathbb{R} \) such that
\[
\Pi(X'_i, (1 - \beta'_i) D_{t'_i}) \in U(N + 1) \cdot B(\Pi(X, (1 - T) D), \epsilon_1)
\]
for any \( t \) satisfying \( 0 < \text{dist}_C(t, 0) < s_i \), since
\[
U(N + 1) \cdot \Pi(X, (1 - \beta'_i) D) \overset{i \to \infty}{\longrightarrow} U(N + 1) \cdot \Pi(X, (1 - T) D)
\]
inside \( \mathbb{H}^{X,N}/U(N + 1) \).

By the \( t \)-continuity of \( \tau(\beta'_i, \cdot) \) for each fixed \( i \gg 1 \), for any \( \varepsilon < \epsilon_1/2 \) there is
\[
s_{i,k} := \sup \left\{ s \in [0, t_i] \left| \tau(\beta'_i, s)B_{C(0,s)} \subset B(O_{\Pi(X, (1 - T) D)}, \varepsilon/2^k) \cup U(N + 1) \cdot B(\Pi(X, (1 - T) D), \epsilon_1) \right. \right\}
\]
where \( |t_i| := \text{dist}_C(t_i, 0) \) and \( B_{C(0,s)} := \{ t \in C \mid \text{dist}_C(t, 0) \leq s \} \). Then \( s_{i,k} = |t_{i,k}| \) is the smallest distance needed for \( t \) so that \( \tau(\beta'_i, t) \) escapes \( B(O_{\Pi(X, (1 - T) D)}, \varepsilon/2^k) \cup U(N + 1) \cdot B(\Pi(X, (1 - T) D), \epsilon_1) \). Clearly, we have \( s_{i,k} + 1 < s_{i,k} \). Now if
\[
\tau(\beta'_i, t_{i,0}) \in \text{SL}(N + 1) \cdot B(\Pi(X, (1 - T) D), \epsilon_1)
\]
we let \( t'_i = t_{i,0} \), otherwise, we let \( t'_i = t'_{i,k} \) where \( t'_{i,k} \) is the first point in \( C \) satisfying
\[
\tau(\beta'_i, t'_{i,k}) \in \text{SL}(N + 1) \cdot B(\Pi(X, (1 - T) D), \epsilon_1).
\]
Such a process must terminate in finite steps by \( \text{(38)} \). Now we define \( M_i \in \mathbb{R} \) to be
\[
M_i := \inf \{ \text{Tr}(g_i^* g_i) + 1 \mid \tau(\beta'_i, t'_i) \in g_i \cdot B(\Pi(X, (1 - T) D), \epsilon_1) \}.
\]

Then again we have two situations exactly the same as in the proof of part one depending on \( \{ M_i \} \) being bounded or not. Replacing \( \beta''_i \) by \( t'_i \) in the argument for Part 1), one see that the rest of the proof is a verbatim, which we will skip. Thus the proof of the Lemma is completed. \( \square \)

**Remark 6.10.** Notice that when \( T = 1 \) and both \( \beta_i, \beta'_i \leq 1 \), \( \forall i \) then Lemma \( 6.9 \) and its proof imply a slight variation of the following form.

Let
\[
\pi_1 : \mathbb{H}^{X,N} = \mathbb{H}^{X,N} \times \mathbb{P}^{\mathbb{R}^{N,N}} \longrightarrow \mathbb{P}^{\mathbb{R}^{N,N}}
\]
be the projection to the first factor.

1. Assume \( \beta_i \to 1 \), \( \beta''_i \to 1 \) and that there is a sequence \( \{ (\alpha_i, \mathcal{D}_{t_i}) \mid (\alpha_i, \mathcal{D}_{t_i}) \) being \( \beta_i \)-K-polystable \) with \( t_i \to 0 \) such that
\[
\pi_1(\Pi(X_i, (1 - \beta_i^*) \mathcal{D}_{t_i})) \overset{i \to \infty}{\longrightarrow} U(N + 1) \cdot \Pi(X) \subset \mathbb{H}^{X,N}
\]
and for \( g_i \in U(N + 1) \)
\[
\pi_1(g_i \cdot \Pi(X_i, (1 - \beta_i) \mathcal{D}_{t_i})) \overset{i \to \infty}{\longrightarrow} \Pi(Y) \in \mathbb{H}^{X,N}.
\]
Then \( \Pi(Y) = g \cdot \Pi(X) \) for some \( g \in U(N + 1) \).

2. Assume \( \beta''_i \nrightarrow 1 \) and that for any fixed \( i \), there is a \( g_i \in U(N + 1) \) such that
\[
\Pi(X_i, (1 - \beta_i^*) \mathcal{D}_{t_i}) \overset{i \to \infty}{\longrightarrow} g_i \cdot \Pi(X, (1 - \beta'_i) D) \subset \mathbb{H}^{X,N}
\]
and
\[
\pi_1(\Pi(X_i, (1 - \beta_i^*) \mathcal{D}_{t_i})) \overset{i \to \infty}{\longrightarrow} \Pi(Y) \in \overline{BO} \setminus O_{\Pi(X)} \subset \mathbb{H}^{X,N}.
\]
If \( X \nsubseteq Y \), then there exists a sequence \( \{ t'_i \} \) satisfying \( 0 < \text{dist}_C(t'_i, 0) < \text{dist}_C(t, 0) \) such that
\[
\Pi(Y) = \lim_{i \to \infty} \pi_1(\Pi(X'_i, (1 - \beta'_i) \mathcal{D}_{t'_i})) \in \left( \overline{O_{\Pi(X)}} \cup \overline{BO} \right) \setminus O_{\Pi(X)} \subset \mathbb{H}^{X,N}.
\]
where \( \text{dist}_C : C \times C \to \mathbb{R} \) is a fixed continuous distance function on \( C \).

Now we are ready to prove the openness.
Proposition 6.12. Let $(X, D; L) \to C$ be Kähler-Einstein degeneration of index $(r, \mathcal{B})$ as in Definition 6.1 with $r = r(X, D)$ being the uniform index as in Theorem 4.1. Then $\mathcal{B}_r(X, D) \subset [e, \mathcal{B}]$ is an open set.

Proof. Let us assume $T \in \mathcal{B}_r(X, D)$, then by fixing a local basis $\{s_i\}$ for $\pi^*\omega_X^{-m}$ we have
\[
\text{dist}_{\mathbb{R}^{N-N}}(\text{Hilb}(X_t, (1 - T)D_t), U(N + 1) \cdot \text{Hilb}(X, (1 - T)D)) \to 0 \quad \text{as} \quad t \to 0.
\]
Now we claim that there is a $\delta > 0$ such that $[e, T + \delta] \subset \mathcal{B}_r(X, D)$. Suppose not, for any $k$, there is a $T < \beta_k < T + 1/k$ and a sequence $\{t_{i,k}\}_{i=1}^{\infty}$:
\[
\text{Hilb}(X_{t_{i,k}}, (1 - \beta_k)D_{t_{i,k}}) \xrightarrow{i \to \infty} \text{Hilb}(Y_k, (1 - \beta_k)E_k) \not\in U \subset \mathbb{H}^{1-N}
\]
with $U \subset \mathbb{H}^{1-N}$ being the $SL(N + 1)$-invariant Zariski open neighborhood of $\text{Hilb}(X, (1 - T)D)$ constructed in Lemma 3.1 since $(X, D)$ is also $\beta_k$-K-polystable because of $\beta_k \in [e, \mathcal{B}]$ and Lemma 2.4. For any fixed $i$, we can pick up $k_i \gg 0$ such that
\[
\text{Hilb}(X_{t_{i,k_i}}, (1 - \beta_{k_i})D_{t_{i,k_i}}) \not\in U(N + 1) \cdot B(\text{Hilb}(X, (1 - T)D), \epsilon_1).
\]
Now let us introduce the diagonal sequence
\[
(\text{Hilb}(X_t, (1 - \beta)D_t)) := \text{Hilb}(X_{t_{i,k_i}}, (1 - \beta_{k_i})D_{t_{i,k_i}})_{i=0}^{\infty}
\]
Then by Theorem 4.1, after passing to a subsequence if necessary, we obtain a new sequence, which by abuse of notation will still be denoted by $\beta_i \not\in T$ and $t_i \to 0$, such that
\[
\text{Hilb}(X_{t_i}, (1 - \beta_i)D_{t_i}) \to \text{Hilb}(Y, (1 - T)E) \not\in O(\text{Hilb}(X, (1 - T)D)).
\]
But this violates the first part of Lemma 6.1 with $\beta^*_i = T \forall i$.

Next we prove the closedness.

Proposition 6.11. Let $(X, D) \to C$ be a family satisfying the condition of Proposition 6.11. Suppose further that $X \to C$ is a family of $\mathcal{B}$-K-polystable varieties. Then $\mathcal{B}_r(X, D) \subset [e, \mathcal{B}]$ is closed with respect to the induced topology, hence $\mathcal{B}(X, D) = [e, \mathcal{B}].$

Proof. By our assumption, for every $t \in C^0$, $(X_t, D_t)$ is a smooth Fano pair with $D_t \geq | - mK_{X_t}|$. Since $X_t$ is $\mathcal{B}$-K-polystable, hence it is $\beta$-K-polystable for $\beta \in [e, \mathcal{B}]$ by Lemma 2.4. As $(X_t, D_t)$ are smooth, by Theorem 4.1 and SW12 Proposition 2.2 [LS14] Proposition 1.7 it admits a unique conical Kähler-Einstein metric $\omega_\tau$ solving
\[
\text{Ric}(\omega(t, \beta)) = \beta \omega(t, \beta) + \frac{1 - \beta}{m} [D_t]
\]
with angle $2\pi(1 - \beta)/m$ along $D_t$ for any $\beta \in [e, \mathcal{B}]$. By Theorem 4.1 and definition of $T$, for any fixed $\beta < T$, we have
\[
(X_t, D_t; \omega(t, \beta) \mid_{\mathcal{B}}(X_0, D_0; \omega(0, \beta))) \text{ as } t \to 0.
\]
By Lemma 6.6 for any sequence $\beta_i \not\in T$ we have
\[
\text{dist}_{\mathbb{R}^{N-N}}(\text{Hilb}(X, (1 - \beta)D), U(N + 1) \cdot \text{Hilb}(X, (1 - T)D)) \to 0
\]
Our goal is to prove that
\[
\text{Hilb}(X_t, (1 - T)D_t) \to U(N + 1) \cdot \text{Hilb}(X, (1 - T)D)
\]
We will argue by contradiction.

Suppose this is the not the case, then there is a subsequence $\{t_i\}_{i=1}^{\infty} \subset C$, $t_i \to 0$ as $i \to \infty$ such that
\[
\text{Hilb}(X_{t_i}, (1 - T)D_{t_i}) \to \text{Hilb}(Y, (1 - T)E) \not\in U(N + 1) \cdot \text{Hilb}(X, (1 - T)D).
\]
By the continuity of $\tau(\cdot; t_i)$ at $T$ for each fixed $i$ (cf. Lemma 6.5), there is a consequence $\{\beta'_i\}_{i=1}^{\infty} \subset (e_0, T)$ such that $\beta'_i \not\in T$ and
\[
\text{Hilb}(X_{t_i}, (1 - \beta'_i)D_{t_i}) \to \text{Hilb}(Y, (1 - T)E) \not\in U(N + 1) \cdot \text{Hilb}(X, (1 - T)D) \text{ as } i \to \infty.
\]
We claim that $\text{Hilb}(Y, (1 - T)E) \not\in T \setminus SL(N + 1) \cdot U$. Otherwise, $\text{Hilb}(Y, (1 - T)E) \in U$ then
\[
\text{Hilb}(X, (1 - T)D) \in SL(N + 1) \cdot \text{Hilb}(Y, (1 - T)E).
\]
But this violates the fact that \((Y, E)\) is \(\mathbf{T}\)-K-polystable by [Be12, Theorem 4.2], since we can construct it as a test configuration of \((Y, E)\) with central fiber \((X, D)\) and vanishing generalized Futaki invariant. Hence our claim is proved.

Now we can apply the second part of Lemma 6.9 to obtain a new sequence \(\{t_i'\} \subset C^0\) satisfying 
\[
\text{Hilb}(Y_i', (1 - T)E') = \lim_{i \to \infty} \text{Hilb}(X_{t_i'}, (1 - \beta_i')D_{t_i'})
\]
which contradicts to the fact that both \((Y', E')\) and \((X, D)\) are \(\mathbf{T}\)-K-polystable by the same reason as above. Thus the proof is completed. \(\square\)

Remark 6.13. We remark an interesting point of the proof is that in the proof of Proposition 6.11, we have only used the continuity of \(\tau(\cdot, t)\) for each fixed \(t\). In particular, its continuity of \(\tau\) with respect to the variable \(t\) is not used. Contrast to this, the continuity of \(\tau(\beta, \cdot)\) with respect to \(t\) is what we use in the proof of Proposition 6.12.

We note that by this point, we have already established the following.

Corollary 6.14. Theorem 1.3 holds under an additional assumption that \(X_t\) is \(\beta\)-K-polystable for all \(t \in C^0\).

7. K-semistability of the nearby fibers

7.1. Orbit of K-semistable points. In this subsection, we extend our continuity method to study the uniqueness of K-polystable Fano varieties that a K-semistable Fano manifold can degenerate to, which will also be needed in the proof of our main theorem.

Let \(X\) be a smooth Fano manifold, and \(D \in |−mK_X|\) be a smooth divisor for \(m \geq 2\). Assume \(X\) is \(\mathbf{T}\)-K-semistable with respect to \(D\). By Theorem 4.1, we know that for any sequence \(\beta_i \nearrow \mathbf{T}\), after possibly passing to a subsequence (which by abusing of notation will still be denoted by \(\beta_i \nearrow \mathbf{T}\)), there exists a log \(\mathbb{Q}\)-Fano pair \((X_0, D_0)\) which is the Gromov-Hausdorff limit of the conical Kähler-Einstein metric \((X, D; \omega(\beta_i))\), that is,
\[
\text{Hilb}(X, (1 - \beta_i)D) \to U(N + 1) \cdot \text{Hilb}(X_0, (1 - T)D_0) \subset \overline{\text{O}_{\text{Hilb}(X, (1 - T)D)}} \quad \text{as } i \to \infty
\]
with \(X_0\) being \(\mathbf{T}\)-K-polystable, where 
\[
\overline{\text{O}_{\text{Hilb}(X, (1 - T)D)}} = \text{the closure of } \text{SL}(N + 1) \cdot \text{Hilb}(X, (1 - T)D) \subset \mathbb{H}^{X,N}.\]

In particular, \((X_0, D_0)\) admits a weak conical Kähler-Einstein metric \(\omega(\mathbf{T})\) with cone angle \(2\pi(1 - (1 - T)/m)\) along the divisor \(D_0 \subset X_0\).

Lemma 7.1. The limit is independent of the choice of the sequence \(\{\beta_i\}\) in the sense that for every sequence \(\beta_i \nearrow \mathbf{T}\),
\[
(X, D; \omega(\beta_i)) \xrightarrow{GH} (X_0, D_0; \omega(\mathbf{T})).
\]

Proof. The existence of a weak conical Kähler-Einstein metric \(\omega(\mathbf{T})\) on \((X_0, D_0)\) allows us to construct a test configuration \((\mathcal{X}, \mathcal{D}; \mathcal{L})\) of \((X, D)\) with central fiber \((X_0, D_0)\) since \(\text{Aut}(X_0, D_0)\) is reductive by Theorem 4.1. Now our claim follows by applying Lemma 6.9(1) to the family \((\mathcal{X}, \mathcal{D}; \mathcal{L})\). \(\square\)

Theorem 7.2. Suppose \(X\) is a smooth K-semistable Fano manifold and \(D_0 \in |−m_0K_X|\) and \(D_1 \in |−m_1K_X|\) are two smooth divisors. Let \(X_0\) and \(X_1\) be the limits defined as in Lemma 7.1 with \(\mathbf{T} = 1\), then \(X_0 \cong X_1\).

Proof. By introducing a third divisor in \(|−mK_X|\) with \(m = \text{lcm}(m_0, m_1)\), we may assume \(r m_0 = m_1\) for a positive integer \(r\). By Bertini’s Theorem, we may choose \(\{D_t\}_{t \in [0, 1]} \subset |−mK_X|\) to be a continuous path joining \(r D_0\) and \(D_1\) such that
- the path \(\{D_t\}\) lies in an algebraic arc \(C \subset |−mK_X|\) with corresponding family \(\mathcal{D} \to C\);
- \(D_t\) is smooth for all \(t \neq 0\).
By assumption, \( X \) is K-semistable, hence \((X, D_t)\) are \(\beta\)-K-stable for all \((\beta, t) \in (0, 1) \times (0, 1)\). In particular, \(\{ (X, D_t) \}\) admit conical Kähler-Einstein metric \(\omega(t, \beta)\), \(\forall (\beta, t) \in (0, 1) \times [0, 1]\) by Corollary 4.2 using Tian’s embedding we can similarly define a map
\[
\sigma : (0, 1) \times (0, 1] \longrightarrow \H^{X/N}
\]
(43)
\[
(\beta, t) \longrightarrow \text{Hilb}(X, (1 - \beta)D_t)
\]
using a prefixed basis of \(H^0(X, O_X(-rK_X))\). By Proposition 4.8 and Don12a Theorem 2, \(\sigma\) is continuous on \((0, 1) \times (0, 1]\). We claim that for \(q \circ \sigma\) is continuous on \((0, 1) \times [0, 1]\) with \(q : \H^{X/N} \rightarrow \H^{X/N}/\U(N + 1)\). For fixed \(\beta \in (0, 1)\), we can deduce the continuity of \(\sigma(\beta, \cdot)\) at 0 by applying Corollary 6.14 to the product family \((X = X \times C, D) \rightarrow C)\) with \((X_t, D_t) = (X, D_t)\).

Thus all we need to show is
\[
\lim_{\beta \to 1} \text{dist}_{\H^{X/N}}(\sigma(\beta, t), U(N + 1) \cdot \text{Hilb}(X_0)) = 0, \quad \forall t \in [0, 1]
\]
where \(\hat{\sigma} := \pi_1 \circ \sigma\) with \(\pi_1\) being given in (34). To achieve that, let \(\hat{q} : \H^{X/N} \rightarrow \H^{X/N}/\U(N + 1)\) then Lemma 7.1 allows us to introduce

\[
\lim_{\beta \to 1} \hat{q} \circ \hat{\sigma}(\beta, t) = U(N + 1) \cdot \text{Hilb}(X_t) \in \H^{X/N}/\U(N + 1), \quad \text{for } t \in [0, 1]
\]
with \(X_t\) being a \(\Q\)-Fano variety admitting weakly Kähler-Einstein metric for each \(t \in [0, 1]\). Let \(X_1 \rightarrow \mathbb{A}^1\) be a test configuration with central fiber \(X_1\) and \(\text{Hilb}(X_1) \in U \subset \H^{X/N}\) be the open neighborhood constructed for the family \(X_1 \rightarrow \mathbb{A}^1\) via Lemma 3.1.

Now suppose (44) does not hold, i.e. there is a \(t_0 \in [0, 1]\) such that
\[
\lim_{\beta \to 1} \hat{\sigma}(\beta, t_0) = \text{Hilb}(X_{t_0}) \not\in U \cdot \text{Hilb}(X_1).
\]
Then by applying the continuity of \(\hat{q} \circ \hat{\sigma}(\beta, \cdot)\) with respect to \(t \in [0, 1]\) for fixed \(\beta\) the same way as in the proof of Lemma 6.9(2), we can construct a new sequence \(\{(\beta_i, t_i)\}_{i=1}^\infty \subseteq (0, 1] \times [t_0, 1]\) such that \(\beta_i \not\to 1\) as \(i \to \infty\) and
\[
\text{Hilb}(Y) = \lim_{i \to \infty} \hat{\sigma}(\beta_i, t_i) \in \left(\frac{\partial\text{Hilb}(X_1)}{\text{Hilb}(X_1)} \bigcup \left(U \cap \partial\text{Hilb}(X_1)\right)\right) \setminus \text{O}_{\text{Hilb}(X_1)} \subset \H^{X/N},
\]
with both \(X_1\) and \(Y(\not\cong X_1)\) being K-polystable, which is impossible. Hence our proof is completed.

\(\square\)

7.2. Zariski Openness of K-semistable varieties. In this section, we will study the Zariski openness of the locus of the smoothable K-semistable varieties inside Hilbert schemes. This needs a combination of the continuity method with the algebraic result in Appendix 9.1.

Let
\[
\left(\mathcal{X} \rightarrow \mathcal{D}\right) \longrightarrow \mathbb{P}^N \times \mathbb{P}^N \times S
\]
be a flat family of \(\Q\)-Fano varieties over a smooth base \(S\) (not necessarily complete) and \(\mathcal{D} \in | -mK_X|\) be an irreducible divisor defined by a section \(s_\mathcal{D} \in \Gamma(S, O_X(-mK_X))\). Let us assume further that \(O_X(-rK_X)\) is relatively very ample and \(\iota\) is the embedding induced by a prefixed basis \(\{s_i(t)\}_{i=0}^N \subseteq \Gamma(S, \pi_*O_X(-rK_X/S))\), in particular \(\iota^*O_{\mathbb{P}^N}(1) \cong O_X(-rK_X/S)\). Then we have the following

**Theorem 7.3.** Let \((X, D) \rightarrow C\) be the family over a smooth curve such that \((X_t, D_t)\) is smooth for \(t \in C^\circ\) and \((X_1, \frac{1}{m}D_1)\) is klt for all \(t \in C\). Assume \((X_0, D_0)\) is \(\mathfrak{B}\)-K-semistable. Then there is a Zariski open neighborhood \(0 \in C^\circ \subseteq C\) such that \((X_t, D_t)\) is \(\mathfrak{B}\)-K-semistable for \(t \in C^\circ\). Furthermore, if \((X_0, D_0)\) is \(\mathfrak{B}\)-K-polystable and has only finitely many automorphisms, then \((X_t, D_t)\) is \(\mathfrak{B}\)-K-polystable after possibly further shrinking of \(C^\circ\).

**Definition 7.4.** For every \(t \in S\), we define the K-semistable threshold as follows
\[
\text{kst}(X_t, D_t) := \sup \{ \beta \in [0, \mathfrak{B}] \mid (X_t, D_t) \text{ is } \beta\text{-K-semistable} \}.
\]
By Theorem 4.1 testing \(\beta\)-K-semistability for \(X_t\), \(\forall t \in S\) is reduced to test for all 1-PS inside \(\text{SL}(N + 1)\) for a fixed sufficiently large \(\mathbb{P}^N\). This implies that \(\text{kst}(X_t, D_t)\) is a constructible function of \(t\) (cf. Proposition 7.5 below). By Remark 6.4 we know \((X_t, D_t)\) is \(\beta\)-K-stable for all \(\beta \in (0, \beta_0]\).
This together with Lemma 2.4 in particular imply that \( \kst(\mathcal{X}_t', \mathcal{D}_t) \) is actually a maximum for every \( t \in S \).

Then we have the following Proposition which is essentially follows from Paul’s work, especially his theory on stability of pairs (see [Paul12 Theorem 1.3]). For reader’s convenience, a proof will be included in the Section 9.1, Proposition 9.4.

**Proposition 7.5.** \( \kst(\mathcal{X}_t, \mathcal{D}_t) \) defines a constructible function on \( S \), i.e. \( S = \sqcup_i S_i \) is a union of finite constructible sets \( \{ S_i \} \), on which \( \kst(\mathcal{X}_t, \mathcal{D}_t) \) are constant.

**Proof of Theorem 7.3.** By Proposition 7.5, \( \kst(\mathcal{X}_t, \mathcal{D}_t) \) is constant when restricted to each strata \( S_i \). So all we need is that if \( t_i \to 0 \) and \((\mathcal{X}_{t_i}, \mathcal{D}_{t_i})\) strictly \( T \)-K-semistable then

\[
T = \kst(\mathcal{X}_{t_i}, \mathcal{D}_{t_i}) \geq \kst(X, D) = B.
\]

Suppose this is not the case, we have \( B > T \) and we seek for a contradiction. First, we claim for any sequence \( t_i \to 0 \), after passing to a subsequence which by abusing of notation still denoted by \( \{ t_i \} \), we can find a sequence \( \{ \beta_i \} \not\supset T \) such that

\[
\dist_{\mathcal{X}_i, B} (\Hilb(\mathcal{X}_{t_i}, (1 - \beta_i^*)D_{t_i}), \U(N + 1) \cdot \Hilb(X, (1 - T)D)) \to 0.
\]

In fact, since we have already established Theorem 7.2 under the extra assumption that the nearby points are all \( \beta \)-K-polystable (see Corollary 5.4), for any fixed \( \beta < T \) we have

\[
\dist_{\mathcal{X}_i, B} (\Hilb(\mathcal{X}_{t_i}, (1 - \beta)D_{t_i}), \U(N + 1) \cdot \Hilb(X, (1 - \beta)D)) \to 0 \text{ as } t \to 0,
\]

thus Lemma 6.9 implies that

\[
\dist_{\mathcal{X}_i, B} (\Hilb(X, (1 - \beta_i^*)D), \U(N + 1) \cdot \Hilb(X, (1 - T)D)) \to 0
\]

for any sequence \( \beta_i^* \not\supset T \). Since \( t_i \to 0 \), for any fixed \( \beta_i^* \) there is a \( k_i \geq i \) such that

\[
\dist_{\mathcal{X}_i, B} (\Hilb(\mathcal{X}_{t_i}, (1 - \beta_i)D_{t_i}), \U(N + 1) \cdot \Hilb(X, (1 - \beta)D)) < 1/i.
\]

Now we pick the subsequence \( \{ t_{k_i} \} \) and define \( \beta_{k_i}^* := \beta_i^* \), then the sequence \( \{ \beta_{k_i}^* \} \to \infty \not\supset T \) is a sequence satisfying (45), hence our claim is justified.

On the other hand, for each fixed \( t_i \), let \( \beta \not\supset T \). By Theorem 4.1 we have

\[
\dist_{\mathcal{X}_i, B} (\Hilb(\mathcal{X}_{t_i}, (1 - \beta)D_{t_i}), \U(N + 1) \cdot \Hilb(\mathcal{X}_{t_i}, (1 - T)D_{t_i})) \to 0
\]

with \( \Hilb(\mathcal{X}_{t_i}, (1 - \beta)D_{t_i}) \in \partial \Hilb(\mathcal{X}_{t_i}, (1 - T)D_{t_i}) \) and \((\mathcal{X}_{t_i}, (1 - T)D_{t_i})\) being a \( T \)-K-polystable variety. Now we claim that

\[
\Hilb(\mathcal{X}_{t_i}, (1 - T)D_{t_i}) \to g \cdot \Hilb(X, (1 - T)D) \text{ for some } g \in \U(N + 1).
\]

To see this, one notices that by Theorem 4.1 and Lemma 4.9 after passing to a subsequence there is a sequence \( \beta_i \not\supset T \) such that

\[
\Hilb(\mathcal{X}_{t_i}, (1 - \beta_i)D_{t_i}) \to \Hilb(Y, (1 - T)E),
\]

such that \( (Y, E) \) is \( T \)-K-polystable. Moreover, we may assume \( \beta_{k_i}^* < \beta_i \) for all \( i \) after rearranging. Combining (46) and Lemma 4.9 we know

\[
(\mathcal{X}_{t_i}, (1 - \beta_i)D_{t_i} ; \omega(t_i, \beta_i)) \xrightarrow{\GR} (Y, E; \omega_Y(T)),
\]

where \( (Y, E) \) is a log \( \mathbb{Q} \)-Fano pair admitting a weak conical Kähler-Einstein metric \( \omega_Y(T) \) with angle \( 2\pi(1 - (1 - T)/m) \) along \( E \). In particular, \( (Y, E) \) is \( T \)-K-polystable. By Lemma 6.9(1), we conclude that

\[
\Hilb(Y, (1 - T)E) = g \cdot \Hilb(X, (1 - T)D) \text{ for some } g \in \U(N + 1).
\]

Hence our claim is proved.

To conclude the proof, we notice that the stabilizer group of \( \Hilb(\mathcal{X}_{t_i}, (1 - T)D_{t_i}^-) \) is of positive dimension for each \( i \). Let \( g = s(N + 1) \) be the Lie algebra. By the upper semicontinuity of the dimension of the stabilizer \( g_{\Hilb(\mathcal{X}_{t_i}, (1 - T)D_{t_i}^-)} \), we must have \( \dim g_{\Hilb(\mathcal{X}_{t_i}, (1 - T)D_{t_i}^-)} > 0 \) contradicting to the fact that the automorphism group of \((X, D)\) is finite for \( T < B \leq 1 \) (see Corollary 6.3). To prove the last part of the statement, we just notice that under our assumption \((\mathcal{X}_t', \mathcal{D}_t')\) has to have finite automorphism groups, which implies

\[
(\mathcal{X}_t', \mathcal{D}_t') \cong (\mathcal{X}_t, \mathcal{D}_t).
\]
Hence our proof is completed for this case.

7.3. **Proof of Theorem 1.1 and 1.2** Before we start the proof, let us fix a divisor $\mathcal{D} \sim_{C} -mK_{X}$ in general position for the flat family $X \to C$ satisfying the assumption of Theorem 5.2 and $(A_{t}, D_{t})$ being smooth for all $t \in C^\circ$.

**Proof of Theorem 1.2** First, we notice that (i) is proved in Section 7.2. To prove (ii), one notices that Theorem 4.1 implies that there exists an $r$, such that the Gromov-Hausdorff limit of the family $(X_{t}, D_{t}; \omega(t, \beta))$ for any $t \in C$ and $\beta < 1$ can all be embedded into $\mathbb{P}^{N}$ for $N = N(r, d)$. By putting Proposition 6.11 and 6.12 together, we obtain that for every $\mathfrak{B} < 1$,

$$B_{r}(X, D) = [\mathfrak{B}]$$

for $(X, D)$ (See Corollary 6.14). Therefore, their union will contain $[\epsilon, 1]$. In particular, it follows from Lemma 6.3 and Remark 6.4 that for $\mathfrak{B} = 1$ that $X = X_{0}$ admits a Kähler-Einstein metric. This in particular verifies the first part of (iii).

Now we finish the proof of part (ii). By part (i), after a possible shrinking of $C$, we may assume that $X_{t}$ is K-semistable for every $t \in C^\circ$. For any $t \neq 0$, there is a unique $K$-polystable $\mathbb{Q}$-Fano $X_{\tilde{t}}$ such that $\text{Hilb}(X_{t}) \in \overline{\text{O}_{\text{Hilb}(X_{0})}}$ by Theorem 7.2, which is the Gromov-Hausdorff limit of $(X_{t}, D_{t}; \omega(\beta))$ as $\beta \to 1$ and hence admits a weak Kähler-Einstein metric $\tilde{\omega}(t)$ by Theorem 1.2.

We claim that

$$\text{dist}_{\text{Hilb}(X)}(U(N + 1) \cdot \text{Hilb}(X), U(N + 1) \cdot \text{Hilb}(X)) \to 0, \text{ as } i \to \infty,$$

and hence part (ii) follows. To prove that, let $t_{i} \to 0$ be any sequence. It follows from the compactness of Hilbert scheme of $\mathbb{P}^{N}$ that after passing to a subsequence if necessary we may assume

$$\text{Hilb}(X_{t_{i}}) \to \text{Hilb}(Y) \text{ as } t_{i} \to 0.$$ 

Since

$$(X_{t_{i}}, D_{t_{i}}; \omega(t_{i}, \beta)) \stackrel{\text{GH}}{\to} (X_{0}; \omega_{X}) \text{ as } \beta \to 1,$$

by Theorem 7.2 there is a sequence $\beta_{i} \to 1$ such that

$$\text{dist}_{\text{Hilb}(X)}(\pi_{i} \circ \text{Hilb}(X_{t_{i}}), (1 - \beta_{i})D_{t_{i}}), U(N + 1) \cdot \text{Hilb}(X_{t_{i}})) < 1/i,$$

where $\pi_{i}$ is given in (34). In particular, by passing to another subsequence if necessary, we may assume

$$(X_{t_{i}}, (1 - \beta_{i})D_{t_{i}}; \omega(t_{i}, \beta_{i})) \stackrel{\text{GH}}{\to} (Y; \omega_{Y})$$

by Lemma 1.9, where $Y$ is a $\mathbb{Q}$-Fano variety admitting a weak Kähler-Einstein metric $\omega_{Y}$. This implies that

$$\text{dist}_{\text{Hilb}(X)}(\pi_{i} \circ \text{Hilb}(X_{t_{i}}), (1 - \beta_{i})D_{t_{i}}), U(N + 1) \cdot \text{Hilb}(Y)) \to 0, \text{ as } i \to \infty.$$

On the other hand, by Lemma 2.4 we know $(X_{t_{i}}, D_{t_{i}})$ is $\beta$-K-polystable for any $\beta < 1$. This together with Corollary 6.14 imply that for every fixed $\beta < 1$

$$\text{dist}_{\text{Hilb}(X)}(\text{Hilb}(X_{t_{i}}), (1 - \beta)D_{t_{i}}), U(N + 1) \cdot \text{Hilb}(X, (1 - \beta)D)) \to 0 \text{ as } i \to \infty.$$

Therefore, for any fixed $\beta$, there is a $k_{i} > i$ such that

$$\text{dist}_{\text{Hilb}(X)}(\text{Hilb}(X_{t_{i}}), (1 - \beta_{i})D_{t_{i}}), U(N + 1) \cdot \text{Hilb}(X, (1 - \beta)D)) < 1/i.$$

On the other hand, Lemma 6.6 implies that

$$\text{dist}_{\text{Hilb}(X)}(\pi_{i} \circ \text{Hilb}(X, (1 - \beta)D), U(N + 1) \cdot \text{Hilb}(X)) \to 0 \text{ as } \beta \to 1.$$

These imply that if we define $\beta_{k}^{*} := \beta_{i} < \beta_{k}$, then $\beta_{k}^{*} \to 1$ and

$$(\pi_{i} \circ \text{Hilb}(X_{t_{i}}), (1 - \beta_{k})D_{t_{i}}), U(N + 1) \cdot \text{Hilb}(X)) \to 0 \text{ as } i \to \infty.$$

By putting together (40) and (49), and applying Remark 6.10 (1), we conclude that $\text{Hilb}(Y) \in U(N + 1) \cdot \text{Hilb}(X)$, and (48) is established. Thus the proof of part (ii) is completed.

Finally, to finish the proof of part (iii), we can assume $X_{t}$ is K-polystable for all $t \in C$ by Theorem 7.3, then by taking $\mathfrak{B} = 1$ we can conclude that $B_{r}(X, D) = [\epsilon, 1]$. In particular, $(X_{t_{i}}; \omega(t_{i})) \stackrel{\text{GH}}{\to} (X_{0}; \omega_{X_{0}})$. Hence our proof is completed.
Proof of Theorem 1.2. Choose a sequence $\beta \not\in B$. Applying Proposition 6.11 and 6.12 we obtain that $B_\epsilon(X, D) = [\epsilon, B]$. Then by repeating the argument completely parallel to the one given above, we obtain the conclusion.

Remark 7.6. We call a $\mathbb{Q}$-Fano variety to be smoothable if there is a projective flat family $X$ over a smooth curve $C$ such that $K_X$ is $\mathbb{Q}$-Cartier, anti-ample over $C$, a general fiber $X_t$ is smooth and $X \cong X_0$ for some $0 \in C$. We note that by a standard argument, we can generalize Theorem 1.1 and 7.2 to the case that the base is of higher dimension. As a consequence, we can just assume in these theorems that the general fibers are smoothable instead of smooth. These extensions will be frequently used in Section 8.

8. Local geometry near a smoothable $K$-polystable $\mathbb{Q}$-Fano variety

In this section, we will study the geometric consequences of Theorem 1.1 especially on how to use it to construct a proper moduli space for smoothable $K$-polystable Fano varieties.

Definition 8.1. We define

$$Z := \begin{cases} \text{Hilb}(Y) & Y \subset \mathbb{P}^N \text{ be a smooth Fano manifold with } N = \dim H^0(Y, O_Y(-rK_Y)), \vspace{1ex} \
\{z \in \mathbb{H}^{X,N} \mid \chi(Y, O_{\mathbb{P}^N}(-rK_Y)) = \chi(k)\} \subset \mathbb{H}^{X,N}. \end{cases}$$

By the boundedness of smooth Fano manifolds with fixed dimension (see [KMM92]), we may choose $r \gg 1$ such that $Z$ includes all such Fano manifolds. Now let $Z \subset \mathbb{H}^{X,N}$ be the closure of $Z \subset \mathbb{H}^{X,N}$ and $Z^*$ be the open set of $Z$ that parametrizes the K-semistable $\mathbb{Q}$-Fano subvariety $Y$ such that $\chi(Y, O_{\mathbb{P}^N}(-rK_Y)) = \chi(k)$. Let $Z^*$ be the semi-normalization of $Z^{\text{red}}$ which is the reduction of $Z^*$.

Remark 8.2. By Theorem 4.1 the Gromov-Hausdorff limit of Fano Kähler-Einstein manifolds is automatically in $Z^*$ and hence so are the smoothable $K$-polystable $\mathbb{Q}$-Fano varieties.

Then we have a commutative diagram

$$\begin{array}{cccc}
\mathcal{X}^* & \xrightarrow{i} & \mathbb{P}^N \times Z^* & \xrightarrow{\pi} & \mathbb{P}^N \\
\downarrow & & \downarrow & & \downarrow \\
Z^* & \xrightarrow{} & Z^* & \xrightarrow{} & Z^{\text{red}}
\end{array}$$

where $\mathcal{X}^*$ is the universal family over $Z^*$ (see [Ko96 Section I.3]).

Before we state the main result of this section, let us first deduce the following boundedness result which is a consequence of our Theorem 1.1

Lemma 8.3. The smoothable $K$-semistable $\mathbb{Q}$-Fano varieties with a fixed dimension form a bounded family.

Proof. We first prove for the statement for $K$-polystable $\mathbb{Q}$-Fano varieties. Let $X$ be an $n$-dimensional smoothable $K$-polystable $\mathbb{Q}$-Fano variety and $\mathcal{X} \to C$ be a smoothing of $X$ with $\mathcal{X}_0 = X$. It follows from Theorem 1.1 that nearby fibers $\mathcal{X}_t$ are all K-semistable, and we can take a $D \sim_{C} -mK_{X/C}$, such that $\lambda_0$ is the Gromov-Hausdorff limit of $(\mathcal{X}_{t_1}, (1 - \beta)D_t)$ for any sequences $t_1 \to 0$ and $\beta \to 1$.

On the other hand, by the boundedness of smooth Fano varieties, we know that there exists $m_0$ depending only on $n$, and a divisor $D_t^* \sim_{C^0} -m_0K_{X^*/C^0}$, such that $D_t^*$ is smooth for any $t \in C^0$. Since all $\mathcal{X}_t$ are K-semistable, they admit conical Kähler-Einstein metrics $\omega(t, \beta_t)$ with cone angle $2\pi(1 - (1 - \beta_t)/m)$ along $D_t^*$. By applying Theorem 1.2(iii) for $(\mathcal{X}_t, (1 - \beta_t)D_t^*)$, we know that the Gromov-Hausdorff limit for this family as $t \to 0$ is also $\mathcal{X}_0$. Thus it is a subvariety of a fixed $\mathbb{P}^N$ for some $N \gg 0$ by Theorem 4.1.

In general, if $X$ is smoothable $K$-semistable $\mathbb{Q}$-Fano variety, then we know that the closure of its orbit contains a $K$-polystable $\mathbb{Q}$-Fano variety $X_0$. And as a consequence of volume convergence for Gromov-Hausdorff limit, we obtain that

$$(-K_{X_0})^n = (-K_X)^n$$
are bounded from above; on the other hand the Cartier index of $K_X$ divides the Cartier index of $K_{X_0}$, which is also bounded from above. Therefore $X$ is contained in a bounded family (see e.g. [HMX14 Corollary 1.8]).

Fix a K-polystable $\mathbb{Q}$-Fano variety $X$ parametrized by a point in $Z^*$, then it admits a weak Kähler-Einstein metric by Theorem 1.1 from which we deduce that $\text{Aut}(X) \subseteq \text{SL}(N+1)$ is reductive. Let $\text{Hilb}(X)$ be the Hilbert point for the Tian’s embedding of $X \subseteq \mathbb{P}^N$ after we fix a basis of $H^0(O_X(−rK_X))$. Let $\mathbb{H}^{x,N} \subseteq \mathbb{P}^M$ be the Plücker’s embedding which is clearly $\text{SL}(N+1)$-equivariant. Then by [Don12b Proposition 1] or the proof of Lemma 3.1 there is an $\text{Aut}(X)$-invariant linear subspace $z_0 := \text{Hilb}(X) \subseteq \mathbb{P}^W \subseteq \mathbb{P}^M$ so that

$$\mathbb{P}^M = \mathbb{P}(W \oplus \mathbb{C}z_0 \oplus \text{aut}(X)^⊥) \text{ with } \text{aut}(X)^⊥ \oplus \text{aut}(X) = \text{sl}(N+1),$$

where $W \oplus \mathbb{C} \cdot z_0 \oplus \text{aut}(X)^⊥ = \mathbb{C}^{M+1}$ is a decomposition as $\text{Aut}(X)$-invariant subspaces.

In particular, this induces a representation $\rho : \text{Aut}(X) \to \text{SL}(W)$. On the other hand, $\text{Hilb}(X)$ is fixed by $\text{Aut}(X)$. We let $\rho_X : \text{Aut}(X) \to \mathbb{G}_m$ denote the character corresponding to the linearization of $\text{Aut}(X)$ on $\mathcal{O}_{\mathbb{P}^W}(1)|_{\text{Hilb}(X)}$ induced from the embedding $\text{Aut}(X) \subseteq \text{SL}(N+1)$. Then we can introduce the following.

**Definition 8.4.** A point $z \in \mathbb{P}^W$ is GIT-polystable (resp. GIT-semistable) if $z$ is polystable (resp. semistable) with respect to the linearization $\rho \otimes \rho_X^{-1}$ on $\mathcal{O}_{\mathbb{P}^W}(1) \to \mathbb{P}^W$ in the GIT sense.

Our main result of this section is the following:

**Theorem 8.5.** There is an $\text{Aut}(X)$-invariant linear subspace $\mathbb{P}W \subseteq \mathbb{H}^{x,N}$ and a Zariski open neighborhood $\text{Hilb}(X) \subseteq U_W \subseteq \mathbb{P}^W \times \mathbb{R}^{N} \cdot Z^*$ such that for any $\text{Hilb}(Y) \subseteq U_W$, $Y$ is K-polystable if and only if $\text{Hilb}(Y)$ is GIT-polystable with respect to $\text{Aut}(X)$-action on $\mathbb{P}^W \times \mathbb{R}^{N} \cdot Z^*$.

Moreover, for all GIT-polystable $\text{Hilb}(Y) \subseteq U_W$, we have $\text{Aut}(Y) < \text{Aut}(X)$, i.e. the local GIT presentation $U_W / \text{Aut}(X)$ is stabilizer preserving in the sense of [AFS16 Proposition 3.1].

**Remark 8.6.** As we will see in Corollary 8.12 that we are able to establish the stabilizer preserving property for all GIT-semistable $\text{Hilb}(Y) \subseteq U_W$. This property is stronger than the condition of being strongly étale introduced in [AFS16 Definition 2.5 ].

Let

$$\Delta : Z^* \to \mathbb{H}^{x,N} \times Z^*$$

be the diagonal morphism, we define $O_{Z} : = \text{SL}(N+1) \cdot \Delta(Z^*) \subseteq \mathbb{H}^{x,N} \times Z^*$ where $\text{SL}(N+1)$ acts trivially on $Z^*$ and acts on $\mathbb{H}^{x,N}$ via the action induced from $\mathbb{P}^N$. This allows us to construct the family of limiting orbits space associated to the family as following:

$$\mathbb{BO}_z \subseteq \overline{\mathbb{O}_z} \xrightarrow{\iota} \mathbb{H}^{x,N} \times Z^*$$

with $\overline{\mathbb{O}_z} \subseteq \mathbb{H}^{x,N} \times Z^*$ be the closure of $O_z$ and $\mathbb{BO}_z$ is the union of limiting broken orbits. Then by Theorem 1.1 we know that there is a unique K-polystable orbit inside $\mathbb{BO}_z$. To see this, one only needs to notice that for any $z \in Z^*$, we can always find a smooth curve $f : C \to Z^*$ that passes through $z$ and the image $f(C)$ meets the dense open locus inside of $Z^*$ corresponding to smooth K-polystable Fano manifolds with the maximal dimension of its $\text{SL}(N+1)$-orbit space. Then our claim follows by applying Theorem 1.1 to the pull back family over $C$.

For a K-polystable point $\text{Hilb}(X) \subseteq Z^*$ (corresponding to the Tian’s embedding of $X \subseteq \mathbb{P}^N$ with respect to the Kähler-Einstein metric), by Lemma 3.1 we can find a Zariski neighborhood $\text{Hilb}(X) \subseteq U \subseteq Z^*$ and after a possible shrinking we may assume

$$U \cap \overline{\mathbb{BO}_{\text{Hilb}(X)}} \text{ contains a unique minimal (cf. Lemma 3.1) orbit } \text{SL}(N+1) \cdot \text{Hilb}(X).$$

By Theorem 7.2 (and its extension in Remark 7.6), every $z \in U$ can be specialized to a K-polystable point $\hat{z}$ unique up to $\text{SL}(N+1)$-translation. Moreover, we have the following

$$U \cap \overline{\mathbb{BO}_{\text{Hilb}(X)}} \text{ contains a unique minimal orbit } \text{SL}(N+1) \cdot \text{Hilb}(X).$$
**Lemma 8.7.** Let $\operatorname{Hilb}(X) \in U \subset Z^*$ be as above, then there is an analytic open neighborhood $\operatorname{Hilb}(X) \in U^{\text{ks}}$ such that for any $K$-semistable points $z \in U^{\text{ks}}$, we can specialize it to a $K$-polystable point $\hat{z} \in U$ via a 1-PS $\lambda \subset \SL(N+1)$. Moreover, if $\lim_{i \to \infty} z_i = \operatorname{Hilb}(X)$, then

$$
\lim_{i \to \infty} \text{dist}_{\mathbb{H}^{N}}(\operatorname{Hilb}(X_i), \omega_{\operatorname{KE}}(\hat{z}_i)), U(N+1) \cdot \operatorname{Hilb}(X)) = 0.
$$

where $\operatorname{Hilb}(X_i, \omega_{\operatorname{KE}}(\hat{z}_i))$ is the Hilbert point corresponding to the Tian’s embedding of $X_i$ with respect to the weak Kähler-Einstein metric $\omega_{\operatorname{KE}}(\hat{z}_i)$.

**Proof.** Suppose this is not the case, there is a sequence $z_i = \operatorname{Hilb}(X_i) \xrightarrow{i \to \infty} \operatorname{Hilb}(X)$ and $O_{\hat{z}} \cap U = \emptyset$ with $O_{\hat{z}} := \SL(N+1) \cdot \hat{z}$. In particular, by equipping each $X_i$ with a weak Kähler-Einstein metric $\omega_{\operatorname{KE}}(\hat{z}_i)$, and taking the Gromov-Hausdorff limit $Y'$, which is still embedded in $\mathbb{P}^N$ by Lemma 8.3, we obtain

$$
\operatorname{Hilb}(X_i, \omega_{\operatorname{KE}}(\hat{z}_i)) \xrightarrow{i \to \infty} g \cdot \operatorname{Hilb}(Y) \in \overline{BO}_{\operatorname{Hilb}(X)} \setminus U \text{ for some } g \in U(N+1)
$$

contradicting to the fact the limiting broken orbits $BO_z$ contains a unique K-polystable orbit. □

Now we are ready to prove Theorem 8.5.

**Proof of Theorem 8.5.** Let $U$ be the open set constructed above satisfying (56) and let $U_{W}^{\text{an}} = (U^{\text{ks}} \cap \mathbb{P}W) \times_{\mathbb{H}^{N}} Z^*$. (cf. Lemma 8.7)

After a possible shrinking, we may assume that all the points in $U_{W}^{\text{an}}$ are GIT-semistable and every GIT-semistable point can be degenerated to a GIT-polystable point in $U_{W}^{\text{an}}$. Suppose $\operatorname{Hilb}(Y) \in U_{W}^{\text{an}}$ is GIT-polystable and strictly K-semistable. Then by Lemma 8.7, we can degenerate it to a variety $Y' \subset \mathbb{P}^N$ which is K-polystable such that $\operatorname{Hilb}(Y') \in U \cap \SL(N+1) \cdot \operatorname{Hilb}(Y) \subset Z^* \subset \mathbb{H}^{N}$.

and $\operatorname{Hilb}(Y')$ is close to $\operatorname{Hilb}(Y)$ in $\mathbb{H}^{N}$ in the sense that there is short (with respect to the metric $\text{dist}_{\mathbb{H}^{N}}$) path inside $\SL(N+1) \cdot \operatorname{Hilb}(Y)$ joining $\operatorname{Hilb}(Y)$ and $\operatorname{Hilb}(Y')$.

Using the transversality of the action of $\mathfrak{aut}(X)^\perp \subset \mathfrak{sl}(N+1)$ on $\mathbb{P}W \subset \mathbb{P}^M$, one can always find a $g \in \SL(N+1)$ close to the identity such that

$$
\operatorname{Hilb}(Y'') := g \cdot \operatorname{Hilb}(Y') \in \mathbb{P}W \times_{\mathbb{H}^{N}} Z^*,
$$

where $Y'' \cong Y'$ is GIT-semistable. This allows us to find a short path inside $\SL(N+1) \cdot \operatorname{Hilb}(Y)$ joining $\operatorname{Hilb}(Y)$ and $\operatorname{Hilb}(Y'')$, which by transversality we may assume to be entirely contained in $\mathbb{P}W$ and satisfies $\operatorname{Hilb}(Y'') \in (\operatorname{Aut}(X) \cdot \operatorname{Hilb}(Y))$. But this is absurd since $\operatorname{Hilb}(Y)$ is already GIT-polystable, no point on the boundary of $(\operatorname{Aut}(X) \cdot \operatorname{Hilb}(Y))$ is semistable.

Conversely, suppose $\operatorname{Hilb}(Y) \in U_{W}^{\text{an}}$ and $Y$ is K-polystable but $\operatorname{Hilb}(Y)$ is not GIT-polystable, then there is a 1-PS $\lambda \subset \operatorname{Aut}(X)$ degenerating $\operatorname{Hilb}(Y)$ to a nearby GIT-polystable

$$
\operatorname{Hilb}(Y') \in (\operatorname{Aut}(X) \cdot \operatorname{Hilb}(Y) \cap U_{W}^{\text{an}})
$$

by the classical GIT. Thus $Y'$ is K-polystable by the previous paragraph, contradicting to the assumption $Y$ being K-polystable. Hence our proof is completed.

To pass from the analytic neighborhood to a Zariski neighborhood, we need to investigate the geometry of $\operatorname{Aut}(X)$-orbits. Let $U_{W}^{\text{an}} \subset \mathbb{P}W$ containing $\operatorname{Hilb}(X)$ be the Zariski open set of GIT-semistable points. By [MFK94] Chapter 2, Proposition 2.14 and [Oda12] Lemma 2.11 and Lemma 2.12, we know that the set of GIT-polystable points in $U_{W}^{\text{an}}$ forms a constructible set. On the other hand, K-polystable points inside $U_{W}^{\text{an}} \cap Z_{\text{red}}$ also form a constructible sets (see Remark 9.5) containing $\operatorname{Hilb}(X)$. These two constructible sets coincide along $U_{W}^{\text{an}}$ after lifting to $\mathbb{P}W \times_{\mathbb{H}^{N}} Z^* \supset U_{W}^{\text{an}}$ by the proof above, so they must coincide on a Zariski open set.

Finally, we establish the last statement. By Theorem 1.1, the set of Hilbert points of the universal family $\mathcal{X}$ of Kähler-Einstein Q-Fano varieties obtained via Tian’s embedding induces a proper (, in
the sense of Assumption 9.6 \( U(N + 1) \)-invariant slice

\[
\Sigma = \{ \text{Hilb}(X_z) \mid z \in (Z^*)^{\text{kps}} \}
\]

(57)

\[
\begin{array}{c}
\text{Hilb}(X_z) \rightarrow \mathbb{P}^M \\\n\text{Plücker} \rightarrow \mathbb{P}^M \cup U(N + 1)
\end{array}
\]

passing through \( z_0 \), where \( (Z^*)^{\text{kps}} \subset Z^* \) denotes the locus of K-polystable points in \( Z^* \). Moreover, the slice \( \Sigma \) satisfies the Assumption 9.6 thanks to the following:

**Claim 8.8.** Let \( X \) be a smoothable \( \mathbb{Q} \)-Fano variety admitting weak Kähler-Einstein metric. Then \( \text{Aut}(X) = (\text{Isom}(X))^C \). In particular, \( \text{Aut}(X) = (\text{Aut}(X) \cap U(N + 1))^C \).

**Proof.** It follows from the proof of Theorem 4 in [CDS15c]. \( \Box \)

By Theorem 9.9, we are able to construct an analytic open set \( U_W \subset \mathbb{P}W \times _{\mathbb{R}^N} \mathbb{Z}^* \) that is stabilizer preserving. To obtain the Zariski openness, one only needs to observe the fact that

\[
\text{Aut}(Z^*) := \{(z,G_z) \in Z^* \times \text{SL}(N+1) \mid G_z < \text{SL}(N+1) \text{ is the stabilizer of } z \text{ in } \text{SL}(N+1)\}
\]

is a constructible set. Hence our proof is completed. \( \Box \)

**Remark 8.9.** One notice that contrast to Theorem 8.5, there exists smooth Fano varieties admitting Kähler-Einstein metrics, which are not asymptotically Chow stable (see [OSY12]). On the other hand, Theorem 8.5 can be regarded as an extension of work [Sze10] to the case of smoothable \( \mathbb{Q} \)-Fano varieties.

Finally to prove Theorem 1.3, we need to show that for each \( C \)-closed point \( [z] \in [Z^*/\text{SL}(N+1)] \), \( \{z\} \) has a good moduli space in the sense of [AFS16] Proposition 3.1. To that to, let us first establish the Assumption 9.10 in Section 9. Let \( z = \text{Hilb}(Y) \in U_W \) specializing to \( z_0 = \text{Hilb}(X) \in U_W \subset \mathbb{H}^{N} \) via a 1-PS \( \lambda(t) : G_m \rightarrow \text{Aut}(X) < \text{SL}(N+1) \). Let \( (Y = X|_C, X) \rightarrow (C = \lambda(t) \cdot z, z_0) \subset U_W \) be the restriction of the universal family \( X \rightarrow Z^* \) to the pointed curve \( (C, z_0) \) and also we prefix a basis \( \{s_i\} \subset \text{O}_Y(-mK_{Y/C}) \).

**Lemma 8.10.** Under the notation introduced above, we have \( \text{Aut}(Y) < \text{Aut}(X) \) for \( z := \text{Hilb}(Y) \) close to \( \text{Hilb}(X) \).

**Proof.** By property 3 in the proof of Lemma 3.1 for \( z = \text{Hilb}(Y) \in U_0 \) we have \( \text{aut}(Y) \subset \text{aut}(X) \), hence the identity component of \( \text{Aut}(Y) \) lies in \( \text{Aut}(X) \). We will assume from now on that \( z = \text{Hilb}(Y) \in \mathbb{P}W \) lies in a small analytic neighborhood of \( z_0 = \text{Hilb}(X) \in U_1 \), i.e. \( z \) is very close to \( z_0 \). This together with the fact that there always exists a finite subgroup \( H < \text{Aut}(Y) \) that meets every connected component of \( \text{Aut}(Y) < \text{SL}(N+1) \) imply that all we need is that: for any finite subgroup \( H < \text{Aut}(Y) \), we have \( H < \text{Aut}(X) \). To achieve that, let us choose \( H \)-invariant smoothable divisor \( E \in | - mK_Y |^H \) so that \( (Y, \frac{E}{m}) \) is klt, the existence of such \( E \subset Y \) is guaranteed by the following result.

**Claim 8.11.** Let \( Y \) be a smoothable \( \mathbb{Q} \)-Fano variety. Fix a finite group \( H \subset \text{Aut}(Y) \). For \( m \) sufficiently divisible there is an invariant section \( E \in | - mK_Y |^H \) such that \( (Y, (1 - \epsilon)E) \) is klt for any \( 0 < \epsilon \leq 1 \) and smoothable. In particular, \( (Y, \frac{E}{m}) \) is smoothable and klt for \( m > 1 \). Moreover, \( m \) can be uniformly bounded provided \( Y \) is inside a bounded family.

**Proof.** Let \( \mu : Y \rightarrow \bar{Y} \) be the quotient of \( Y \) by \( H \), and \( D \) be the branched divisor, so \( \mu^*(K_{\bar{Y}} + D) = K_Y \). In particular, \( (\bar{Y}, D) \) is klt (as klt is preserved under finite quotient [KM98, Theorem 5.20] ) and \( -(K_{\bar{Y}} + D) \) is ample. Thus for a sufficiently divisible \( m \) satisfying \( -m(K_{\bar{Y}} + D) \) being very ample, we can choose a general section \( F \in | - m(K_{\bar{Y}} + D) | \) so that \( (\bar{Y}, D + (1 - \epsilon)F) \) is klt for any \( 0 < \epsilon \leq 1 \). Then \( E := \mu^*(F) \) is \( H \)-invariant and \( (Y, (1 - \epsilon)E) \) is klt for any \( 0 < \epsilon \leq 1 \). Finally, we justify that \( (Y,E) \) is actually smoothable as long as \( Y \) is. Since \( Y \) is a degeneration a smooth family \( \{Y_i\}_t \), and every element in \( | - mK_{Y} | \) can be represented as a degeneration of general members of \( | - mK_{Y_i} | \), from which we conclude \( (Y,E) \) is a degeneration of smooth pairs \( \{(Y_i, E_i)\}_t \). \( \Box \)
Then by Theorem 1.2 and 5.2 \( (Y, \mathbb{P}^m) \) admits a continuous family of Kähler metric \( \{\omega_Y(\beta)\} \) solving

\[
\text{Ric}(\omega_Y(\beta)) = \beta \omega_Y(\beta) + \frac{1 - \beta}{m} [E] \text{ on } Y,
\]

from which we obtain

\[
(58) \quad \text{Hilb}(Y, \omega_Y(\beta)) \xrightarrow{\beta \to 1} U(N + 1) \cdot \text{Hilb}(X) \subset \mathbb{P}^{N^2}
\]

thanks to Theorem 7.2 and the fact that \( m \) is uniformly bounded by Lemma 8.3 where \( \text{Hilb}(Y, \omega_Y(\beta)) \) is the Hilbert point corresponding to the Tian’s embedding of \( Y \subset \mathbb{P}^N \) with respect to the metric \( \omega_Y(\beta) \) on \( Y \subset \mathbb{P}^N \) and any prefixed basis \( \{s_i \} \subset H^0(\mathcal{O}_Y(-rK_Y)) \). This allows us to introduce a continuous family of Hermitian metric \( h_{KE}(\beta(t)) \) with \( \beta(t) := 1 - \|t\| \) on \( \mathcal{O}_{\mathcal{Y}_t}(-K_{\mathcal{Y}_t}) \rightarrow \mathcal{Y}_t \) for \( t \) small, \( 0 < \|t\| := \text{dist}_{\mathcal{C}}(t, 0) < 1 \), such that \( \omega_Y(\beta(t)) = -\sqrt{-1} \partial \bar{\partial} \log h_{KE}(\beta(t)) \). By (58), the metric \( h_{KE}(\beta(t)) \) can be continuously extended to \( t \in C \). Now let \( \{s_i\} \) be the local basis of \( \mathcal{O}_{\mathcal{Y}_t}(-rK_{\mathcal{Y}_t/C}) \mid_{\{t \mid [t] \} \subset C} = \mathcal{O}_{\mathcal{P}_{\mathcal{X}_t}}(1) \) corresponding to the coordinate sections of \( \mathcal{O}_{\mathcal{P}_{\mathcal{X}_t}}(1) \) such that \( \{s_i(0)\} \) induces Tian’s embedding for \( z_0 = \text{Hilb}(X) \) and define

\[
A_{KE}(t, \beta(t)) = \left[ (s_i, s_j)_{KE,\beta(t)}(t) \right]
\]

with

\[
(s_i, s_j)_{KE,\beta(t)}(t) = \int_{\mathcal{Y}_t} \langle s_i(t), s_j(t) \rangle h_{KE}(\beta(t))^{\omega^*_N}(\beta(t)),
\]

then we obtain a family of Tian’s embedding

\[
T : (\mathcal{Y}_t, E_t; \omega_Y(\beta(t))) \rightarrow \mathbb{P}^N \text{ with } (\mathcal{Y}_t, E_t) \cong (Y, E) \text{ for } t \neq 0.
\]

given by \( \{g(t) \circ s_i(t)\}_{i=0}^n \) with \( g(t) = A_{KE}^{1/2}(\beta(t)) \). The map \( T \) extends to \( \mathcal{Y}_0 = X \) thanks to the continuity of the metric \( h_{KE}(\beta(t)) \) at \( 0 \in C \).

Now by our choice of \( z_0 \) and basis \( \{s_i(t)\} \), we have \( A_{KE}(0, 1) = I_{N+1} \in \text{SL}(N + 1) \), and hence

\[
(60) \quad g(t) \sim I_{N+1} + O(t).
\]

This implies that

\[
(61) \quad \hat{z}(t) := \text{Hilb}(\mathcal{Y}_t, \omega_Y(\beta(t))) = g(t) \cdot z_t \in U_{1,\epsilon} := \exp(\text{aut}(X)_{z_t}^{1}) \cdot U_1
\]

for \( 0 < \|t\| \ll 1 \), where \( z_t = \lambda(t) \cdot \text{Hilb}(X) \). Since \( \omega_Y(\beta(t)) \) is a conical Kähler-Einstein metric on \( \mathcal{Y}_t \), it follows from the log version of Claim 8.8 (cf. CDS15c, Theorem 4.1) that

\[
H_{\mathcal{Y}_t}(t) = g(t) \cdot H_{\mathcal{Y}_t} \cdot g(t)^{-1} < U(N + 1) \text{ where } H_{\mathcal{Y}_t}(t) = \lambda(t) \cdot H \cdot \lambda(t)^{-1}.
\]

By Lemma 0.8 and (60), we obtain that \( H_{\mathcal{Y}_t}(t) < \text{Aut}(X) \) and hence \( H < \text{Aut}(X) \) as \( \lambda(t) < \text{Aut}(X) \) by our choice. On the other hand, by transversality of \( \text{aut}(X)^{1} \)-action on \( U_{1,\epsilon} \), for \( 0 < \|t\| < 1 \) we have \( \text{Aut}_0(Y) < \text{Aut}(X) \) where \( \text{Aut}_0(Y) \) is the identity component of \( \text{Aut}(Y) \). This implies that \( \text{Aut}(Y) = \langle \text{Aut}_0(Y), H \rangle < \text{Aut}(X) \) where \( \langle \text{Aut}(Y), H \rangle \) is the subgroup generated by \( H \) and \( \text{Aut}_0(Y) \) and our proof is completed.

As a direct consequence of Lemmas 8.3 and 8.10 and 8.7, we have the following:

**Corollary 8.12.** After a possible shrinking of the Zariski open neighborhood \( z_0 \subset U_\mathcal{W} \subset \mathbb{P}^N \times \mathbb{R}^{X,N} \) \( Z^* \), we have

\[
\text{SL}(N + 1)_{z} < \text{Aut}(X), \forall z \in U_\mathcal{W}
\]

where \( \text{SL}(N + 1)_{z} \) is the stabilizer of \( z \) inside \( \text{SL}(N + 1) \). In particular, Assumption 9.10 holds in this case.

Next in order to apply Lemma 0.14 in Section 9, we now establish Assumption 9.13. Let us fix \( G = \text{SL}(N + 1) \) and \( G_{z_0} = \text{Aut}(X) \).

**Lemma 8.13.** Let \( z_0 \subset U_r \subset \mathbb{P}^N \) be defined in Definition 9.12 and

\[
U_{Z^*,r} := U_r \times_{\mathbb{R}^{X,N}} Z^*.
\]

Then for \( 0 < r \) sufficiently small, we have \( U_{Z^*,r} \subset U^{1,\epsilon} \), i.e. Assumption 9.13 is satisfied on \( U_{Z^*,r} \).
Proof. In order to better illustrate the idea, let us first deal with the case that $z_0$ is K-stable, hence $G_{z_0} < \infty$. As we have seen in the proof of Theorem 8.5 there is a proper $U(N + 1)$-invariant slice $z_0 \in \Sigma \subset \mathbb{P}^N$ obtained via Tian’s embedding. By the continuity of $\Sigma$ and transversality of the $g_{z_0}$-action on $U_0$ (cf. the proof Lemma 8.1), for some $0 < r'' < r' \ll 1$ and $0 < \epsilon < 1$ we have
\begin{equation}
B_{2r'}(z_0, r'') \subset U_0 \cap \exp g_{z_0}^\perp \cdot \Sigma,
\end{equation}
where $g_{z_0}^\perp := \{ \xi \in g_{z_0} \mid |\xi| < \epsilon \}$ and $B_{2r'}(z_0, r'')$ denotes the ball of radius $\epsilon$ centered at $z_0 \in Z^*$ with respect to a prefixed continuous metric on $Z^*$. Moreover, by choosing a small $r$ if necessary, we may assume $X_0$ is K-stable for all $z \in B_{2r'}(z_0, r'')$. To see the lemma, let $\{s_i\}$ be the local basis of $\pi_*([\mathcal{O}_z(1)];\epsilon)$ corresponding to the coordinate sections of $\mathbb{P}^N$ such that the induced embedding of $X = X_{z_0} \subset \mathbb{P}^N$ gives rise to the unique Kähler-Einstein metric when restricted to each $X_0$ with $z \in B_{2r'}(z_0, r'')$, and we can introduce the matrix $A_{\mathbb{KE}}(z)$ as in the proof of Lemma 8.10. Then (62) follows from the continuity of $A_{\mathbb{KE}}(z)$ with respect to $z \in Z^*$ and $A_{\mathbb{KE}}(z_0) = I_{N+1}$ (as $X \subset \mathbb{P}^N$ is a Tian’s embedding).

As a consequence, for any pair $(z, g) \in B_{2r'}(z_0, r'') \times G$ satisfying $g \cdot z \in B_{2r'}(z_0, r'')$, there are $h', h'' \in G$ such that under the quotient map
\begin{equation}
[\cdot] : G \to \frac{G}{G_{z_0}},
\end{equation}
$[h'], [h''] \in G/G_{z_0}$ are perturbations of $[1] \in G/G_{z_0}$ and $h' \cdot z, h'' \cdot g \cdot z \in \Sigma$. Since both $h \cdot z$ and $h' \cdot q \cdot z$ are the Hilbert points of Tian’s embedding of the same $\mathbb{Q}$-Fano variety, we know that $u := h^{-1} \cdot h' \cdot g \in U(N + 1)$. This implies that $g \cdot z = h \cdot z$ with $h = h^{-1} \cdot h'$ and $[h]$ being uniformly bounded (which depend only on $B_{2r'}(z_0, r'')$ and $z_0$) in $G/G_{z_0}$.

Since the property whether or not $z$ lies in $t^{th}$ is independent of the $G_{z_0}$-translation, we conclude that Assumption 9.13 holds for K-polystable points lies in $U_{2r'} \subset G_{z_0} \cdot B_{2r'}(z_0, r'')$ for some $0 < r < r''$.

For the general case, let us introduce a general divisor $D \in | - mK_X|$ for sufficiently divisible $m$ such that
\begin{enumerate}
\item $(X, D)|_{U_W}$, where $U_W$ is given in the proof of Theorem 8.5, are family of $\mathbb{Q}$-Fano variety;
\item $D_{z_0}$ is smooth whenever $X_0$ is for $z \in U_W$.
\end{enumerate}

Then by Theorem 1.2 we can construct a thickened (due to the introducing of $D$) proper $U(N + 1)$-invariant slice $\Sigma_{z_0} = \Sigma_{z_0, D}$ using Tian’s embedding of $X_0 \subset \mathbb{P}^N$ with respect to the the unique conical Kähler-Einstein metric
\[\text{Ric}(\omega_{X_0}(\beta)) = \beta \omega_{X_0}(\beta) + \frac{1 - \beta}{m}[D_{z_0}] \text{ on } X_z\]
for all $z \in U_W$ near $z_0$. In particular, Theorem 1.2 and 7.2 imply that $\Sigma_{z_0, D} \rightarrow \Sigma$ in the sense that $\forall \epsilon > 0$, $\Sigma_{z_0, D} \cap \mathbb{B}(z, \epsilon) \rightarrow \Sigma$ (the $\epsilon$-tubular neighborhood of $\Sigma$ as $\beta \rightarrow 1$). This implies that for $0 < r' \ll 1$ and $z, z' \in B_{2r'}(z_0, r'')$ that are contained in
\[\left( G \cdot \text{Hilb}(X_z) \right) \bigcap (U(N + 1) \cdot \exp \sqrt{-1} g_{z_0}^\perp \cdot B_{2r'}(z_0, r') \cap \{ z \in \mathbb{P}^W \text{ determined in 53} \})\]
with $g_{z_0}^\perp := \{ \xi \in g_{z_0} \mid |\xi| < \epsilon \}$, the $U(N + 1)$-orbits for Tian’s embedding of $(X_z, D_z)$ and $(X_{z'}, D_{z'})$ are very close in the sense that they can be translated to each other by an element $h \in U(N + 1) \cdot \exp \sqrt{-1} g_{z_0}^\perp \cdot Z^* \subset G$ (i.e. $[h] \in G/G_{z_0}$ is bounded in the sense of 71). In particular, this allows us to treat these two $U(N + 1)$-orbits as almost identical one and we argue exactly the same way as the K-stable case. This completes the justification of Assumption 9.13 for a neighborhood of $z_0 \in U_{2r'}$, for some sufficient small $r > 0$. □

**Proof of Theorem 8.5.** By [AFS16, Proposition 3.1], proving our statement boils down to establishing the following: for any $\mathbb{C}$-closed point $[z_0] \in [Z^*/\text{SL}(N + 1)]$ there is an affine neighbourhood $z_0 \in U_W \subset \mathbb{P}^W$ determined in 53 such that
\begin{enumerate}
\item The morphism $[U_W/G_{z_0}] \rightarrow [Z^*/G]$ is affine and strongly étale (i.e. stabilizer preserving and sending closed point to closed point), and
\item For any $z \in Z^*$ specializing to $z_0$ under $G$-action, the closure of substack $[z]$ inside $[Z^*/G]$, $[\{ z \}] \subset [Z^*/G]$ admits a good moduli space.
\end{enumerate}
Here we fix $G = \text{SL}(N + 1)$ and $G_\text{red} = \text{Aut}(X)$.

We have shown the morphism is strongly étale by Theorem 8.5. Next we confirm the affineness. Since $Z^* \to [Z^*/\text{SL}(N + 1)]$ is faithfully flat, it suffices to show that
\[
\phi: G \times_{G_{\text{red}}} U_W \to Z^*
\]
is affine. Since $\phi$ is quasi-finite and $Z^*$ is separated, it suffices to choose $U_W$ such that $G \times_{G_{\text{red}}} U_W$ is affine. Let $U_W \subset Z^* \cap \mathbb{P}(W)$ be a $G_\text{red}$-invariant affine open set then we know $G \times_{G_{\text{red}}} U_W$ is affine since it is a quotient of the affine scheme $G \times U_W$ by the free action of the reductive group $G_{\text{red}}$.

Furthermore, we have isomorphism
\[
(G \times_{G_{\text{red}}} U_W)/G \cong U_W/G_{\text{red}}
\]
and $G \times_{G_{\text{red}}} U_W$ is the inverse image of the affine neighborhood
\[
\pi_W|_{U_W}(z_0) = 0 \in U_W/G_{\text{red}} \text{ with } \pi_W \text{ defined in (70)}
\]
under the GIT quotient by $G$.

Now we establish the second condition. Since we have already established the uniqueness of minimal orbit contained in $\overline{\mathcal{B}(O)}$ stated after diagram (55), all we need is the affineness of $G \cdot \pi_W^{-1}(0)$ as it implies that for any $z \in Z^*$ satisfying $G \cdot z \ni z_0$ the closure of $[z] \in [Z^*/G]$ is a closed substack of $[G \cdot \pi_W^{-1}(0)/G]$, which can be written as the form $\text{Spec}(A)/G$ for some affine scheme $\text{Spec}(A)$, hence $[z]$ admits a good moduli space.

To obtain the affineness, one notices that Theorem 8.5 and Corollary 8.12 guarantee the Assumption 9.10, also we have already established Assumption 9.13 by Lemma 8.13. Thus the morphism
\[
\phi|_{G \times_{G_{\text{red}}} U_r} : G \times_{G_{\text{red}}} U_r \to G \cdot U_r \text{ (cf. Definition 9.12 for } U_r\text{)}
\]
is a finite morphism for $0 < r \ll 1$ by Lemma 9.14 in Section 9. By choosing $0 < r$ even smaller, we may conclude that $\phi|_{G \times_{G_{\text{red}}} U_r}$ is an analytic isomorphism, since $\phi|_{G \times_{G_{\text{red}}} U_r}$ is an isomorphism and immersion near $G \cdot z_0$. Now we restrict $\phi$ to the fiber over $[z_0] \in [Z^*/G]$, we have a finite morphism
\[
G \times_{G_{\text{red}}} \pi_W^{-1}(0) \to G \cdot \pi_W^{-1}(0).
\]
Since $G \times_{G_{\text{red}}} \pi_W^{-1}(0)$ is a fiber of a GIT quotient morphism, we conclude that $G \cdot \pi_W^{-1}(0)$ is affine.

As a consequence, the étale chart $\phi/G : (G \times_{G_{\text{red}}} U_W)/G \to G \cdot U_W/G$ is actually a finite morphism, which implies $G \cdot U_W/G$ is affine. This gives an affine neighborhood of $[z_0] \in \mathcal{K}_N$. This proves that the algebraic space $\mathcal{K}_N$ is actually a scheme. Finally to prove the last statement of Theorem 1.3, we observe that Lemma 8.3 implies that the closed points of $\mathcal{K}_N$ stabilizes. However, since $\mathcal{K}_N$ is semi-normal, we indeed know that they are isomorphic (see [Ko96, 7.2]).

**Remark 8.14.** We want to point out that by shrinking $U_W$ if necessary the map $\phi : G \times_{G_{\text{red}}} U_W \to G \cdot U_W$ is actually strongly étale in the sense of [MFK94, page 198], i.e. $U_W$ is a Luna’s étale slice. To see that one notices that we have already established in the above that the categorical quotient $(G \cdot U_W)/G$ is in fact a good quotient (see also [Dr04, Definition 2.12]) and moreover the map $\phi$ induces an étale morphism
\[
\phi/G : (G \times_{G_{\text{red}}} U_W)/G \to (G \cdot U_W)/G.
\]
So all we need to show is
\[
(\phi, \pi_{G \times_{G_{\text{red}}} U_W}) : (G \times_{G_{\text{red}}} U_W) \to (G \cdot U_W) \times_{(G \cdot U_W)/G} (G \times_{G_{\text{red}}} U_W)/G
\]
is an isomorphism, where $\pi_{G \times_{G_{\text{red}}} U_W} : (G \times_{G_{\text{red}}} U_W) \to (G \times_{G_{\text{red}}} U_W)/G \cong U_W/G_{\text{red}}$ is the GIT quotient map. But this follows from the fact that $\phi|_{G \times_{G_{\text{red}}} U_r}$ in (63) is an analytic isomorphism for small $r$ and $\phi$ is finite.

**Remark 8.15.** Notice that we can take the local GIT quotient of a similarly defined $Z_{\text{red}}^\circ$ for each $N_r = \chi(X, O_X(-rK_X)) - 1$. Although we are unable to conclude that those local GIT quotients we constructed in this section will be stabilized for $N \gg 1$, their semi-normalizations indeed will be. Another reason we work over a seminormal base is that the condition of being smoothable does not yield a reasonable moduli functor for schemes, e.g., in general there is no good definition of smoothable varieties over an Artinian ring.
Lemma 9.2
(Chapter 2, Proposition 2.7, [MFK94])

First, let us recall some basics from [MFK94, section 2 of Chapter 2]. Let the
is the unique
relation:

Definition 9.1.
Let

where

Lemma 9.3.
Let

The next Lemma is a slight extension of [MFK94, Chapter 2, Proposition 2.14] essentially con-

Moreover, if

The next Lemma is a slight extension of [MFK94, Chapter 2, Proposition 2.14] essentially con-

Lemma 9.4.
Let

Proposition 9.4.
Let

9. Appendix
In this section, we will prove Proposition 7.5 in a more general setting.
First, let us recall some basics from [MFK94, section 2 of Chapter 2]. Let
be a reductive group
acting on a (quasi-)projective variety
acting on

such that

Appendix

ON PROPER MODULI SPACE OF SMOOTHABLE KÄHLER-EINSTEIN FANO VARIETIES 33
and define

$$\varpi_{G}^{M_{1}, M_{2}}(z) := \sup \left\{ \beta \in (0, 1) \left| \inf_{\delta \in D(G)} \mu_{L_{1} - \beta}(z, \delta) \geq 0, \forall \beta' \in [0, \beta] \right. \right\}$$

or 0 if the set on the right hand side is an empty set. Suppose \( S \subset Z \) is a constructible set such that \( \varpi_{G}^{M_{1}, M_{2}} \mid_{S} \geq 0 \). Then \( \varpi(M_{1}, M_{2}) \) defines a \( Q \)-valued constructible function on \( S \), i.e. \( S = \sqcup_{i} S_{i} \) is a union of finite constructible sets with \( \varpi(M_{1}, M_{2}) \) being constant on each \( S_{i} \).

**Proof.** We replace \( L \to Z \) by its power such that \( L_{1} := L \otimes M_{1} \) and \( L_{2} := L \otimes M_{2} \) are both ample. Then we fix a maximal torus \( T \subset G \) and let \( \{ I^{l_{i}} \} \) and \( \{ I^{l_{2}} \} \) be the rational linear functionals on \( \text{Hom}_{G}(\mathbb{G}_{m}, T) \) associated to \( L_{i}, i = 1, 2 \). By Lemma 9.3, for any \( I \in 2^{1, \ldots, r_{I_{1}}} \sqcup 2^{1, \ldots, r_{I_{2}}} \), \( S^{T}_{I} := \psi^{-1}(I) \cap S \) is a constructible set. Now we define

$$\varpi_{T}^{M_{1}, M_{2}}(z) := \sup \left\{ \beta \in (0, 1) \left| \inf_{\delta \in D(T)} \mu_{L_{1} - \beta}(z, \delta) \geq 0, \forall \beta' \in [0, \beta] \right. \right\}$$

or 0 if the right hand side is an empty set. In other words, it is the *first* time such that the difference of two rational piecewise linear convex functions

$$\mu_{L_{1}}(z, \cdot) - (1 - \beta) \mu_{L_{2}}(z, \cdot) - \beta \mu_{L}(z, \cdot) = \mu_{M_{1}}(z, \cdot) - (1 - \beta) \mu_{M_{2}}(z, \cdot)$$

vanishes along a ray in \( \text{Hom}_{G}(\mathbb{G}_{m}, T) \) or in \( \{0,1\} \). Clearly, we have \( \beta_{1} \in Q \) and they are independent of the choice of \( L \).

Now in order to pass from \( \varpi_{T}^{M_{1}, M_{2}} \) to \( \varpi_{G}^{M_{1}, M_{2}} \), let us recall Chevalley’s Lemma [Har77, Chapter II, Exercise 3.19] which states that the image of constructible set under an algebro-geometric morphism is again constructible. By applying it to the group action morphism

$$G \times Z \to Z,$$

we obtain that \( S^{G}_{J} := G \cdot (\psi^{-1}(I) \cap S) \subseteq S^{T}_{J} \) for all constructible \( \forall I \in 2^{1, \ldots, r_{I_{1}}} \sqcup 2^{1, \ldots, r_{I_{2}}} \).

Now for any 1-PS \( \lambda \), there is a \( \gamma \in G \) such that \( \gamma \lambda \gamma^{-1} \subset T \). By Lemma 9.2, we have \( \mu_{L_{i}}(z, \lambda) = \mu_{L_{i}}(\gamma z, \gamma \lambda \gamma^{-1}) \), \( i = 1, 2 \), which implies that

$$\varpi_{G}^{M_{1}, M_{2}}(z) = \min \left\{ \beta \left| S^{G}_{J} \cap G \cdot z \neq \emptyset \text{ for } J \in 2^{1, \ldots, r_{I_{1}}} \sqcup 2^{1, \ldots, r_{I_{2}}} \right. \right\}.$$

To see it is a constructible function on the constructible set \( G \cdot S \), one notices that all possible finite intersections of \( \{ S^{G}_{J} \}_{J} \) form a stratification of \( G \cdot S \) into constructible sets and \( \varpi_{G}^{M_{1}, M_{2}} \) is constant on each stratum.

Now to apply the above set up to the \( \beta \)-K-stability of \((X, D) \subset \mathbb{P}^{N} \) with respect to the SL\((N + 1)\) action. Let \( N + 1 = \dim H^{0}(X, K_{X}^{-m}) \) and we define an open subscheme

\[
Z := \left\{ \text{Hilb}(X, D) \mid (X, D) \subset \mathbb{P}^{N} \times \mathbb{P}^{N} \text{ be a } k \text{-th pair with Hilbert polynomial } x = (x, \bar{x}) \right. \}
\]

satisfying \( D \subset X, D \in [-mK_{X}] \) and \( \mathcal{O}_{z,N}(1)|_{X} \cong K_{X}^{-m} \).

Let \( \lambda_{CM} \to Z \) (cf. [FR06], Definition 2.3 or [PT06], equation (2.4)) be the CM-line bundle over \( Z \) normalized in such a way that the corresponding weight for any one parameter subgroup of SL\((N + 1)\) is exactly the DF introduced in Definition 2.3 and

$$\lambda_{\text{Chow}}(X) := \lambda_{\text{Chow}}(X, \mathcal{O}_{X}(-mK_{X})) \to Z \text{ and } \lambda_{\text{Chow}}(D) := \lambda_{\text{Chow}}(D, \mathcal{O}_{X}(-rK_{X}))[D] \to Z$$

be the Chow line bundles introduced in [FR06], equation (3.3)] for the flat family \( X \to Z \) and \( D \to Z \) respectively.

**Proof of Proposition 7.3** Let us introduce

$$M_{1} := \lambda_{CM} \text{ and } M_{2} := \left( \lambda_{\text{Chow}}(X)^{\otimes \frac{m}{r+m} \otimes \lambda_{\text{Chow}}(D)} \right)^{\otimes \frac{1}{r+m} \otimes K_{X}^{-m}} \text{ (cf. 2)}.$$ 

By Theorem 5.2 we know \( (X_{i}, D_{i}) \) is \( \beta \)-K-stable \( \forall t \in C \) and \( \beta \in [0, \beta_{0}] \). After removing finite number of points from \( C \), we obtain a quasi-projective \( 0 \in S \subset C \) over which \( \pi_{*, \mathcal{O}_{X}(-rK_{X}/C)}|_{S} \cong \mathcal{O}_{S}^{\otimes N + 1} \). By fixing a basis of \( \pi_{*, \mathcal{O}_{X}(-rK_{X}/C)}|_{S} \), we obtain an embedding

$$i : (X_{i}, D_{i}, \mathcal{O}_{X}(-rK_{X}))/C \times S \to \mathbb{P}^{N} \times S \times S$$
which in turn induces an embedding $S \subset Z$ with $S$ being constructible and $\varpi_{SL(N+1)}^{M_1,M_2} \geq \beta_0 > 0$. By applying Proposition 9.4 to $S \subset Z$, we obtain $\kst(\mathfrak{X}_t, D_t) = \varpi_{SL(N+1)}^{M_1,M_2}(t), \forall t \in S$ is a constructible function. Our proof is completed.

**Remark 9.5.** The above argument first appeared in [Pau12] and [Oda12] independently, they observed that one can also conclude that the K-polystable locus in $S$ is also constructible.

9.2. Stabilizer Preserving Property.

9.2.1. Richardson’s example. Consider $SL(2)$-action on

$$\text{Sym}^{\otimes 3} \mathbb{C}^2 = H^0(\mathbb{F}_2(3)) = \text{Span}_\mathbb{C}\{X^3, X^2Y, XY^2, Y^3\}$$

induced by the standard action on $\mathbb{C}^2$. Then the stabilizer of $p_0(X, Y) = (X - Y)(X + Y)^2$ is trivial and the stabilizer of $p(X, Y) = (X - Y)(X - \omega Y)(X - \omega^2 Y)$ is given by

$$\begin{bmatrix} 0 & \omega \\ \omega & -1 \end{bmatrix} \in SL(2, \mathbb{C}) \text{ with } \omega^3 = 1.$$ 

Let

$$\alpha(t) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} t^2 & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in GL(2, \mathbb{C}) = \frac{1}{2} \begin{bmatrix} t^2 + 1/t & -t^2 + 1/t \\ t^2 + 1/t & t^2 + 1/t \end{bmatrix} \in GL(2, \mathbb{C})$$

then

$$\alpha(t) : \begin{cases} X - Y \rightarrow t^2(X - Y) \\ X + Y \rightarrow t^{-1}(X + Y) \end{cases}$$

hence fixes $p_0(X, Y) = \frac{3}{4}(X - Y)(X + Y)^2 \in \text{Sym}^{\otimes 3} \mathbb{C}^2$. Now let us define

$$p_t(X, Y) = p(\alpha(t) \cdot X, \alpha(t) \cdot Y)$$

$$= \frac{1}{4} t^3(X - Y)(t^2(1 + \omega) + t^{-1}(1 - \omega))X + (t^2(1 + \omega^2) + t^{-1}(1 - \omega^2))Y)$$

$$\cdot (t^2(1 + \omega^3) + t^{-1}(1 - \omega^3))X + (t^2(1 + \omega^2) + t^{-1}(1 - \omega^2))Y)$$

$$= \frac{1}{4} (X - Y)(t^3(1 + \omega) + (1 - \omega))X + (t^3(1 + \omega^2) + (1 - \omega))Y) \cdot$$

$$\cdot (t^3(1 + \omega^3) + (1 - \omega^3))X + (t^3(1 + \omega^2) + (1 - \omega^2))Y),$$

then we have

$$\lim_{t \to 0} p_t(X, Y) = \frac{3}{4}(X - Y)(X + Y)^2$$

and the stabilizer of $p_t$ is the subgroup $\langle \zeta_t := \zeta_{p_t} \rangle \subset SL(2)$ with

$$\zeta_{p_t} := \alpha(t^{-1}) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \in SL(2, \mathbb{C})$$

In particular, the family of stabilizers $\langle \zeta_t \rangle \subset SL(2, \mathbb{C})$ is unbounded as $t \to 0$ unless $\omega = 1$.

9.2.2. Main results. Before we proceed to the proof, let us collect some basic facts on compact Lie groups acting on $\mathbb{P}^M$. Let $K$ be a compact Lie group and $\rho : K \to SU(M + 1)$ be a linear representation and $\rho^C : G = K^C = SL(M + 1)$ be its complexification. Let $z_0 \in \mathbb{P}^N$ with stabilizer $G_{z_0} = (K_{z_0})^C := (G_{z_0} \cap K)^C$. Let $\mathfrak{v}_{z_0} = \text{Lie}(K_{z_0})$ be the Lie algebra and we fix a bi-invariant inner product $\langle \cdot, \cdot \rangle_t$ on $\mathfrak{v}$ and let $\mathfrak{v}^\perp_{z_0} \subset \mathfrak{v}$ be its orthogonal complement with respect to $\langle \cdot, \cdot \rangle_t$. Then the infinitesimal action $\sigma_{z_0} : g \mapsto T_{z_0}^\mathbb{P}^M$ is $G_{z_0}$-equivariant in the sense that

$$\sigma_{z_0}(\text{Ad}_g \xi) = g \cdot \sigma_{z_0}(\xi) \text{ for all } g \in G_{z_0},$$

and there is a $G_{z_0}$-invariant linear subspace $z_0 \in \mathbb{P} W := \mathbb{P}(W' \oplus \mathbb{C} z_0) \subset \mathbb{P}^M$ such that

$$\mathbb{P}^M = \mathbb{P}(W' \oplus (t_{z_0}^\perp)^C) = \mathbb{P}(W \oplus \mathbb{C} z_0' \oplus (t_{z_0}^\perp)^C) \text{ with } (t_{z_0}^\perp)^C := t_{z_0}^\perp \oplus \mathbb{C},$$

where $0 \neq z_0 \in \mathbb{C}^{M+1}$ is a lift of $z_0 \in \mathbb{P}^M$ and $\mathbb{C}^{M+1} = W' \oplus \mathbb{C} z_0 \oplus (t_{z_0}^\perp)^C$ is a decomposition as $G_{z_0}$-module.
Assumption 9.6 (Boundedness). There is a $K$-invariant slice
\[ \Sigma \xrightarrow{\phi} \mathbb{P}^M \]
containing $z_0$ such that $\Sigma$ is proper over $\mathbb{P}^M/K$ (i.e. the restriction of the quotient map $q|_{\Sigma}: \Sigma \to \mathbb{P}^M/K$ is proper) and $G_z = (G_z \cap K)^C$ for all $z \in \Sigma$.

Consider the map
\[ G \times \mathbb{P}W \xrightarrow{(g,w)} G \cdot \mathbb{P}W \subset \mathbb{P}^M \]
then for $\xi \in \mathfrak{g}_{z_0}$ and $\delta w \in T_{z_0}\mathbb{P}W$ we have
\[ d\phi|_{(\epsilon, \mathfrak{g})}(\xi, \delta w) = \sigma_{\mathfrak{g}_0}(\xi) + \delta w \in T_{z_0}\mathbb{P}W \]
where $\sigma_{\mathfrak{g}_0}: \mathfrak{g} \to T_{z_0}\mathbb{P}W$ denotes the infinitesimal action and $e \in G$ denotes the identity, and as a consequence $\ker d\phi|_{(\epsilon, \mathfrak{g})} = \mathfrak{g}_{z_0}$. Now let us define an open set
\[ U_0 := \{ w \in \mathbb{P}W \, | \, \text{rk} \left( q \circ d\phi|_{(\epsilon, \mathfrak{g})} : \mathfrak{g} \times T\mathbb{P}W \to (\mathbb{P}^N|_W)/(\mathbb{P}W) \right) = \dim \mathfrak{g}_{z_0} \} \subset \mathbb{P}W \]
with $q: \mathbb{P}^N|_W \to (\mathbb{P}^N|_W)/\mathbb{P}W$ being the quotient morphism between vector bundles over $\mathbb{P}W$. Then we have

**Lemma 9.7.** $U_0 \subset \mathbb{P}W$ is a $G_{z_0}$-invariant Zariski open set.

**Proof.** Note that the Zariski openness follows from the fact that $q \circ d\phi \in H^0(\mathbb{P}W, T(G \times \mathbb{P}W)|_{\mathfrak{g}_0} \times \mathbb{P}W \otimes (\mathbb{P}^N|_W)/\mathbb{P}W)$. So all we need is the $G_{z_0}$-invariance. To achieve that, one notices that for any $g \in G_{z_0}$, $\xi \in \mathfrak{g}$ and $w \in \mathbb{P}W$ we have
\[ (g \cdot \sigma_w(\xi) = \sigma_{g \cdot w}(\Ad g \xi), \]
which implies that
\[ \sigma_w(\xi) \in T_w\mathbb{P}W \iff \sigma_{g \cdot w}(\Ad g \xi) \in T_{g \cdot w}\mathbb{P}W. \]

Now $w \in U_0$ can be characterized as $q \circ d\phi$ being of full rank which is also equivalent to
\[ \sigma_w(\xi) \in T_w\mathbb{P}W \iff \xi \in \mathfrak{g}_{z_0}. \]
If $g \cdot w \notin U_0$ then there is a $0 \neq \Ad g \xi \in \mathfrak{g}_{z_0}$ such that $\sigma_{g \cdot w}(\Ad g \xi) \in T_{g \cdot w}\mathbb{P}W$, and hence $\sigma_{w}(\xi) \in T_w\mathbb{P}W$. On the other hand, we have decomposition $\mathfrak{g} = \mathfrak{g}_{z_0} \oplus \mathfrak{g}_{z_0}$ as a $G_{z_0}$-module via the Adjoint action thanks to the reductivity of $G_{z_0}$. This implies that $0 \neq \xi \in \mathfrak{g}_{z_0}$, contradicting to (68) and the assumption that $w \in U_0$. Thus our proof is completed. \hfill $\Box$

Now $\phi$ is $G_{z_0}$-invariant with respect to the action $h \cdot (g, w) = (gh^{-1}, h \cdot w)$, hence it descends to a $K$-invariant map, which by abusing of notation is still denoted by
\[ G \times G_{z_0} \mathbb{P}W \xrightarrow{\phi} G \cdot \mathbb{P}W \subset \mathbb{P}^M. \]
Moreover, it is a bi-holomorphism (see the proof of [3a05] Theorem 1.12) from a $K$-invariant tubular neighborhood
\[ U_\epsilon := \{ (g \exp \sqrt{-1} \xi, w) \in G \times G_{z_0} \mathbb{V} \mid g \in K, \xi \in \mathfrak{t}_c, \epsilon \} \]
of the orbit $K \cdot z_0 \cong K/K_{z_0}$ onto $\phi(U_\epsilon) = K \cdot \exp \mathfrak{t}_{z_0} \cdot \mathbb{V}$ for $0 < \epsilon \ll 1$, where $z_0 \in V \subset \mathbb{P}W$ is a $K$-invariant analytic open neighborhood.

Now suppose $\check{g} = g \cdot \exp \sqrt{-1} \xi$ satisfies $g \in K$ and $\xi \in \mathfrak{t}$ with $|\xi| < \epsilon$ such that $\check{g} \cdot w = w$ then:
\[ \phi(g \cdot \exp \sqrt{-1} \xi, w) = \phi(\check{g}, w) = \check{g} \cdot w = w = \phi(e, w) \text{ and } (\check{g}, w) \in U_\epsilon \]
these together with the fact that $\phi|_{U_\epsilon}$ is bi-holomorphic imply that
\[ (\check{g}, w) \sim (e, w) \in G \times \mathbb{P}W. \]
i.e. there is a $h \in G_{z_0}$ such that $(\tilde{g}h^{-1}, hw) = (e, w)$, hence $\tilde{g} = h \in G_{z_0} \cap G_w$. In conclusion, we obtain the following:

**Lemma 9.8 (Local Rigidity).** Let $w \in V \subset \mathbb{P}W$ (defined in (69)) and suppose $\tilde{g} \in G_w$ is of the form $\tilde{g} = g \cdot \exp \xi$ with $g \in K$ and $\xi \in \mathfrak{g}$ satisfies $|\xi| < \epsilon$. Then $\tilde{g} \in G_{z_0}$.

**Theorem 9.9.** Let $K$ be a compact Lie group acting on $\mathbb{P}M$ via a representation $K \to U(M+1)$ and $G = K^C$ be its complexification. Let $z_0 \in \mathbb{P}N$ with its stabilizer $G_{z_0}$ satisfying $G_{z_0} = (G_{z_0} \cap K)^C$ and $z_0 \in \Sigma \subset \mathbb{P}M$ satisfying Condition 9.6. Then there is an $G_{z_0}$-invariant Zariski open neighborhood $z_0 \in U^{op} \subset \mathbb{P}W$ such that for $\forall w \in U^{op} \cap G \cdot \Sigma$ we have $G_w < G_{z_0}$.

Proof. We will first prove it for an analytic neighborhood, then by the constructibility we can pass it to a Zariski open neighborhood.

Suppose Assumption 9.6 holds, then the continuity of the slice $\Sigma$ implies that there is a sufficiently small analytic $K_{z_0}$-invariant neighborhood $z_0 \in \tilde{V} \subset V \subset \mathbb{P}W$ such that for any $w \in \tilde{V}$, there is a $\xi \in (t^*_z)^C$ satisfying $|\xi| < \delta < \epsilon$ and $z \in \Sigma$ such that $w = \exp \xi \cdot z$. In particular, $\exp \xi \cdot K_z \cdot \exp(-\xi) \subset G_w$ is a maximal compact subgroup of $G_w$. Since $K_z < K$ is compact we have

$$\exp \xi \cdot K_z \cdot \exp(-\xi) = \{ h \cdot \exp(Ad_{h^{-1}}\xi) \cdot \exp(-\xi) \mid h \in K_z \} \subset \{ g \cdot \exp \sqrt{-1}\zeta \mid \xi \in \mathfrak{g}, |\xi| < \epsilon \text{ and } g \in K \}.$$

By Lemma 9.8 we must have $\exp(-\xi) \cdot K_z \cdot \exp \xi \subset G_{z_0}$. Hence

$$G_{z_0} \supset \left( \exp(-\xi) \cdot K_z \cdot \exp \xi \right)^C = G_w,$$

since $G_{z_0}$ is reductive. Finally, one notices that the set

$$\{ w \in \mathbb{P}W \mid G_w < G_{z_0} \} \supset G_{z_0} \cdot \tilde{V}$$

is $G_{z_0}$-invariant and constructible. This allows us to choose a $G_{z_0}$-invariant Zariski open subset $U^{op} \supset G_{z_0} \cdot \tilde{V}$, and our proof is completed. □

**Assumption 9.10 (Stabilizer Preserving).** There is a $G_{z_0}$-invariant Zariski open neighborhood of $z_0 \in U^{op} \subset \mathbb{P}W$ such that $G_w < G_{z_0}$ for all $w \in U^{op}$.

**Example 9.11.** Notice that Assumption 9.10 does not hold in general, even in the situation that a 1-PS $\alpha(t)$ degenerating $\lim_{t \to 0} \alpha(t) \cdot z = z_0$, we cannot conclude that $G_{z_0} < G_{z_0}$. Consider the $G = \text{SL}(2)$-action on $\mathbb{P}(\text{Sym}^3 \mathbb{C}^2)$ as in Example 9.2.1. The 1-PS

$$\alpha(t) = \begin{bmatrix} 1 & \frac{t}{2} & 1 \\ -1 & 0 & -1 \\ 0 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & t \\ 1 & t & -1 \end{bmatrix} = \begin{bmatrix} t + 1/t & -t + 1/t \\ -t + 1/t & t + 1/t \end{bmatrix} \in \text{SL}(2, \mathbb{C})$$

degenerates $p(X, Y)$ to $p_0(X, Y) \in \mathbb{P}(\text{Sym}^3 \mathbb{C}^2)$. Then $\mathbb{Z}/3\mathbb{Z} \cong \text{SL}(2)_{p_l} \not\subset \text{SL}(2)_{p_0} \cong \langle \alpha(t) \rangle \cong \mathbb{G}_m$, and the map

$$\text{SL}(2) \times_{\mathbb{G}_m} \mathbb{P}W \to \text{SL}(2) \cdot \mathbb{P}W$$

is not finite.

Twisting the linearization of $G_{z_0}$ on $\mathcal{O}_{\mathbb{P}M}(1)|_{\mathbb{P}W}$ by the inverse of the character corresponding to the action $G_{z_0} \cap \mathcal{O}_{\mathbb{P}M}(1)|_{\mathbb{P}W}$ as in the proof of Lemma 3.1 we obtain that $z_0 \in \mathbb{P}W$ is GIT-polystable with respect to the new $G_{z_0}$-linearization on $\mathcal{O}_{\mathbb{P}M}(1)|_{\mathbb{P}W}$. Let $U^{ss} \subset \mathbb{P}W$ denote the GIT-semistable points with respect to this linearization and

$$\pi_w : \mathbb{P}W \supset U^{ss} \to \mathcal{M} := \mathbb{P}W/G_{z_0} \text{ with } \pi_w(z_0) = 0 \in \mathcal{M}$$

denote the GIT quotient map. Let $0 \in B_{\mathcal{M}}(0, r) \subset \mathcal{M}$ be the open ball of radius $r$ with respect to a prefixed continuous metric. Then for each $r > 0$, we introduce

**Definition 9.12.** Let $U_r$ be the connected component of

$$(G \cdot \pi_w^{-1}(B(0, r))) \cap \mathbb{P}W \subset U^{ss}$$

containing $z_0$. In particular, $U_r$ is $G_{z_0}$-invariant.
Let \([\cdot] : G \to G/G_{z_0}\) denote the quotient map. We say a sequence \([h_i]\) \(\subset G\) is bounded in \(G/G_{z_0}\) if and only if \(\{\psi^{-1}(h_i)\}\) is contained in a bounded subset of \(K \times_{\nu_{z_0}} (\sqrt{-1}\xi)\), where \(\psi\) is the Cartan decomposition (cf. [Sja95, equation (1.8)])

\[
\psi : K \times_{\nu_{z_0}} (\sqrt{-1}\xi) \longrightarrow G/G_{z_0},
\]

which is a \(K\)-equivariant diffeomorphism.

**Assumption 9.13 (Finite Distance).** An analytic open neighborhood of \(z_0 \in U^{\text{fd}} \subset \mathcal{P}W\) is of finite distance if there is a bounded (in the sense above) set \(G_{U^{\text{fd}}} \subset G/G_{z_0}\) depending only on \(U^{\text{fd}}\) and \(z_0\) such that for any pair \((z, g) \in U^{\text{fd}} \times G\) satisfying \(g \cdot z \in U^{\text{fd}}\), there is an \(h \in G\), \([h] \in G_{U^{\text{fd}}} \subset G/G_{z_0}\) such that \(g \cdot z = h \cdot z\), where \([\cdot] : G \to G/G_{z_0}\) is the quotient map. It follows from the definition that \(U^{\text{fd}}\) is \(G_{z_0}\)-invariant.

**Lemma 9.14.** Suppose both Assumption 9.10 and 9.13 are satisfied. Then there is a positive \(\epsilon > 0\) such that for any \(0 < r < \epsilon\), \(U_r\) satisfies the following: for any sequence \([\{g_i, y_i\}]\) \(\subset G \times G_{z_0} U_r\) satisfying \(z_i = g_i \cdot y_i \to z_\infty\), as \(i \to \infty\), after passing to a subsequence, there is a \((g_\infty, y_\infty) \in \{(g_i, y_i)\} \subset G \times G_{z_0}\) such that \(g_\infty \cdot y_\infty = z_\infty\).

In particular, the map \(\phi|_{G \times G_{z_0} U_r} : G \times G_{z_0} U_r \to G \cdot U_r\) is a finite morphism.

**Proof.** First, we notice that after translating \(z_\infty\) by a \(g \in G\) if necessary, we may assume that \(z_\infty \in U_r\). Since we can always pass to a subsequence, we may and will assume \(y_i \to y_\infty \in U_r\) after a possible decreasing of \(r\).

By Assumption 9.13 that there is a sequence \([h_i]\) \(\subset G\), with \([h_i]\) bounded in \(G/G_{z_0}\) and satisfies \(g_i \cdot y_i = h_i \cdot y_i\), hence \(h_i^{-1} \cdot g_i \in G_{y_i}\), \(\forall i\). Now by Assumption 9.10 we have \(h_i^{-1} \cdot g_i \in G_{y_i} < G_{z_0}\), \(\forall i\),

from which we conclude that \([\{g_i\}]\) is bounded in \(G/G_{z_0}\) and hence the set \([\{g_i, y_i\}] \subset G \times G_{z_0} U_r\) is precompact. Thus the morphism \(\phi|_{G \times G_{z_0} U_r} : G \times G_{z_0} U_r \to G \cdot U_r\) is a proper and étale morphism hence finite. \(\square\)

**References**


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