ALGEBRAIC GEOMETRY 2013 FALL

Name ____________________ Student I.D. ____________________.

Instruction:

Please show all work for full credit. You are **not supposed to discuss with any one nor consult with the internet** while working on the final. There are 100 points possible, as indicated below and in the exam. The question marked with (*) are possibly more difficult than others.

Please send back the **latexed** answer in .pdf file by 21:00 June 27 2015 to

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If you have any questions, send them to the same email address. I will try to answer them as soon as possible.

Denote by k the underground field, which is always algebraically closed and of characteristic 0.

Problem 1 (40 Points)

a) (10) Consider the singular hypersurface $X \subset \mathbb{C}^n$ with an equation

$$X : \left(x_1^d + x_1^d + \cdots + x_n^d = 0\right) \subset \mathbb{C}^n.$$

Prove that $0 \in X$ is log canonical if and only if $d \leq n$. Find all possible d such that $0 \in X$ is canonical.

b) (10) Let $E \subset \mathbb{P}^2$ be a smooth curve. Blow up 9 distinct points $p_1, ..., p_9$ on E, we get a smooth surface S. Show that if $\mathcal{O}(3)|_E - \sum_{i=1}^{9} p_i$ is not a torsion element in $\mathrm{Pic}^0(E)$, then $-K_S$ is nef but not semi ample.

c) (10) Let Y be a smooth surface. Let $X \to Y$ be degree 2 finite morphism branch over a curve D. Show that X is smooth if D is smooth; and X has canonical singularities if (Y, D) is simple normal crossing.

d) (10) Let X_i ($i = 1, 2$) be two normal projective with ample divisors H_i. Let $U_i \subset X_i$ be open subsets such that $X_i \setminus U_i$ has codimension ≥ 2 in X_i. Let $f_U \colon U_1 \to U_2$ be an isomorphism such that $f_U(H_1|_{U_1}) = H_2|_{U_2}$. Then f_U extends to an isomorphism $f_X \colon X_1 \to X_2$.

Problem 2 (15 Points) We call (X, Δ) to be a weakly log Fano pair, if (X, Δ) is klt and $-K_X - \Delta$ is big and nef. If $-K_X - \Delta$ is indeed ample, then we call (X, Δ) to be log Fano.

a) (10) Show the étale fundamental group $\pi_1^{\mathrm{alg}}(X) = \{e\}$, i.e., if there is an irreducible normal variety Y, with $f\colon Y \to X$ being finite and étale, then $f = id$.

b) (5) Assume the following fact: for any log Fano pair (X, Δ), we know that $\pi_1^{\mathrm{alg}}(X^{\mathrm{sm}})$ is finite, where X^{sm} is the smooth locus of X. Then show that for any weakly log Fano pair (X, Δ), $\pi_1^{\mathrm{alg}}(X^{\mathrm{sm}})$ is also finite. (If you are not familiar with étale fundamental group, you can also assume $k = \mathbb{C}$ and show the above for topological fundamental groups.)

Problem 3 (20 Points) Assume $k = \mathbb{C}$. Let C be a smooth curve. Let $(X, \Delta) \to C$ be a flat morphism and (X, Δ) is klt. Let $0 \in C$ be a point, and we denote by X_0 the fiber of X over C. Let us assume X_0 to be reduced and irreducible.

a) (5) Assume for all $E \neq X_0$ with $\text{Center}_X(E) \subset X_0$, we have $a(E; X, \Delta + X_0) > -1$. Then show this implies that $(X_0, \Delta|_{E_0})$ is klt.

(*) b) (5) Show the converse of the above is also true.

c) (10) Assume X satisfies the assumption of a). Assume $(Y, \Gamma) \to C$ to be another family, such that if we denote by $C^0 = C \setminus \{0\}$, then

$$(X, \Delta) \times_C C^0 \cong (Y, \Gamma) \times_C C^0.$$

Assume the fiber Y_0 of Y over 0 is reduced and $(Y, \Delta + Y_0)$ is log canonical. And assume

$$K_X + \Delta \sim_{\mathbb{Q},C} K_Y + \Gamma \sim_{\mathbb{Q},C} 0.$$

Show $(X, \Delta) \cong (Y, \Gamma)$.

Problem 4 (15 points) Let $f\colon X \to Y$ be a projective morphism between two normal $\mathbb{Q}$-factorial varieties such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, $\dim Y < \dim X$ and $\rho(X/Y) = 1$. Show that for any prime divisor D on X, if $f(D) \neq Y$, then $f(D)$ is a divisor on Y; and if $f(D_1) = f(D_2)$ for two prime divisors D_1 and D_2 on X, then we have $D_1 = D_2$.

(*) **Problem 5** (10 Points) Let X be a smooth n-dimensional projective variety with $\omega_X \cong \mathcal{O}_X$. Show that the period map for the deformation space of X is locally an immersion.