EXERCISES FOR 'HIGHER DIMENSIONAL GEOMETRY'

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1. Ampleness criterion

Question 1.1. *Read the materials in* [KM98, 1.41-1.46].

Question 1.2. *Let S be the blow up of a point $p \in \mathbb{P}^2$.*
 (1) *Compute the generators of $NE(S)$.*
 (2) *Compute the shape of $Nef(S)$ under the identification of $N_1(S) = N^1(S)$.*

Question 1.3. *Let X be the blow up of a point $p \in \mathbb{P}^3$.*
 (1) *Compute the generators of $NE(X)$.*
 (2) *Compute the shape of $Nef(X)$ under suitable natural basis.*

Question 1.4. *Prove Nakai-Moishezon criterion for proper scheme of finite type over k.*

Hint: Chow Lemma.

2. Vanishing Theorems

We assume $\mathrm{Char}(k) = 0$.

Question 2.1. *Let X be a surface such that $K_X \cong \mathcal{O}_X$ e.g., X is K3-surface or Abelian surface. L is a big and nef line bundle on X. Prove $h^0(X, L) \geq 2$.*

Question 2.2 (Grauert-Riemenschneider vanishing theorem)**.** *Let $f : Y \to X$ be a birational morphism between projective varieties such that Y is smooth and X is normal. Then $R^i f_*(\omega_Y) = 0$.*

Question 2.3. *Let X be a projective variety with Gorenstein singularities. Assume X has a crepant resolution, i.e. there is a resolution $f : Y \to X$ such that $f^*(\omega_X) = \omega_Y$. Then prove that for a big and nef line bundle L on X, $H^i(X, \omega_X \otimes L) = 0$.*

Question 2.4 (Examples of round cones)**.** *Exercise 32 in* [Kol08].

3. Singularities in MMP

Question 3.1. *Prove the surface singularities $xy = z^{n+1}$ of type A_n is canonical.*

Question 3.2. *Let (X, Δ) be a pair such that X is normal variety and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ is a finite dominant morphism from a normal variety. Define Δ_Y by*

$$f^*(K_X + \Delta) = K_Y + \Delta_Y.$$

Then $a(E, X, \Delta) \geq -1$ (resp. > -1) for any divisor E if and only if $a(E_Y, Y, \Delta_Y) \geq -1$ (resp. > -1) for any divisor E_Y.

Question 3.3. *Let* (X,Δ) *be a klt pair, then show the set*

$$\{E \mid a(E,X,\Delta)<0\}$$

is a finite set.

Question 3.4 (Cone singularities)**.** *Exercise 70 in* [Kol08].

Question 3.5. *Let* (X,Δ) *be a* n*-dimensional pair such that* X *is normal variety and* $K_X+\Delta$ *is* $\mathbb{Q}$*-Cartier. Fix a point* x *on* X*, we define* $\mathrm{mld}_x(X,\Delta)$ *to be*

$$\min\{a(X,\Delta,E)+1|\mathrm{Center}_X(E)=x\}.$$

1. Prove if $x\in X$ *is smooth, then* $\mathrm{mld}_x(X)=n$*.*
2^**. Prove that for any* $\Delta\geq 0$ *and* $x\in X$*,* $\mathrm{mld}_x(X,\Delta)\leq n$ *and the equality holds if and only if* X *is smooth at* x *and* $\mathrm{Supp}(\Delta)$ *does not contain* x*.*

Remark: Part 2 is open except in lower dimension. In fact, for a fixed dimension n, it's not even know that $\mathrm{mld}_x(X,\Delta)$ is bounded from above by a number which only depends on n.

4. MMP for smooth surfaces

Question 4.1. *Let* $f\colon X\to Y$ *be a birational morphism from a projective smooth surface to a normal surfae. Let* $p\in X$ *and* $f^{-1}(p)=\sum_{i=1}^m E_i$*, then prove the matrix* $(g_{ij})_{1\leq i\leq m}$ *whose entries* $g_{ij}=E_i\cdot E_j$ *is negative definite.*

Question 4.2. *(1) Let* $f\colon X\to Y$ *be a projective birational morphism from a smooth rational surface to a normal surface. Assume* $f(E)$ *is a point for a curve* E *on* Y*, assume* $K_X\cdot E<0$*, then prove* E *is a* (-1)*-curve.*
(2) Use this to prove that given a normal surface singularity $y\in Y$*, among all its resolutions, there is a unique minimal resolution* $f\colon X\to Y$ *such that* $K_X\cdot E\geq 0$*, for any* E *with* $f(E)=y$*.*

Question 4.3. *Prove Theorem 3 in* [Kol08].

Question 4.4. *Exercise 19 in* [Kol08].

5. Main theorems: Nonvanishing theorem, base point free theorem, rationality theorem, cone theorem

Question 5.1. *Exercise 38-46 in* [Kol08].

The following facts have been used a few times in our argument. It has a central importance in higher dimensional geometry.

Question 5.2. *Let* (X,Δ) *be a lct pair and* $M\neq 0$ *be a Cartier divisor.*

(1) *if* M *is a general member in a base point free linear system* $|L|$*, then for any* $c<1$*,* $(X,\Delta+cM)$ *is klt.*
(2) *Define the* **log canonical threshold** *of* (X,Δ) *with respect to* M *to be*

$$\mathrm{lct}(X,\Delta;M)=\max\{t|K_X+\Delta+tM \text{ is log canonical}\}.$$

Show $\mathrm{lct}(X,\Delta;M)\in(0,\infty)$.

(3) *Assume a linear system $|L|$ has base locus B. Let $M_1, ..., .M_m$ be m general divisors in $|L|$. Let $c = \mathrm{lct}(X, \Delta; \sum_{i=1}^m M_i)$. Show that for $m \gg 0$, the divisor E such that*

$$a(E, X, \Delta + c\sum_{i=1}^{m} M_i) = -1$$

has its center contained in B. (Such center is called a **log canonical center***).*

Question 5.3 (Tie-and-break). *Let (X, Δ) be a klt pair, M a big divisor. Let $c = \mathrm{lct}(X, \Delta; M)$. Prove that for any $\epsilon > 0$, we can always find a $\mathbb{Q}$-divisor M_1 such that $M_1 \sim_{\mathbb{Q}} c_1 M$ with $|c - c_1| < \epsilon$ and $(X, \Delta + c_1 M_1)$ is log canonical and has precisely one divisor E such that $a(E, X, \Delta + c_1 M_1) = -1$.*

Question 5.4. *Read* [KM98, Section 3.6].

6. KSBA Theory

Question 6.1. *Show that if X and Y are birational, and both of them have canonical singularities, then*

$$H^0(X, mK_X) \cong H^0(Y, mK_Y).$$

Question 6.2 (Kollár-Shokurov Connectedness Theorem). *Let (X, D) be a log pair, i.e., X is normal, D is an effective $\mathbb{Q}$-divisor. Assume $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f: Y \to (X, D)$ be a log resolution and write*

$$f^*(K_X + D) + \sum_{i, a_i > -1} a_i E_i + \sum_{j, b_j \leq -1} b_j F_j = K_Y,$$

where E_i and F_j does not have common components. Then prove that $\mathrm{Supp} F = \sum_j F_j$ *is connected in a neighborhood of any fiber of f.*

7. Variation of Hodge Structure

Question 7.1 (Local Torelli of CY). *Let X be a smooth n-dimensional projective variety with $\omega_X \cong \mathcal{O}_X$. Show that the period map for the deformation space of X is locally an immersion.*

Hint: Using Serre duality to interpret the Kodaira-Spencer map.

8. Degeneration of Hodge Structure

Question 8.1. *Let f_3 be a general homogenous degree three polynomial with three variables. Consider the family $\mathcal{X} \subset \mathbb{P}(x, y, z) \times \Delta$ with t the coordinate of Δ, where*

$$\mathcal{X} := \left(tf_3(x, y, z) + (x^2 z + (y + z)y^2) = 0\right).$$

Over $\Delta \setminus \{0\}$, it is a smooth family. Compute the limiting mixed Hodge structure over $t = 0$.

Question 8.2. *Let $f: X \to C$ be a projective morphism, smooth over $C^0 = C \setminus \{0\}$. Assume $(X, X_0 = f^{-1}(0))$ is simple normal crossing. Let $W(M)_\bullet$ be the monodromy weight filtration. Compute the weight 0 part*

$$Gr_0^{W(M)} H^k(X_0, \Omega^\bullet_X(\log X_0) \otimes \mathcal{O}_{X_0}).$$

References

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original.

[Kol08] János Kollár, *Exercises in the birational geometry of algebraic varieties*, arXiv:0809.2549 (2008).

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