EXAM 2016 SPRING

Name ____________________ Student I.D. ____________________.

To solve the problems, you are **not** allowed to discuss with anybody. You are **not** allowed to use the internet or any textbook to search the answers either.

However, you **can** use your lecture notes.

Problem 1 We consider the representations of a finite group G on finite dimensional $\mathbb{C}$-vector spaces.

a) Let V and W be irreducible representations of G, and $L_0 \colon V \to W$ any linear mapping. Define $L \colon V \to W$ by $L(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} L_0(g \cdot v)$.

Show that $L = 0$ if V and W are not isomorphic, and that L is multiplication by $\frac{\mathrm{Trace}(L_0)}{\dim V}$ if $V = W$.

b) Show that if all irreducible representations of G are represented by unitary matrices, the matrix entries of these representations form an orthogonal basis for space of *all* functions on G.

Problem 2 Let $\Lambda \subset \mathbb{C}$ be a lattice. The Weierstrass function is defined to be

$$\mathfrak{p}(z,\Lambda) = \frac{1}{z^2} + \sum_{\omega\in\Lambda,\omega\neq 0} \big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\big).$$

Prove the indefinite integral

$$\int \mathfrak{p}(z)^2 dz = \frac{1}{6}\mathfrak{p}'(z) + \frac{1}{12} g_2 \cdot z + C,$$

where $g_2 = 60 \cdot \sum_{\omega\in\Lambda,\omega\neq 0} \omega^{-4}$.

Problem 3 Let $\mathfrak{g} \subset \mathrm{gl}(V)$ ($\mathrm{gl}(V)$ is the Lie algebra consisting of all endomorphisms of V) be a Lie subalgebra such that every $X \in \mathfrak{g}$, X is a nilpotent endomorphism on V. Then prove there exists a non-zero vector $v \in V$ such that $X(v) = 0$ for all $X \in \mathfrak{g}$.

Problem 4 Let $K = \mathbb{C}(x)$ the field consisting of all rational functions with complex coefficients, i.e. $a \in \mathbb{C}(x)$ if and only if $a = \frac{f}{g}$, where $f, g \in \mathbb{C}[x]$ and $g \neq 0$. Let $L \subset K$ a subfield containing $\mathbb{C}$, such that $[K\colon L] < \infty$. Prove $L \cong \mathbb{C}(x)$.