XVA Analysis From the Balance Sheet

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Main references (see https://math.maths.univ-evry.fr/crepey)

- C. Albanese and S. Crépey. XVA Analysis From the Balance Sheet.
- C. Albanese, S. Caenazzo and S. Crépey. Credit, Funding, Margin, and Capital Valuation Adjustments for Bilateral Portfolios. *Probability, Uncertainty and Quantitative Risk* 2017 (forthcoming).

See also

- On the funding issues:
  - Andersen, L., D. Duffie, and Y. Song (2016). Funding value adjustments. ssrn.2746010.
- On the capital issues:
  - The Solvency II actuarial literature
  - Green and Kenyon and Elouerkhaoui KVA papers

Introduction

- In the aftermath of the financial crisis, regulators launched in a major banking reform effort aimed at securing the financial system by raising collateralisation and capital requirements, as if the costs of capital and of funding for collateral were irrelevant.
- The quantification by banks of market incompleteness based on various XVA metrics, in particular KVA (capital valuation adjustment) and MVA (margin valuation adjustment), is emerging as the unintended consequence of the banking reform.

- The presence of the KVA and the MVA breaks several of the conclusions of Modigliani-Miller theory.
- The purpose of this work is to explain and amend the banking XVA metrics in the light of a capital structure model acknowledging the impossibility for a bank to replicate jump-to-default related cash-flows.

Outline

Counterparty Risk,

with its funding and capital implications, is at the origin of all XVAs:

CVA Credit valuation adjustment

- The value you lose due to the defaultability of your counterparties

DVA Debit valuation adjustment

- The value your counterparties lose due to your own defaultability
- The symmetric companion of the CVA
- The value you gain due to your own defaultability? (2011 DVA debate)

FVA Funding valuation adjustment

- Cost of funding variation and initial margin: MVA merged with FVA in these slides to spare one "VA"
- But what about the Modigliani-Miller theorem?? (2013 FVA debate)

DVA2 Funding windfall benefit at own default

KVA Cost of capital

- The price for the bank of having to reserve capital at risk (ongoing KVA debate)

Contra-assets and Contra-liabilities

CA: Contra-assets, entail the valuation of all cash-flows related to the credit risk of either the counterparties or the bank and occurring before the default of the bank itself, i.e. having an impact on shareholder value.

- CVA, FVA, ...

CL: Contra-liabilities, entail the valuation of all the cash-flows received by the bank during the resolution process starting at its default time, i.e. only having an impact on bank creditors, by modifying the recovery rate of the bank, but not on shareholders.

- DVA, FDA, ...

CVA−DVA+FVA−DVA2 [+KVA]: The XVA debates

- FVA and DVA2 cash flows NPV-match each other

→ CVA-DVA yields the fair, symmetrical adjustment between two counterparties of equal bargaining power

- But "Contra-liabilities" DVA and DVA2 are only a benefit to the creditors of the bank, whereas only the interest of shareholders matters in bank managerial decisions

→ DVA and DVA2 should be ignored in entry prices

→ CVA+FVA

Funds Transfer Price

- Moreover, counterparty default losses cannot be replicated and a bank must reserve shareholder capital to cope with residual risk
- Shareholders that put capital at risk deserve a remuneration at a hurdle rate, which corresponds to the KVA

$\rightarrow$ FTP=CVA+FVA+KVA

Connection with the Modigliani and Miller (1958) Theorem

- The Modigliani-Miller theorem includes two key assumptions.
  - One is that, as a consequence of trading, total wealth is conserved.
  - The second assumption is that markets are complete.
- In our setup we keep the wealth conservation hypothesis but we lift the completeness.
- Hence the conclusion of the theorem, according to which the fair valuation of counterparty risk to the bank as a whole should not depend on its funding policy, is preserved.

- However, due to the incompleteness of counterparty risk, derivatives trigger wealths transfers from bank shareholders to creditors
  - The interests of bank shareholders and creditors are not aligned with each other
- Which, in the case of a market maker such as a bank, can only be compensated by add-on to entry prices

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More precisely, quoting Villamil (2008):

> *In fact what is currently understood as the Modigliani-Miller Proposition comprises four distinct results from a series of papers (1958, 1961, 1963). The first proposition establishes that under certain conditions, a firms debt-equity ratio does not affect its market value. The second proposition establishes that a firms leverage has no effect on its weighted average cost of capital (i.e., the cost of equity capital is a linear function of the debt-equity ratio). The third proposition establishes that firm market value is independent of its dividend policy. The fourth proposition establishes that equity-holders are indifferent about the firms financial policy.*

- The proof of the fourth proposition is based on the ability of shareholders to redeem all debt of the bank in order to prevent wealth transfers to creditors.

However:

- Redeeming the debt means hedging its own default, which is not possible for a bank.
  - Banks are special firms in that they are intrinsically leveraged and cannot be transformed into a pure equity entity.
  - This is also related to an argument of scale.
    - Banks liabilities are overwhelming with respect to all other wealth numbers.
    - It has been estimated that if all European banks were to be required to have capital equal to a third of liabilities, the total capitalization of banks would be greater than the total capitalization of the entire equity market as we know it today.

Hence:

- Shareholders cannot redeem all debt of the bank.
- The assumption of the fourth proposition of the Modigliani-Miller theorem does not apply to a bank.

- Quoting the conclusion of Modigliani and Miller (1958)

  "These and other drastic simplifications have been necessary in order to come to grips with the problem at all. Having served their purpose they can now be relaxed in the direction of greater re and relevance, a task in which we hope others interested in this area will wish to share."

- And Miller (1988) in *The Modigliani-Miller Proposition after Thirty Years*

  "Showing what doesn't matter can also show, by implication, what does."

Outline

- In this section we present the main ideas of our XVA approach in an elementary static one-year setup, with r set equal to 0.
- Assume that at time 0 a bank, with equity $\mathrm{E} = w_0$ corresponding to its initial wealth, enters a derivative position (or portfolio) with a client.
- Let $P = \mathbb{E}\mathcal{P}$ denote the mark-to-market of the deal ignoring counterparty risk and assuming risk-free funding.

- We assume that the bank and its client are both default prone with zero recovery.
- We denote by J and J_1 the survival indicators of the bank and its client at time 1
  - Both being assumed alive at time 0
  - With default probability of the bank $\mathbb{Q}(J = 0) = \gamma$
  - And no joint default for simplicity, i.e $\mathbb{Q}(J = J_1 = 0) = 0$.

XVA Cost of Capital Pricing Approach

- In order to focus on counterparty risk and XVAs, we assume that the market risk of the bank is perfectly hedged by means of perfectly collateralized back-to-back trades
    - The back-to-back hedged derivative portfolio reduces to its counterparty risk related cash flows
- At the bottom of this work lies the fact that a bank cannot replicate jump-to-default exposures
- Cost of capital pricing approach in incomplete counterparty risk markets

- Standing risk-neutral valuation measure $\mathbb{Q}$
- Derivative entry prices in our sense include, on top of the valuation of the corresponding cash-flows, a KVA risk premium
  - Risk margin (RM) in a Solvency II terminology
  - Computed assuming $\mathbb{P} = \mathbb{Q}$, as little of relevance can be said about the historical probability measure for XVA computations entailing projections over decades
  - The discrepancy between $\mathbb{P}$ and $\mathbb{Q}$ is left to model risk
- Cost of capital pricing approach applied to the counterparty risk embedded into the derivative portfolio of a bank

Cash Flows

- The counterparty risk related cash flows affecting the bank before its default are its counterparty default losses and funding expenditures, respectively denoted by $\mathcal{C}^\circ$ and $\mathcal{F}^\circ$.
- The bank wants to charge to its client an add-on, or obtain from its client a rebate, denoted by CA, accounting for its expected counterparty default losses and funding expenditures.
- Accounting for the to-be-determined add-on CA, in order to enter the position, the bank needs to borrow $(P - \mathrm{CA})^+$ unsecured or invest $(P - \mathrm{CA})^-$ risk-free, depending on the sign of $(P - \mathrm{CA})$, in order to pay $(P - \mathrm{CA})$ to its client.

- At time 1:
  - If alive (i.e. $J = 1$), then the bank closes the position while receiving $\mathcal{P}$ if its client is alive (i.e. $J_1 = 1$) or pays $\mathcal{P}^-$ if its client is in default (i.e. $J_1 = 0$).
    - Note $J_1\mathcal{P} - (1 - J_1)\mathcal{P}^- = \mathcal{P} - (1 - J_1)\mathcal{P}^+$. Hence the counterparty default loss of the bank appears as the random variable

$$\mathcal{C}^\circ = (1 - J_1)\mathcal{P}^+. \tag{1}$$

    In addition, the bank reimburses its funding debt $(P - \mathrm{CA})^+$ or receives back the amount $(P - \mathrm{CA})^-$ it had lent at time 0.
  - If in default (i.e. $J = 0$), then the bank receives back $\mathcal{P}^+$ on the derivative as well as the amount $(P - \mathrm{CA})^-$ it had lent at time 0.

- We assume that unsecured borrowing is fairly priced as $\gamma \times$ the amount borrowed by the bank, so that the funding expenditures of the bank amount to

$$\mathcal{F}^{\circ} = \gamma(P - \mathrm{CA})^{+},$$

deterministically in this one-period setup.

- We assume further that a fully collateralized back-to-back market hedge is set up by the bank in the form of a deal with a third party, with no entrance cost and a payoff to the bank $-(\mathcal{P} - P)$ at time 1, irrespective of the default status of the bank and the third party at time 1.

- Collecting cash flows, the wealth of the bank at time 1 is

$$
\begin{aligned}
w_1 &= \mathrm{E} - \mathcal{F}^\circ + (1-J)\big(\mathcal{P}^+ + (P - \mathrm{CA})^-\big) \\
&\quad + J\big(J_1\mathcal{P} - (1-J_1)\mathcal{P}^- - (P-\mathrm{CA})^+ + (P-\mathrm{CA})^-\big) - (\mathcal{P} - P) \\
&= \big(\mathrm{E} - (\mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CA})\big) + (1-J)(\mathcal{P}^- + (P-\mathrm{CA})^+), \qquad (2)
\end{aligned}
$$

  - as easily checked for each of the three possible values of the pair (J, J_1)

- The result of the bank over the year is

$$w_1 - w_0 = w_1 - \mathrm{E} = -(\mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CA}) + (1 - J)(\mathcal{P}^- + (P - \mathrm{CA})^+).$$

- However, the cash flow $(1 - J)(\mathcal{P}^- + (P - \mathrm{CA})^+)$ is only received by the bank if it is in default at time 1, so that it only benefits bank creditors.
- Hence, the profit-and-loss of bank shareholders reduces to $-(\mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CA})$, i.e. the trading loss-and-profit of the bank, which we denote by L, appears as

$$L = \mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CA}. \tag{4}$$

Remark 1

- The derivation (2) allows for negative equity, which is interpreted as recapitalization.
- In a variant of the model excluding recapitalization, where the default of the bank would be modeled in a structural fashion as $\mathrm{E} - L < 0$ and negative equity is excluded, we would get instead of (2)

$$w_1 = (\mathrm{E} - L)^+ + \mathbb{1}_{\{\mathrm{E}<L\}}(\mathcal{P}^- + (P - \mathrm{CA})^+). \tag{5}$$

- In our approach we consider a model with recapitalization for reasons explained later.

Funds Transfer Price

- In order to account for expected counterparty default losses and funding expenditures, the bank charges to its client the add-on

$$\mathrm{CA} = \underbrace{\mathbb{E}\mathcal{C}^\circ}_{\mathrm{CVA}} + \underbrace{\mathbb{E}\mathcal{F}^\circ}_{\mathrm{FVA}}. \tag{6}$$

- Note that, since

$$\mathrm{FVA} = \mathbb{E}\mathcal{F}^\circ = \mathcal{F}^\circ = \gamma(P - \mathrm{CA})^+$$

(all deterministically in a one-period setup), (6) is in fact an equation for CA.

- Equivalently, we have the following semi-linear equation for $\mathrm{FVA} = \mathrm{CA} - \mathrm{CVA}$:

$$\mathrm{FVA} = \gamma(P - \mathrm{CVA} - \mathrm{FVA})^{+},$$

  which has the unique solution

$$\mathrm{FVA} = \frac{\gamma}{1+\gamma}(P - \mathrm{CVA})^{+}. \tag{7}$$

- Substituting this and (1) into (6), we obtain

$$\mathrm{CA} = \underbrace{\mathbb{E}[(1 - J_1)\mathcal{P}^{+}]}_{\mathrm{CVA}} + \underbrace{\frac{\gamma}{1+\gamma}(P - \mathrm{CVA})^{+}}_{\mathrm{FVA}}. \tag{8}$$

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- Note that the realized recovery is $(1-J)(\mathcal{P}^- + (P - \mathrm{CA})^+)$ because of the trade that occurred, but this was not anticipated and not reflected in the price of borrowing when the bank issued its funding debt.
- As the funding debt was fairly valued ignoring this, the value $\mathrm{FDA} = \mathbb{E}[(1-J)(P - \mathrm{CA})^+]$ of the default funding cash flow $(1-J)(P-\mathrm{CA})^+$ equals the cost $\mathrm{FVA} = \gamma(P - \mathrm{CA})^+$ of funding the position.
- But the FVA and the FDA do not impact the same economic agent, namely the FVA hits bank shareholders whereas the FDA benefits creditors.
- Hence, the net effect of funding is not nil to shareholders, but reduces to an FVA cost.

- In view of (4) and (6), observe that charging to the client a CA add-on corresponding to expected counterparty default losses and funding expenditures is equivalent to setting the add-on CA such that, in expectation, the trading loss-and-profit of bank shareholders is zero ($\mathbb{E}L = 0$), as it would also be the case without the deal.
- However, without the deal, the loss-and-profit of bank shareholders would be zero not only in expectation, but deterministically.

- Hence, to compensate shareholders for the risk on their equity triggered by the deal, under our cost of capital approach, the bank charges to its client an additional amount (risk margin)

$$\mathrm{KVA} = h\mathrm{E}, \tag{9}$$

where h is some hurdle rate, e.g. 10%.

- Moreover, since E can be interpreted as capital at risk earmarked to absorb the losses $(\mathcal{C}^\circ + \mathcal{F}^\circ)$ of the bank above CA, it is natural to size E by some risk measure of the bank shareholders loss-and-profit L.
- The all-inclusive XVA add-on to the entry price for the deal, which we call funds transfer price (FTP), is

$$\mathrm{FTP} = \underbrace{\mathrm{CA}}_{\text{Expected costs}} + \underbrace{\mathrm{KVA}}_{\text{Risk premium}}. \tag{10}$$

Monetizing the Contra-Liabilities?

- Let us now assume, for the sake of the argument, that the bank would be able to hedge its own jump-to-default risk through a further deal, whereby the bank would deliver a payment $(1-J)(\mathcal{P}^- + (P - \mathrm{CA})^+)$ at time 1 in exchange of an upfront fee fairly valued as

$$\mathrm{CL} = \underbrace{\mathbb{E}[(1-J)\mathcal{P}^-]}_{\mathrm{DVA}} + \underbrace{\mathbb{E}[(1-J)(P-\mathrm{CA})^+]}_{\mathrm{FDA}=\gamma(P-\mathrm{CA})^+=\mathrm{FVA}}, \tag{11}$$

  - DVA and FDA stand for debt valuation adjustment and funding debt adjustment.

- Let CR denote the modified CA charge to be passed to the client when the hedge is assumed.
- Accouting for the hedging gain $\mathcal{H}^{cl} = \mathrm{CL} - (1-J)(\mathcal{P}^- + (P - \mathrm{CA})^+)$, the wealth of the bank at time 1 now appears as (cf. (2))

$$\begin{aligned}\widetilde{w}_1 &= (\mathrm{E} - (\mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CR})) + (1-J)(\mathcal{P}^- + (P - \mathrm{CA})^+) + \mathcal{H}^{cl} \\ &= \mathrm{E} - (\mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CR}) + \mathrm{CL}. \qquad (12)\end{aligned}$$

- By comparison with (2), the CL originating cash flow $(1-J)(\mathcal{P}^- + (P - \mathrm{CA})^+)$ is hedged out and monetized as an amount CL received by the bank at time 0.
- The trading loss-and-profit of bank shareholders now appears as

$$\widetilde{L} = w_0 - \widetilde{w}_1 = \mathrm{E} - \widetilde{w}_1 = \mathcal{C}^\circ + \mathcal{F}^\circ - \mathrm{CR} - \mathrm{CL}. \qquad (13)$$

- The amount CR making $\widetilde{L}$ centered is

$$\begin{aligned}\mathrm{CR} &= \mathbb{E}(\mathcal{C}^\circ + \mathcal{F}^\circ) - \mathrm{CL} \\ &= (\mathrm{CVA} + \mathrm{FVA}) - (\mathrm{DVA} + \mathrm{FDA}) = \mathrm{CVA} - \mathrm{DVA},\end{aligned} \tag{14}$$

  because FVA=FDA (cf. (11)).
- Hence, if the bank was able to hedge its own jump-to-default risk, in order to satisfy its shareholders in expectation, it would be enough for the bank to charge to its client an add-on $\mathrm{CR} = \mathrm{CVA} - \mathrm{DVA}$.

- The amount $\mathrm{CR} = \mathrm{CVA} - \mathrm{DVA}$ also coincides with the fair valuation of counterparty risk when market completeness and no trading restrictions are assumed (cf. Duffie and Huang (1996)).
- However, under our approach, in the present setup, the bank would still charge to its client a KVA add-on $h\widetilde{\mathrm{E}}$ as risk compensation for the non flat loss-and-profit $\widetilde{L}$ triggered by the deal (unless $\widetilde{L}$ can be hedged out as well).
- But $\widetilde{\mathrm{E}}$ would be sized by some risk measure of $\widetilde{L}$, instead of L for E in (9).

Wealth Transfer Interpretation

- As mentioned before, a bank cannot hedge its own jump-to-default risk in practice. But the above findings are important from an interpretive point of view.
- We see from (6) and (11) that CA can be viewed as the sum between CL and the fair valuation $\mathrm{CR} = \mathrm{CVA} - \mathrm{DVA}$ of counterparty risk.
- In view of the above, CL can be interpreted as an add-on that the bank needs to source from the client, on top of the fair valuation of counterparty risk, in order to compensate the loss of value to shareholders due to the inability of the bank to hedge its own jump-to-default risk.
- In other words, due to this market incompleteness (or trading restriction), the deal triggers a wealth transfer from bank shareholders to creditors equal to CL, which then needs be sourced by the bank from its client in order to put shareholders back at value equilibrium in expected terms.

Funds Transfer Price Decomposition

- In conclusion, in a one-period setup, the FTP can be represented as

$$\begin{aligned}\mathrm{FTP} &= \underbrace{\mathrm{CVA} + \mathrm{FVA}}_{\text{Expected costs } \mathrm{CA}} + \underbrace{\mathrm{KVA}}_{\text{Risk premium}} \\ &= \underbrace{\mathrm{CVA} - \mathrm{DVA}}_{\text{Fair valuation } \mathrm{CR}} + \underbrace{\mathrm{DVA} + \mathrm{FDA}}_{\text{Wealth transfer } \mathrm{CL}} + \underbrace{\mathrm{KVA}}_{\text{Risk premium}},\end{aligned} \tag{15}$$

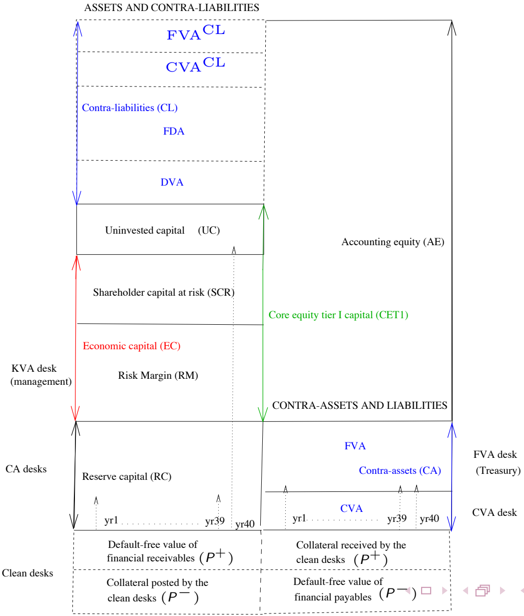
  where CA is given by (8) and where the random variable L used to size the equity E in the KVA formula (9) is the bank shareholders loss-and-profit L as per (4).

Outline

- Need
  - of a multi-period model, which involves rebalancing between various banking accounts, for dealing with incremental portfolios,
  - to put the so-called contra-assets and contra-liabilities of the bank in a balance sheet perspective, for identifying the structural connection between the different XVA metrics.

$\rightarrow$ We introduce a capital structure model of a bank that shows the different bank accounts involved.

Balance Sheet of a Bank

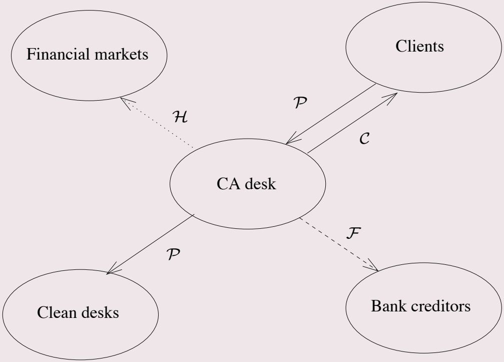


A Bank With Three Floors

- The "CA desk" of the bank sells the contra-assets to the clients of the bank and is exposed to the corresponding payoffs
  - Counterparty default losses and risky funding expenditures of the bank
  - The CA desk may also setup a CA hedge, i.e. a (partial) hedge of these payoffs.
- After the contracts have thus been cleaned of their counterparty risk and (other than risk-free) funding implications by the CA desk, the other trading desks of the bank, which we call clean desks (or "bottom floor") of the bank, are left with the the management of the market risk of the contracts in their respective business lines, ignoring counterparty risk.
- The top (third) floor is the management in charge of the KVA payments, i.e. of the dividend distribution policy of the bank.

- In fact, we deal with two portfolios, the client portfolio between the clients of the bank and the CA desk and the cleaned portfolio between the CA desk and the clean desks.
- The corresponding (cumulative streams of) contractually promised cash flows are the same, denoted by $\mathcal{P}$. But, as intuitively clear and detailed in the sequel, counterparty risk only really impacts the client portfolio.

CA desk cash flows graph: contractually promised $\mathcal{P}$, risky funding $\mathcal{F}$, and hedging $\mathcal{H}$ cash flows



Run-off Assumption

- In what follows the derivative portfolio of the bank is assumed held on a run-off basis until its final maturity T.
  - Accounting for the bank default time τ, the time horizon of the model is then $\bar{\tau} = \tau \wedge T$.
- The (realistic) case of incremental portfolios will be considered in a second stage.

Continuous Reset Assumption

- Losses-and-earnings realization times are typically quarter ends for bank profits, released as dividends, vs recapitalization managerial decision times for losses. However, there is no way to "calibrate" losses-and-earnings realization times in a pricing or risk model.
- In our model, we assume that losses-and-earnings are marked to model and realized in real time.

Bank Default Model

- Instead of viewing losses as money flowing away from the balance sheet, we view them as money flowing into it as refill, i.e. replenishment of the different bank accounts at their theoretical target level, until the point of default where the payers cease willing to do so.
- When this happens is modeled as a totally unpredictable time τ calibrated to the bank CDS spread, which we view as the most reliable and informative credit data regarding anticipations of markets participants about future recapitalization, government intervention and other bank failure resolution policies.

Comparison with the Merton model

- In a Merton mindset, the default of the bank in our setup could be modeled as the first time when CET1 becomes negative.
- Merton (1974)'s purpose was to develop an option-theoretic view on equity and corporate debt. For this of course a structural model of the default time of a firm is required.
- In the case of a bank, given recapitalisation and managerial resolution schemes, it is more realistic to model the default as a totally unpredictable (liquidity or operational) event at some exogenous time τ, calibrated to the bank CDS spread.
  - Duffie (2010)'s analysis of major bank defaults during the crisis
- The purpose of our capital structure model of the bank is not to model the default of the bank as the point of negative equity, which would be unrealistic...
- ... But to put in a balance sheet perspective the contra-assets and contra-liabilities of the bank, items which are not present in the Merton model.

Accouting and Dividend Policies of the Bank

Continuous reset implies that:

- The RC account is continuously reset to its theoretical target CA level by the CA desk
  - Much like with futures, the position of the CA desk is reset to zero at all
    but it generates gains $(-dL_t^{ca})$.
- The RM account is continuously reset by the management of the bank to its theoretical target KVA level.
  - $(-d\mathrm{KVA}_t)$ amounts continuously flow from the RM account to the shareholder dividend stream

$\rightarrow$ Balance conditions

$$\mathrm{RC} = \mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}\,, \;\; \mathrm{RM} = \mathrm{KVA}. \tag{16}$$

Invariance Valuation Setup

We denote by $J = \mathbb{1}_{[0,\tau)}$ the survival indicator process of the bank.

Assumption 1

- Clean desks price and hedge ignoring the default of the bank, using some reference filtration $\mathbb{F}$ such that τ is not an $\mathbb{F}$ stopping time.
  - The OIS rate (publically available risk-free interest rate) process r, hence the corresponding discount factor β, as well as the contractually promised cumulative cash flow stream $\mathcal{P}$, are assumed to be $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ adapted.
- But the bank is defaultable, hence the full model information used by the CA desk, as well as by the management of the bank in charge of the KVA payments, is a larger filtration $\mathbb{G} = (\mathfrak{G}_t)_{t\geq 0}$ such that τ is a $\mathbb{G}$ stopping time, endowed with a $(\mathbb{G}, \mathbb{Q})$ intensity γJ_-.
- Any $\mathbb{G}$ stopping time η admits an $\mathbb{F}$ stopping time η' such that $\eta \wedge \tau = \eta' \wedge \tau$; any $\mathbb{G}$ semimartingale Y admits a unique $\mathbb{F}$ semimartingale Y', called the reduction of Y, that coincides with Y before τ.

- For any left-limited process Y, we denote by $\Delta_\tau Y = Y_\tau - Y_{\tau-}$ the jump of Y at τ and by $Y^{\tau-} = JY + (1-J)Y_{\tau-}$ the process Y stopped before time τ, so that

$$dY_t = dY_t^{\tau-} + (-\Delta_\tau Y)\, dJ_t\,,\;\; 0 \le t \le \bar{\tau}. \tag{17}$$

Definition 1

- By trading loss L^{cl} of the clean desks, we mean the negative of their wealth process $\mathcal{W}^{cl}$ as it results from their trading by an application of a self-financing assumption, defined with respect to the reference filtration $\mathbb{F}$.
- By trading loss L^{ca} of the CA desk, we mean the negative of its wealth process $\mathcal{W}^{ca}$ as it results from its trading by an application of a self-financing assumption with respect to the filtration $\mathbb{G}$, stopped before τ for alignment with shareholder interest. That is,

$$L^{ca} = -(\mathcal{W}^{ca})^{\circ}. \tag{18}$$

- By trading loss L of the bank as a whole, we mean

$$L = -(\mathcal{W}^{cl} + \mathcal{W}^{ca})^{\circ} = (L^{cl})^{\circ} + L^{ca}. \tag{19}$$

Consistency of valuation across the perspectives of the different desks of the bank is granted by the following:

Assumption 2

- Clean and CA traders use not only different filtrations $\mathbb{F}$ and $\mathbb{G}$, but also different pricing measures $\mathbb{P}$ on $\mathcal{F}_T$ and $\mathbb{Q}$ on $\mathfrak{G}_T$, equivalent on $\mathcal{F}_T$ and such that $(\mathbb{F}, \mathbb{P})$ martingales stopped before τ are $(\mathbb{G}, \mathbb{Q})$ martingales.
  - Conversely, the reductions of $(\mathbb{G}, \mathbb{Q})$ martingales stopped before τ are $(\mathbb{F}, \mathbb{P})$ martingales.
- The process L^{cl} is an $(\mathbb{F}, \mathbb{P})$ martingale on $[0, T]$. The process L^{ca} is a $(\mathbb{G}, \mathbb{Q})$ martingale stopped before τ.

- In other words τ is an invariance time as per Crépey and Song (2017a).
- The most standard situation is a basic immersion setup where $(\mathbb{F}, \mathbb{Q})$ local martingales are $(\mathbb{G}, \mathbb{Q})$ local martingales without jump at τ, in which case τ is an invariance time with $\mathbb{P} = \mathbb{Q}$.
- We also cover the case of a default-free bank as a (simpler but unrealistic) situation where $\tau = +\infty$ holds $\mathbb{Q}$ a.s. and $(\mathbb{F}, \mathbb{P}) = (\mathbb{G}, \mathbb{Q})$.
- Conditional expectation with respect to $(\mathfrak{G}_t, \mathbb{Q})$ (respectively $(\mathcal{F}_t, \mathbb{P})$) is denoted by $\mathbb{E}_t$ (respectively $\mathbb{E}'_t$), or simply $\mathbb{E}$ (respectively $\mathbb{E}'$) if $t = 0$.

As an immediate consequence of (19) and Assumption 2:

Corollary 1

The trading loss L of the bank as a whole is a $(\mathbb{G}, \mathbb{Q})$ *martingale without jump at time* τ *and its reduction* L' *is an* $(\mathbb{F}, \mathbb{P})$ *martingale.*

Given Y representing a process of cumulative cash flows or trading or hedging losses, respectively an XVA process, we denote by $\widetilde{Y}$ the corresponding process of cumulative OIS discounted cash flows or trading or hedging losses, respectively the corresponding OIS discounted XVA process.

Example 1

$\widetilde{L} = \int_0^{\cdot} \beta_t dL_t$, $\widetilde{\mathrm{CVA}} = \beta\mathrm{CVA}$.

Corollary 2

In the case of a cumulative cash flow or loss process Y, the process Y is a martingale if and only if $\widetilde{Y}$ is a martingale.

Lemma 1

Shareholder cumulative discounted dividends are given by

$$-\widetilde{L} - \widetilde{\mathrm{KVA}}^{\circ}. \tag{20}$$

We emphasize that, in our model, negative dividends are possible. They are interpreted as recapitalisation (or equity dilution).

- All our XVA processes will be sought for in a suitable Hilbert space $\mathcal{S}_2$ of square integrable $\mathbb{G}$ adapted processes containing the null process, defined until time $\bar{\tau}$ (note that $(\mathbb{G}, \mathbb{Q})$ valuation is never needed beyond that point).
- We denote by $\mathcal{S}_2^\circ$ the corresponding subspace of processes Y without jump at τ and such that $Y_T = 0$ on $\{T < \tau\}$.

Outline

Clean Vs. Risky Valuation

Definition 2

- Given an $\mathbb{F}$ adapted cumulative cash flow stream $\mathcal{D}$, the OIS discounted $(\mathbb{F}, \mathbb{P})$ value process of $\mathcal{D}$ is the $(\mathbb{F}, \mathbb{P})$ conditional expectation process of the future OIS discounted cash flows in $\mathcal{D}$.
- By mark-to-market or clean valuation P of the (client or cleaned) portfolio, we mean the $(\mathbb{F}, \mathbb{P})$ value process of the contractually promised cash flow stream $\mathcal{P}$.

Proposition 1

- *Clean valuation is additive over contracts, i.e. the mark-to-market of a portfolio of contracts is the sum of the mark-to-markets of the individual contracts.*
- *Clean valuation is also intrinsic to the contracts themselves. In particular, it is independent of the involved parties and of their collateralization, funding and hedging policies.*

Proof. The promised cash flows of a portfolio simply consist in the disjoint union of the promised cash flows of the contracts. Hence the result follows by linearity of the cash flow $(\mathbb{F}, \mathbb{P})$ valuation rule of Definition 2.

Definition 3

By value of the cleaned portfolio (process used by the CA and clean desks for marking to the model the cleaned portfolio between them), we just mean the clean valuation P of the portfolio.

Definition 4

Given a $\mathbb{G}$ adapted cumulative cash flow stream $\mathcal{D}$, the OIS discounted $(\mathbb{G}, \mathbb{Q})$ (or risky) value process of $\mathcal{D}$ is the $(\mathbb{G}, \mathbb{Q})$ conditional expectation process of the future OIS discounted cash flows in $\mathcal{D}$.

Assumption 3

The funding costs of the CA desk are of the form

$$(-\text{OIS accrual of the RC account}) + \mathcal{F}, \tag{21}$$

for some $(\mathbb{G}, \mathbb{Q})$ martingale $\mathcal{F}$ starting from 0, interpreted as the risky funding costs of the CA desk.

- The first term in (21) is the funding benefit to which funding would boil down if risk-free funding was available to the bank.
- The rationale underlying Assumption 3 is that funding is implemented in practice as the stochastic integral of predictable hedging ratios against funding assets.
- Under the cash flow $(\mathbb{G}, \mathbb{Q})$ valuation rule of Definition 4, the value process of each of these assets is a martingale modulo risk-free accrual.
- Therefore the funding costs of the bank accumulate into a $(\mathbb{G}, \mathbb{Q})$ martingale $\mathcal{F}$, coming on top of a risk-free accrual (actual benefit, i.e. negative cost) of the RC cash account of the CA desk.

Example 2

- Let

$$dB_t = r_t B_t dt$$
$$dD_t = (r_t + \lambda_t) D_t dt + (1 - R) D_{t-} dJ_t = r_t D_t dt + D_{t-}\big(\lambda_t dt + (1 - R)$$

represent the risk-free OIS deposit asset and a risky bond issued by the bank for its investing and unsecured borrowing purposes.

- The risk-neutral martingale condition that applies to (βD) under our standing valuation framework implies that $\lambda = (1 - R)\gamma$, hence

$$\lambda_t dt + (1 - R)\, dJ_t = (1 - R) d\mu_t,$$

where $d\mu_t = \gamma dt + dJ_t$ is the $(\mathbb{G}, \mathbb{Q})$ compensated jump-to-default martingale of the bank

Example 3 (Cont'd)

- We assume all re-hypothecable collateral and we denote by Q the amount of collateral posted by the CA desk to the clean desks net of the amount received by the CA desk from the clients.
- The funding policy of the CA desk is represented by a splitting of the amount CA_t on the RC account of the bank as

$$\begin{aligned}\mathrm{CA}_t = & \underbrace{Q_t}_{\text{Collateral remunerated OIS}} \\ & + \underbrace{(\mathrm{CA}_t - Q_t)^+}_{\text{Cash in excess invested at the risk-free rate as } \nu_t B_t} \\ & - \underbrace{(\mathrm{CA}_t - Q_t)^-}_{\text{Cash needed unsecurely funded as } \eta_t D_t} \\ = & \underbrace{\left(Q_t + (\mathrm{CA}_t - Q_t)^+\right)}_{\text{Invested at the risk-free rate as } \nu_t B_t} - \underbrace{(\mathrm{CA}_t - Q_t)^-}_{\text{Unsecurely funded as } \eta_t D}\end{aligned}$$

Example 3 (Cont'd)

- A standard continuous-time self-financing equation expressing the conservation of cash flows at the level of the bank as a whole yields

$$
\begin{aligned}
d\left(\nu_t B_t - \eta_t D_t\right) &= \nu_t dB_t - \eta_{t-} dD_t \\
&= \nu_t r_t B_t dt - \eta_t (r_t + \lambda_t) D_t dt - (1-R)\eta_{\tau-} D_{\tau-} dJ_t \\
&= r_t \mathrm{CA}_t dt - (1-R)\eta_{t-} D_{t-} d\mu_t\,,\;\; 0 \le t \le \bar{\tau}
\end{aligned} \tag{24}
$$

  - A left-limit in time is required in η because D jumps at time τ, so that the process η, which is defined implicitly through CA^- in (23), is not predictable.

- Equivalently viewed in terms of costs, i.e. flipping signs in the above, we obtain

$$
-\,d\left(\nu_t B_t - \eta_t D_t\right) = -r_t \mathrm{CA}_t dt + d\mathcal{F}_t \tag{25}
$$

where $d\mathcal{F}_t = (1-R)(Q_{t-} - \mathrm{CA}_{t-})^+ d\mu_t$.

Regarding now hedging losses:

Assumption 4

The hedging loss $\mathcal{H}$ of the CA desk, including the cost of setting the hedge, is a $(\mathbb{G}, \mathbb{Q})$ martingale starting from 0.

- The rationale here is that hedging gains or losses arise in practice as the stochastic integral of predictable hedging ratios against wealth processes of individual hedging assets.
- Note that we are considering wealth processes inclusive of the associated funding costs here, which corresponds to the most common situation of hedges that are either swapped or traded through a repo market, without upfront payment.
- Under the cash flow $(\mathbb{G}, \mathbb{Q})$ valuation rule of Definition 4, each hedging asset is valued as risk-free discounted expectation of its future cash flows.
- Hence the wealth processes related to long positions in any of the hedging assets are $(\mathbb{G}, \mathbb{Q})$ martingales, as are stochastic integrals against them.

As explained before:

Assumption 5

The bank cannot hedge its own jump-to-default exposure, hence $\mathcal{H}^{\bullet}=0$.

Example 3

Assuming the CA hedge implemented through a repo market on a Black-Scholes stock S with volatility σ, then, supposing no dividends and no repo basis on S:

$$d\mathcal{H}_t = d\mathcal{H}_t^{\circ} = -\zeta_t\big(dS - rS_t dt\big) = -\zeta_t \sigma S_t dW_t\,,\;\; d\mathcal{H}_t^{\bullet} = 0, \tag{26}$$

where W is the $(\mathbb{G}, \mathbb{Q})$ Brownian motion driving S and ζ is the hedging ratio used in S.

- The instantaneous cost of funding the hedge is $(-\zeta_t rS_t dt)$, which is included in (26).

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Remark 2

The valuation impact of a theoretical (but impractical) hedge by the bank of its contra-liabilities will be considered separately in Proposition 2(ii).

As immediate consequences of Assumptions 3 through 5:

Corollary 3

The processes $\mathcal{F}$ and $\mathcal{H} = \mathcal{H}^{\circ}$ are $(\mathbb{G}, \mathbb{Q})$ martingales with zero risky value.

Definition 5

- We call CA (contra-asset value process), CVA (credit valuation adjustment), and FVA (funding valuation adjustment), the solutions to the following fixed-point problems, assumed well-posed in $\mathcal{S}_2^\circ$: For $t \leq \bar{\tau}$,

$$\begin{aligned}\widetilde{\mathrm{CA}}_t &= \mathbb{E}_t\big(\widetilde{\mathcal{C}}^\circ_{\bar{\tau}} + \widetilde{\mathcal{F}}^\circ_{\bar{\tau}} - \widetilde{\mathcal{C}}^\circ_t - \widetilde{\mathcal{F}}^\circ_t + \mathbb{1}_{\{\tau<T\}}\widetilde{\mathrm{RC}}^\circ_\tau\big) \\ &= \mathbb{E}_t\big(\widetilde{\mathcal{C}}^\circ_{\bar{\tau}} + \widetilde{\mathcal{F}}^\circ_{\bar{\tau}} - \widetilde{\mathcal{C}}^\circ_t - \widetilde{\mathcal{F}}^\circ_t + \mathbb{1}_{\{\tau<T\}}\widetilde{\mathrm{CA}}^\circ_\tau\big),\end{aligned} \tag{27}$$

by (16), and

$$\widetilde{\mathrm{CVA}}_t = \mathbb{E}_t\big(\widetilde{\mathcal{C}}^\circ_{\bar{\tau}} - \widetilde{\mathcal{C}}^\circ_t + \mathbb{1}_{\{\tau<T\}}\widetilde{\mathrm{CVA}}^\circ_\tau\big) \tag{28}$$

$$\widetilde{\mathrm{FVA}}_t = \mathbb{E}_t\big(\widetilde{\mathcal{F}}^\circ_{\bar{\tau}} - \widetilde{\mathcal{F}}^\circ_t + \mathbb{1}_{\{\tau<T\}}\widetilde{\mathrm{FVA}}^\circ_\tau\big). \tag{29}$$

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Definition 5 (Cont'd)

- We define the contra-liabilities value process CL by

$$\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA} + \mathrm{CVA}^{\mathrm{CL}} + \mathrm{FVA}^{\mathrm{CL}}, \tag{30}$$

  where:
  - The DVA (debt valuation adjustment) is the $(\mathbb{G}, \mathbb{Q})$ value of $\mathcal{C}^{\bullet}$,
  - The FDA (funding debt adjustment) is the $(\mathbb{G}, \mathbb{Q})$ value of $\mathcal{F}^{\bullet}$, and
  - $\mathrm{CVA}^{\mathrm{CL}}$ and $\mathrm{FVA}^{\mathrm{CL}}$ are the $(\mathbb{G}, \mathbb{Q})$ values of terminal cash flows $\mathbb{1}_{\{\tau<T\}}\mathrm{CVA}^{\circ}_{\tau}$ and $\mathbb{1}_{\{\tau<T\}}\mathrm{FVA}^{\circ}_{\tau}$ at time $\bar{\tau}$.
- We call fair valuation of counterparty risk, denoted by CR, the $(\mathbb{G}, \mathbb{Q})$ value of $\mathcal{C}$.

Lemma 2

We have

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} \tag{31}$$

and

$$\mathrm{CR} = \mathrm{CA} - \mathrm{CL}, \tag{32}$$

which is also the $(\mathbb{G}, \mathbb{Q})$ *value of* $(\mathcal{C} + \mathcal{F} + \mathcal{H})$.

Regarding the FVA

- The industry terminology tends to distinguish an FVA, in the technical sense of the cost of funding cash collateral for variation margin, from an MVA, defined as the cost of funding segregated collateral posted as initial margin (see Albanese et al. (2017)).
- The academic literature, as in this paper, tends to merge the two in an overall FVA meant in the broad sense of the cost of funding the derivative trading strategy of the bank.

Regarding the contra-liabilities

- The DVA is the value that the bank clients lose due to the possible default of the bank in the future.
- The FDA is the value of the amount of its funding debt that the bank fails to reimburse if it defaults.
- $\mathrm{CVA}^{\mathrm{CL}}$ and $\mathrm{FVA}^{\mathrm{CL}}$ are contra-liability components of the CVA and the FVA, valuing the residual amounts $\mathbb{1}_{\{\tau<T\}}\mathrm{CVA}^{\circ}_{\tau}$ and $\mathbb{1}_{\{\tau<T\}}\mathrm{FVA}^{\circ}_{\tau}$, summing up to $\mathbb{1}_{\{\tau<T\}}\mathrm{CA}^{\circ}_{\tau} = \mathbb{1}_{\{\tau<T\}}\mathrm{RC}^{\circ}_{\tau}$ (cf. (16)), which are transferred from the RC account to bank creditors at time τ.

Lemma 3

Denoting by Π the value of the client portfolio (process used by the CA desk for marking to the model the client portfolio), defining $\mathrm{VA} = P - \Pi$, the trading loss processes L^{ca} of the CA desk and L of the bank as a whole satisfy

$$\begin{aligned}
\widetilde{L}^{ca} &= \widetilde{\mathrm{VA}}^{\circ} + \widetilde{\mathcal{C}}^{\circ} + \widetilde{\mathcal{F}}^{\circ} + \widetilde{\mathcal{H}} \\
\widetilde{L} &= \widetilde{\mathrm{VA}}^{\circ} + \widetilde{\mathcal{C}}^{\circ} + \widetilde{\mathcal{F}}^{\circ} + \widetilde{\mathcal{H}} + (\widetilde{L}^{cl})^{\circ}.
\end{aligned} \tag{33}$$

Proposition 2

(i) *Assuming* VA *in* $\mathcal{S}_2^\circ$, *we have* $\mathrm{VA} = \mathrm{CA}$, *hence*

$$\begin{aligned} \widetilde{L}^{ca} &= \widetilde{\mathrm{CA}} + \widetilde{\mathcal{C}}^\circ + \widetilde{\mathcal{F}}^\circ + \widetilde{\mathcal{H}} \\ \widetilde{L} &= \widetilde{\mathrm{CA}} + \widetilde{\mathcal{C}}^\circ + \widetilde{\mathcal{F}}^\circ + \widetilde{\mathcal{H}} + (\widetilde{L}^{cl})^\circ. \end{aligned} \tag{34}$$

(ii) *If, in opposition to our assumptions so far, the bank could hedge its own jump-to-default risk, i.e. assuming that the bank could and would additionally sell on the financial markets a contract paying* CL_τ *at time* τ *(e.g. through repurchasing of its own bond as contemplated in Burgard and Kjaer (2011a, 2011b)), assuming further* VA *in* $\mathcal{S}_2^\circ$ *and* CL *in* $\mathcal{S}_2$, *then, before* τ, *we would have*

$$\mathrm{VA} = \mathrm{CA} - \mathrm{CL} = \mathrm{CR}. \tag{35}$$

Corollary 4

CL *is interpreted as the wealth transfer triggered by the deals from shareholders to creditors, due to the inability of the bank to hedge its own jump-to-default exposure.*

Outline

- Since a bank cannot hedge its own jump-to-default, Proposition 2 pleads in favor of an XVA add-on defined by CA $=$ CVA $+$ FVA.
- However, not only a bank cannot hedge its own jump-to-default: It cannot replicate its counterparty default losses either.
- An XVA add-on defined by CA $=$ CVA $+$ FVA ensures that the trading loss L of the bank is zero in expectation.
- But the impossibility of replicating counterparty default losses implies that the trading of the bank generates a non-vanishing loss-and-profit process L.
- Then the regulator comes and requires that capital be set at risk by the shareholders, which therefore require a risk premium.

- Valuation is risk-neutral with respect to the stochastic bases $(\mathbb{F}, \mathbb{P})$ or $(\mathbb{G}, \mathbb{Q})$.
- Economic capital and KVA assess risk and its cost, which refer to the historical probability measure.
- In our setup, the duality of perspective of the clean vs. CA desks, on pricing as reflected by Assumptions 1–2, also applies to risk measurement.
    - Capital calculations are always made "on a going concern", i.e. assuming that the bank is alive, and therefore with respect to the reference filtration $\mathbb{F}$.
    - Instead, cost of capital calculations are made by the management of the bank in a model including the default of the bank.

- However, in the context of XVA computations entailing projections over decades, the main source of information is market prices of liquid instruments, which allow the bank to calibrate the pricing measure, and there is little of relevance that can be said about the historical probability measure.
- Hence, in our model:

Assumption 6

The estimates $\widehat{\mathbb{P}}$ and $\widehat{\mathbb{Q}}$ of the historical probability measure respectively used in economic capital and cost of capital computations coincide with the pricing measures $\mathbb{P}$ and $\mathbb{Q}$.

- Any discrepancy between $\mathbb{P}$ and $\widehat{\mathbb{P}}$ or $\mathbb{Q}$ and $\widehat{\mathbb{Q}}$ is left to model risk, meant to be included in an AVA (additional valuation adjustment) in an FRTB terminology, which is left for future research.

Economic Capital

- The economic capital (EC) of the bank is its resource devoted to cope with losses beyond their expected levels that are already taken care of by reserve capital (RC).
- Basel II Pillar II defines economic capital as the 99% value-at-risk of the negative of the variation over a one-year period of core equity tier I capital (CET1), the regulatory metric that represents the wealth of the shareholders within the bank.
- Recently, the FRTB required a shift from 99% value-at-risk to 97.5% expected shortfall.
- In our setup, capital depletions correspond to the trading loss process L.

- Accordingly, also accounting for discounting (and recalling that L' is the $\mathbb{F}$ reduction of L):

Definition 6

Our reference definition for the (discounted) economic capital of the bank at time t is the $(\mathcal{F}_t, \mathbb{P})$ conditional 97.5% expected shortfall of $(\widetilde{L}'_{t+1} - \widetilde{L}'_t)$, which we denote by $\widetilde{\mathrm{ES}}'_t(L)$.

- Solvency II introduces a further modification of economic capital, which is required to be in excess of the risk margin (RM), i.e. of the KVA (cf. (16)). This modification is considered later.

- For the purpose of economic capital and cost of capital computations, the trading loss process L of the bank can be considered as an exogenous process ($(\mathbb{G}, \mathbb{Q})$ martingale without jump at τ, by Lemma 1).
- Accordingly we just write $\widetilde{\mathrm{ES}}'_t$ for $\widetilde{\mathrm{ES}}'_t(L)$, and ES'_t for $\mathrm{ES}'_t(L)$, the undiscounted version of $\widetilde{\mathrm{ES}}'_t(L)$.

Lemma 4

ES' *is nonnegative.*

KVA as a Risk Premium

- Counterparty default losses, as also funding payments, are materialities for default if not paid, hence true liabilities to shareholders.
- In contrast, KVA payments are at the discretion of the bank management and released to bank shareholders themselves.

Accordingly:

Assumption 7

The risk margin is loss-absorbing, hence part of economic capital.

Corollary 5

Shareholder capital at risk (SCR) *is the difference between the economic capital* (EC) *of the bank and its risk margin* (RM), *i.e.*

$$\mathrm{SCR} = \mathrm{EC} - \mathrm{RM}. \tag{36}$$

Assumption 8

An exogenous and constant hurdle rate h prevails, in the sense that bank shareholders are constantly maintained by the KVA payments on an "efficient frontier" such that, at any time t

$$\text{“Shareholder instantaneous average return}_t = h \times \mathrm{SCR}_t\text{.”} \tag{37}$$

- In practice the level of compensation required by shareholders on their capital at risk is driven by market considerations. Typically, investors in banks expect a hurdle rate h of about 10% to 12%.
- In this paper we assume a constant h for simplicity.
- An endogenous and stochastic hurdle rate would arise in a model of competitive equilibrium, where different banks compete for clients.
  - As opposed to our setup where only one bank is considered.

- In view of Lemma 1 and Corollary 5, where $\mathrm{RM} = \mathrm{KVA}$ holds at all times by (16), and since $\widetilde{L}$ is a $(\mathbb{G}, \mathbb{Q})$ martingale by Lemma 1, the informal statement (37) is formulated in mathematical terms by the requirement that

$$\begin{gathered}(-\widetilde{\mathrm{KVA}}) \text{ has a } (\mathbb{G}, \mathbb{Q}) \text{ drift given as} \\ \text{the time-integrated process } h(\widetilde{\mathrm{EC}} - \widetilde{\mathrm{KVA}}),\end{gathered} \tag{38}$$

  assumed to define a unique KVA process in $\mathcal{S}_2^{\circ}$
  - This includes that the KVA process is defined until $\bar{\tau}$ and without jump at τ.

- However, the KVA equation (38) is only preliminary if EC there is just meant as ES', which would then be forgetful of a consistency condition $\mathrm{SCR} \geq 0$.
- This is fixed in the next section by pushing EC above ES' until the constraint is satisfied.

The KVA Constrained Optimization Problem

- Assume that, for any tentative economic capital process C in a suitable Hilbert space $\mathcal{L}_2$ of square integrable processes containing $\mathcal{S}_2$ and the process ES', the equation (cf. (38))

  $(-\widetilde{K})$ has a $(\mathbb{G}, \mathbb{Q})$ drift given as the time-integrated process $h(\widetilde{C} - \widetilde{K})$

  defines a unique process $K = K(C)$ in $\mathcal{S}_2^{\circ}$.

Definition 7

The set of admissible economic capital processes is defined as

$$\mathcal{C} = \{C \in \mathcal{L}_2; C \geq \max\big(K(C), \mathrm{ES}'\big)\}, \tag{40}$$

where (b) $C \geq \mathrm{ES}'$ is the risk acceptability condition and (a) $C \geq K(C)$ is the self-consistency condition.

The KVA Constrained Optimization Problem

- In view of (39) and (40), the natural guess for the smallest and cheapest admissible economic capital process is

$$\mathrm{EC} = \max(\mathrm{ES}', \mathrm{KVA}), \tag{41}$$

for a process KVA in $\mathcal{S}_2^\circ$ such that

$$\begin{aligned} &(-\widetilde{\mathrm{KVA}}) \text{ has a } (\mathbb{G}, \mathbb{Q}) \text{ drift given as the} \\ &\text{time-integrated process } h\big(\max(\widetilde{\mathrm{ES}}', \widetilde{\mathrm{KVA}}) - \widetilde{\mathrm{KVA}}\big). \end{aligned} \tag{42}$$

$\rightarrow$ The discounted KVA is a $(\mathbb{G}, \mathbb{Q})$ supermartingale.

Remark 3

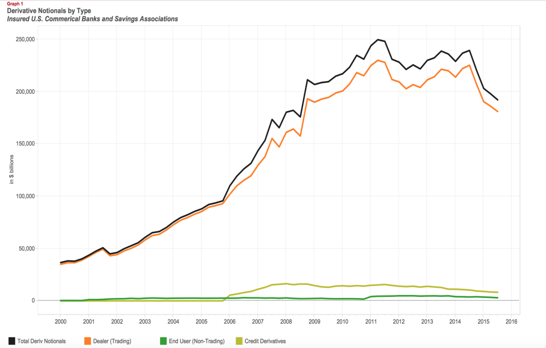
In the case of perfect clean and CA hedges where the process L (hence L') is constant, then ES vanishes and $\mathrm{KVA} = 0$ obviously solves (42) in $\mathcal{S}_2^\circ$.

Incremental XVA Approach

- Given the actual (incremental) derivative porfolio of a bank, the above can be applied to the version of the portfolio that would be run-off by the bank from time 0 onward until its final maturity T.
- The ensuing XVA numbers are interpreted as the amounts CA $= \mathrm{CVA} + \mathrm{FVA}$ to maintain on the reserve capital (RC) account and KVA to maintain on the risk margin (RM) account, which would allow the bank to go into run-off in line with shareholder interest.

- Such a "soft landing option" is key from a regulator point of view, as it guarantees that the bank should not be tempted to go into snowball or Ponzi kind of schemes where always more trades are entered for the sole purpose of funding previously entered ones.

Ponzi scheme in the last financial crisis (source: Office of the Comptroller of the Currency, Q3 2015 Quarterly Bank Trading Revenue Report).



- Moreover, since we rely on a dynamic analysis, this possibility, for a bank respecting the balance conditions (16), of going run-off in line with shareholder interest, is granted not only from time 0 onward, but from any future time onward, as long as there are no new deals in the portfolio.

- A new trade has two impacts: it triggers a wealth transfer from shareholders to bondholders and alters the risk profile of the portfolio.
- This is reflected by a jump "$\Delta\cdot$" of the balance sheet, from the one related to the endowment (pre-trade portfolio) right before the time t the new deal is considered, to the one related to the portfolio including the new deal at time t (both portfolios being assumed held on a run-off basis).

- Hence the balance conditions (16) and the associated soft landing option of the bank are impaired, unless the missing RC and RM amounts are sourced from the client of the deal in order to restore them.

$\rightarrow$ $\Delta \mathrm{RC} = \Delta \mathrm{CA}, \Delta \mathrm{RM} = \Delta \mathrm{KVA}.$

→ The all-inclusive XVA add-on to the entry price for a new deal, called fund transfer pricing (FTP), is

$$\mathrm{FTP} = \Delta\mathrm{CA} + \Delta\mathrm{KVA} = \Delta\mathrm{CVA} + \Delta\mathrm{FVA} + \Delta\mathrm{KVA} \tag{43}$$

- Obviously, the endowment has a key impact on the FTP of a new trade. For instance, it can happen that a new deal is risk-reducing with respect to the pre-existing portfolio, in which case $\mathrm{FTP} < 0$.

- The preservation of the balance conditions in between and throughout deals yields a sustainable strategy for profits retention, which is already the key principle behind Solvency II.
- From this "soft landing" perspective it is natural to perform the XVA computations under the following assumption, in line with a run-off procedure where market risk is first hedged out, but we conservatively assume no XVA hedge, and the portfolio is then let to amortize until its final maturity T:

Assumption 9

We assume a perfect clean hedge by the clean traders, i.e. L^{cl} constant

- can be taken as zero as it is only the fluctuations of the loss processes that matter

and no CA hedge, i.e. $\mathcal{H} = 0$.

As it then immediately follows from Lemma 3 and Proposition 2(i):

Corollary 6

We have

$$\widetilde{L} = \widetilde{L}^{ca} = \widetilde{\mathrm{CA}} + \widetilde{\mathcal{C}}^{\circ} + \widetilde{\mathcal{F}}^{\circ}. \tag{44}$$

Hence the process L that is used as input to capital and KVA computations (cf. Definition 6 and (42), where $\mathrm{ES}' = \mathrm{ES}'(L)$) is the output of the CA computations, making the XVA problem as a whole self-contained.

Outline

Spaces

Assuming that the pre-intensity γ of τ is $\mathbb{F}$ predictable (without loss of generality by reduction), we denote by:

- $\mathcal{S}_2$, the space of càdlàg $\mathbb{G}$ adapted processes Y over $[0, \bar{\tau}]$ such that, denoting $Y_t^* = \sup_{s\in[0,t]} |Y_s|$:

$$\mathbb{E}\Big[Y_0^2 + \int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} d(Y_s^*)^2\Big] < \infty; \tag{45}$$

- $\mathcal{S}_2^\circ$, the subspace of the processes Y in $\mathcal{S}_2$ such that Y is without jump at τ on $\{\tau < T\}$ and $Y_T = 0$ on $\{T < \tau\}$;
- $\mathcal{S}_2^\bullet$, the subspace of the processes Y in $\mathcal{S}_2$ such that $Y_{\bar{\tau}} = 0$;
- $\mathcal{L}_2$, the space of $\mathbb{G}$ progressively measurable processes X over $[0, T]$ such that

$$\mathbb{E}\big[\int_0^T e^{\int_0^s \gamma_u du} \mathbb{1}_{\{s<\tau\}} X_s^2 ds\big] < +\infty; \tag{46}$$

- $\mathcal{S}_2'$, the space of càdlàg $\mathbb{F}$ adapted processes Y' over $[0, T]$ such that

$$\mathbb{E}'\Big[\sup_{t\in[0,T]} (Y_t')^2\Big] < \infty \tag{47}$$

and $Y_T' = 0$;
- $\mathcal{L}_2'$, the space of $\mathbb{F}$ progressively measurable processes X' over $[0, T]$ such that

$$\mathbb{E}'\Big[\int_0^T (X_t')^2 dt\Big] < +\infty. \tag{48}$$

KVA in the Case of a Default-Free Bank

- Note that the primary reason for the KVA to exist is the default of the bank clients, as opposed to the default of the bank itself
  - which on the other hand is the key of the contra-liabilities related wealth transfer issue.
- In this part we suppose the bank default free, i.e.

$$\tau = +\infty\,,\;\; (\mathbb{F},\mathbb{P}) = (\mathbb{G},\mathbb{Q}) \text{ and } \gamma = 0.$$

  - This is then extended to the case of a defaultable bank in the next part.

- At that stage in this part we use the "$\cdot'$" notation, not in the sense of reduction (as $\mathbb{F} = \mathbb{G}$), but simply in order to distinguish the equations in this part, where $\mathbb{F} = \mathbb{G}$, from the ones in the next part, where $\mathbb{F} \neq \mathbb{G}$
  - The data of this subsection will then be interpreted a posteriori as the reductions of the corresponding data in the next subsection.

Given $C' \geq \mathrm{ES}'$ representing a putative economic capital process for the bank, consider the following BSDEs (cf. (39) and (42) when $\tau = +\infty$):

$$K'_t = \mathbb{E}'_t \int_t^T \big(hC'_s - (r_s + h)K'_s\big)ds\,, \quad t \in [0, T], \tag{49}$$

$$\mathrm{KVA}'_t = \mathbb{E}'_t \int_t^T \big(h \max(\mathrm{ES}'_s, \mathrm{KVA}'_s) - (r_s + h)\mathrm{KVA}'_s\big)ds\,, \quad t \in [0, T], \tag{50}$$

to be solved for respective processes K' and KVA'.

Lemma 5

Assuming that r is bounded from below and that r , C', and ES' are in $\mathcal{L}'_2$, then the BSDEs (49) *and* (50) *are well posed in $\mathcal{S}'_2$, where well-posedness includes existence, uniqueness and comparison. We have, for $t \in [0, T]$,*

$$\mathrm{KVA}'_t = h\mathbb{E}'_t \int_t^T e^{-\int_t^s (r_u+h)du} \max(\mathrm{ES}'_s, \mathrm{KVA}'_s)ds. \tag{51}$$

Proof. By application of monotonic coefficient BSDE results (see e.g. Kruse and Popier (2016, Sect. 4)).

Proposition 3

Assuming that r is bounded from below and that r and ES' are in $\mathcal{L}_2'$, we have:

(i) $\mathrm{EC}' = \min \mathcal{C}', \mathrm{KVA}' = \min_{C' \in \mathcal{C}'} K'(C')$;

(ii) *The process KVA' is nonnegative and it is nondecreasing in h.*

Proof. By applications of BSDE comparison theorem (see e.g. Kruse and Popier (2016, Proposition 4).

KVA in the Case of a Defaultable Bank

In the case of a defaultable bank, "$\cdot'$" now denoting reduction, then, by the results of Crépey and Song (2017b):

- For any $C \in \mathcal{L}_2$, we have $C' \in \mathcal{L}_2'$ and the $(\mathbb{G}, \mathbb{Q})$ BSDE (39) in $\mathcal{S}_2^\circ$ is equivalent to the $(\mathbb{F}, \mathbb{P})$ BSDE (49) in $\mathcal{S}_2'$ through the correspondence $K = (K')^{\tau -}$ on $[0, \bar{\tau}]$;
- Assuming ES' in $\mathcal{L}_2'$, the $(\mathbb{G}, \mathbb{Q})$ KVA BSDE (42) in $\mathcal{S}_2^\circ$ is equivalent to the $(\mathbb{F}, \mathbb{P})$ KVA' BSDE (50) in $\mathcal{S}_2'$ through the correspondence $\mathrm{KVA} = (\mathrm{KVA}')^{\tau -}$ on $[0, \bar{\tau}]$.

Hence, by application of Lemma 5 and Proposition 3 through the above correspondences:

Lemma 6

Assuming that r is bounded from below and that r, C', and ES' are in $\mathcal{L}_2'$, then the $(\mathbb{G}, \mathbb{Q})$ linear BSDEs (39) for $K = K(C)$ and the $(\mathbb{G}, \mathbb{Q})$ KVA BSDE (42) are well posed in $\mathcal{S}_2$, where well-posedness includes existence, uniqueness and comparison.

Theorem 1

Assuming that r is bounded from below and that r and ES' are in $\mathcal{L}_2'$:
(i) $\mathrm{EC} = \min \mathcal{C}, \mathrm{KVA} = \min_{C \in \mathcal{C}} K(C)$;
(ii) *The process KVA is nonnegative and it is nondecreasing in h.*

Contra-assets and Contra-liabilities

- The counterparty exposure and funding cumulative cash flow streams $Y = \mathcal{C}$ and $\mathcal{F}$ (recall Assumption 9 set $\mathcal{H} = 0$) are given as $\mathbb{G}$ finite variation processes.
- C° and $\mathcal{F}^{\circ}$ can be assumed to be $\mathbb{F}$ finite variation processes, without loss of generality by reduction.
- Regarding the funding cash flows, we assume more specifically:

$$d\mathcal{F}^{\circ}_t = f_t(\mathrm{FVA}_t)dt \text{ until } \tau, \tag{52}$$

for some predictable coefficient (random function) f.

- A structure (52) for $\mathcal{F}$ is a slight departure from our abstract setup, where, for simplicity of presentation in a first stage, $\mathcal{F}$ was introduced as an exogenous process.
- But, as already found in the one-period setup, the dependence of $\mathcal{F}$ on the FVA is only semi-linear (i.e. f in (52) is Lipschitz or monotonous) in practice.
- Provided the corresponding FVA fixed-point problem is well-posed, one can readily check, by revisiting all the above, that such dependence does not affect any of the qualitative conclusions in the above.

Theorem 2

For $\mathcal{C}$ and $\mathcal{F}$ thus specialized, the CVA *and* FVA *equations* (28) *and* (29) *in $\mathcal{S}_2^\circ$ are equivalent to the following equations in $\mathcal{S}_2'$:*

$$\mathrm{CVA}'_t = \mathbb{E}'_t \int_t^T \beta_t^{-1}\beta_s d\mathcal{C}_s^\circ\,, \quad t \in [0, \bar{T}], \tag{53}$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t \int_t^T \beta_t^{-1}\beta_s f_s(\mathrm{FVA}'_s) ds\,, \quad t \in [0, \bar{T}], \tag{54}$$

equivalent through the following correspondence:

$$\mathrm{CVA} = (\mathrm{CVA}')^{\tau-} \text{ and } \mathrm{FVA} = (\mathrm{FVA}')^{\tau-} \text{ on } [0, \bar{\tau}]. \tag{55}$$

Proof. By application of the results of Crépey and Song (2017b), the $(\mathbb{G}, \mathbb{Q})$ CVA equation (28) in $\mathcal{S}_2^\circ$ is equivalent through the first identity in (55) to the $(\mathbb{F}, \mathbb{P})$ CVA' formula (53) in $\mathcal{S}_2'$. Likewise, for $\mathcal{F}^\circ$ as per (52), the $(\mathbb{G}, \mathbb{Q})$ FVA equation (29) in $\mathcal{S}_2^\circ$ is equivalent through (55) to the $(\mathbb{F}, \mathbb{P})$ FVA' BSDE (54) in $\mathcal{S}_2'$.

Example 4

In the setup of Example 2, we have, for $0 \leq t \leq \bar{\tau}$:

$$\begin{aligned} d\mathcal{F}^{\circ}_t &= \lambda_t (Q_t - \mathrm{CA}_t)^+ dt \\ d\mathcal{F}^{\bullet}_t &= (1-R)(Q_{t-} - \mathrm{CA}_{t-})^+ (-dJ_t). \end{aligned} \tag{56}$$

Hence $\mathcal{F}^{\circ}$ is of the form (52) for $f_t(y) = \lambda_t (Q_t - \mathrm{CVA}_t - y)^+$. and

$$dL^{ca}_t = d\mathrm{CA}_t - r_t \mathrm{CA}_t dt + d\mathcal{C}^{\circ}_t + \lambda_t (Q_t - \mathrm{CA}_t)^+ dt. \tag{57}$$

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Example 4 (Cont'd)

- Assume further that the bank portfolio involves a single client with default time denoted by τ_1, that $\mathbb{Q}(\tau_1 = \tau) = 0$, that the liquidation of a defaulted party is instantaneous and that no derivative cash flows are due at the exact times τ and τ_1.
- Let J and J^1, respectively R and R_1, denote the survival indicator processes and the recovery rates of the bank and its client.

Example 4 (Cont'd)

- Then Q is of the form $J^1 Q^1$, where Q^1 is the difference between the mark-to-market P of the variation margin provided by the CA desk to the clean desks and the mark-to-market, denoted by Γ, of the variation margin provided to the CA desk by the client.
- Moreover, for $0 \le t \le \bar{\tau}$,

$$\begin{aligned} d\mathcal{C}^{\circ}_t &= (1 - R_1)(Q^1_{\tau_1})^+(-dJ^1_t) \\ d\mathcal{C}^{\bullet}_t &= \mathbb{1}_{\{\tau \le \tau_1\}}(1 - R)(Q^1_{\tau})^-(-dJ_t). \end{aligned} \tag{58}$$

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Proposition 4

In the setup of Example 4, assuming that r is bounded from below and that the processes r, λ, and $\lambda(J^1 Q^1 - \mathrm{CVA}')^+$ are in $\mathcal{L}'_2$, and that CVA' in (60) is in $\mathcal{S}'_2$, then the CVA and FVA equations (28) and (29) are well-posed in $\mathcal{S}^\circ_2$ and we have, for $0 \le t \le \bar{\tau}$:

$$\mathrm{CVA}_t = (\mathrm{CVA}')^{\tau-}_t \text{ and } \mathrm{FVA}_t = (\mathrm{FVA}')^{\tau-}_t, \text{ where for } 0 \le t \le T \tag{59}$$

$$\mathrm{CVA}'_t = \mathbb{E}'_t\big[\mathbb{1}_{\{t<\tau_1<T\}}\beta_t^{-1}\beta_{\tau_1}(1-R_1)(Q^1_{\tau_1})^+\big]; \tag{60}$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t \int_t^T \beta_t^{-1}\beta_s\lambda_s(J^1_s Q^1_s - \mathrm{CVA}'_s - \mathrm{FVA}'_s)^+ ds. \tag{61}$$

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Proposition 4 (Cont'd)

$$\mathrm{CL}_t = \underbrace{\mathbb{E}_t\left[\mathbb{1}_{\{\tau\leq\tau_1\wedge T\}}\beta_t^{-1}\beta_{\tau_1}(1-R)(Q^1_\tau)^-\right]}_{\mathrm{FTDDVA}_t} \quad (62$$

$$+\underbrace{\mathbb{E}_t\left[\beta_\tau/\beta_t\mathbb{1}_{\{\tau<T\}}(J^1_{\tau-}Q^1_{\tau-}-\mathrm{CA}_{\tau-})^+\right]}_{\mathrm{FDA}_t} \quad (63$$

$$+\underbrace{\mathbb{E}_t\left[\beta_t^{-1}\beta_\tau\mathbb{1}_{\{\tau<T\}}\mathrm{CVA}'_{\tau-}\right]}_{\mathrm{CVA}^{\mathrm{CL}}_t}+\underbrace{\mathbb{E}_t\left[\beta_t^{-1}\beta_\tau\mathbb{1}_{\{\tau<T\}}\mathrm{FVA}'_{\tau-}\right]}_{\mathrm{FVA}^{\mathrm{CL}}_t}; \quad (64$$

$$\mathrm{CR}_t = \underbrace{\mathbb{E}_t\left[\mathbb{1}_{\{t<\tau_1\leq\tau\wedge T\}}\beta_t^{-1}\beta_{\tau_1}(1-R_1)(Q^1_{\tau_1})^+\right]}_{\mathrm{FTDCVA}_t} \quad (65$$

$$-\underbrace{\mathbb{E}_t\left[\mathbb{1}_{\{t<\tau\leq\tau_1\wedge T\}}\beta_{\tau_1}/\beta_t(1-R)(Q^1_\tau)^-\right]}_{\mathrm{FTDDVA}_t}$$

$$dL_t = d\mathrm{CA}_t - r_t\mathrm{CA}_t dt + (1-R_1)(Q^1_{\tau_1})^+(-dJ^1_t) + \lambda_t(J^1_tQ^1_t-\mathrm{CA}_t)^+ \quad (66$$

Proof. Under the assumptions of the proposition, the $(\mathbb{F}, \mathbb{P})$ FVA' BSDE (54) is a monotonous coefficient BSDE well-posed in $\mathcal{S}_2'$, based on the results of Kruse and Popier (2016, Sect. 4).
In view of Theorem 2, this proves the CVA and FVA related statements, whereas the CL and CR formulas (64) and (65) readily follow from (30), (56) and (58) for CL and Definition 5 and (58) for CR.
The dynamics (66) for L are obtained by plugging into (57) the first line in (58).

- Proposition 4 is easily extended to bilateral trade portfolios with several counterparties.
  - cf. Albanese, Caenazzo, and Crépey (2017) and (73)-(74) below

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- Proposition 4 is derived in a pure valuation perspective.
- In most other former XVA references in the literature, XVA equations are based on hedging arguments.
    - Most previous XVA works were not considering KVA yet.
    - Under our approach, the KVA is the risk premium for the market incompleteness related to contra-assets.
    - Hence, for consistency, our KVA treatment requires a pure valuation (as opposed to hedging) view on contra-assets

- The formula (65) for the valuation of counterparty risk is derived in Duffie and Huang (1996) in the limit case of a perfect market (complete counterparty risk market without trading restrictions).
- Formula (65) is symmetrical, i.e. consistent with the law of one price, in the sense that $(\mathrm{FTDCVA} - \mathrm{FTDDVA})$ corresponds to the negative of the analogous quantity considered from the point of view of the counterparty.
- It only involves the first-to-default CVAs and DVAs, where the default losses are only considered until the first occurrence of a default of the bank or its counterparty in the deal.
    - This is consistent with the fact that later cash flows will, as first emphasised in Duffie and Huang (1996), Bielecki and Rutkowski (2002) and Brigo and Capponi (2008), not be paid in principle.

- Proposition 4 extends the validity of the formula (65) for the valuation (CR) of counterparty risk from the point of view of the bank of the whole in our incomplete market setup.
- Since the presence of collateral has a direct reducing impact on FTDCVA/DVA, this formula may give the impression that collateralization achieves a reduction in counterparty risk at no cost to either the bank or the clients.
- However, in the present incomplete market setup, the value CR from the point of view of the bank as a whole ignores the misalignement of interest between the shareholders and the creditors of a bank.

- Proposition 4 gives explicit decompositions of the respective cost of counterparty risk to shareholders (CA) and of the wealth transfer (CL) triggered from the shareholders to the creditors by the impossibility for the bank to hedge its own jump-to-default exposure.
- Due to the latter and to the impossibility for the bank to replicate counterparty default losses, these contra-liabilities (CL) as well as the cost of capital (KVA) are material to shareholders and need to be reflected in entry prices on top of the fair valuation (CR) of counterparty risk.

- Only the fluctuations of L matter in economic capital calculations, hence the (unknown) value of L_0 is immaterial in all XVA computations.

Outline

- Even though our setup includes the default of the bank itself, which is the essence of the contra-liabilities related wealth transfer issue, we end up with unilateral CVA, FVA and KVA formulas such as (60), (61) and (50) pricing the related cash flows until the end of times T (as opposed to $\bar{\tau} = \tau \wedge T$).
  - And these equations only involve the original discount factor β, without any credit spread.
- This is indeed what follows from a careful analysis of the wealth transfers involved.
- However this also makes the ensuing XVAs more expensive than the bilateral XVAs that appear in most of the related literature.

- A unilateral CVA is actually required for being in line with the regulatory requirement that reserve capital should not diminish as an effect of the sole deterioration of the bank credit spread.
- But a bilateral FVA already satisfies the regulatory monotonicity requirement
  - Essentially, as, when the bank credit spread deteriorates, the shortest duration of a bilateral FVA is compensated by the higher funding spread.
- And the KVA is not concerned by this requirement.
  - Actually, a unilateral KVA might arguably be unjustified, with regard to the fact that bank insolvency means depletion of the whole economic capital of the bank, which includes the risk margin. Hence the notion of transfer of the residual risk margin to creditors at bank default would be pointless.
  - However, the default of a bank does not mean insolvency, but illiquidity mainly.

Assuming all the risk margin already gone at time $\bar{\tau}$ through some additional model feature, such as an operational loss that would occur at τ and trigger instantaneous depletion of economic capital, would result in the following modified KVA equation in $\mathcal{S}_2^{\bullet}$:

$$\begin{aligned}&(-\widetilde{\mathrm{KVA}}^{\circ}) \text{ has a } (\mathbb{G},\mathbb{Q}) \text{ drift given as the}\\ &\text{time-integrated process } h\big(\max(\widetilde{\mathrm{ES}}', \widetilde{\mathrm{KVA}}) - \widetilde{\mathrm{KVA}}\big),\end{aligned} \tag{67}$$

i.e.

$$\mathrm{KVA}_t = h\mathbb{E}_t \int_t^{\bar{\tau}} e^{-\int_t^s (r_u+h)du} \max(\mathrm{ES}'_s, \mathrm{KVA}_s)ds, \; t \in [0,\bar{\tau}], \tag{68}$$

or, in an an equivalent $(\mathbb{F},\mathbb{P})$ formulation, $\mathrm{KVA} = (\mathrm{KVA}')^{\tau}$ on $[0,\bar{\tau}]$, where (compare with (51), noting in particular the "$+\gamma_u$" in the discount factor in (69))

$$\mathrm{KVA}'_t = h\mathbb{E}'_t \int_t^{T} e^{-\int_t^s (r_u+h+\gamma_u)du} \max(\mathrm{ES}'_s, \mathrm{KVA}'_s)ds, \; t \in [0,T]. \tag{69}$$

From Unilateral to Bilateral FVA

- A bilateral FVA, which already satisfies the regulatory monotonicity requirement on the related reserve capital, might be advocated as follows.
- Assume for the sake of the argument that the portfolio of the defaulted bank with clients is unwounded with risk-free counterparties, called novators.
- The residual amount of CVA reserve capital is required by the novators to deal with the residual counteparty risk on the deals.
- But the residual amount of FVA reserve capital is useless to the novators.
- In view of this one could decide that, upon bank default, the residual FVA capital reserve flows back into equity capital and not to creditors.

- For formalizing this mathematically, one needs to disentangle the CA desk into a CVA desk and an FVA desk, each endowed with their own reserve capital account (and hedge).
- This would result in an FVA equation stated in $\mathcal{S}_2^\bullet$ as

$$\widetilde{\mathrm{FVA}}_t = \mathbb{E}_t\big(\widetilde{\mathcal{F}}^\circ_{\bar{\tau}} - \widetilde{\mathcal{F}}^\circ_t\big)\,, \quad t \leq \bar{\tau}, \tag{70}$$

instead of the FVA equation (29) in $\mathcal{S}_2^\circ$.
- That is (compare with (54))

$$\mathrm{FVA}_t = \mathbb{E}_t \int_t^{\bar{\tau}} \beta_t^{-1}\beta_s f_s(\mathrm{FVA}_s)ds,\; t \in [0,\bar{\tau}], \tag{71}$$

or, equivalently, $\mathrm{FVA} = (\mathrm{FVA}')^\tau$ on $[0,\bar{\tau}]$, where

$$\mathrm{FVA}'_t = \mathbb{E}'_t \int_t^T e^{-\int_t^s (r_u+\gamma_u)du} f_s(\mathrm{FVA}'_s)ds,\; t \in [0,T]. \tag{72}$$

- Note again the blended discount factor in (72), as opposed to the risk-free discount factor β in (54).

Using Economic Capital as Variation Margin

- Next we account for the additional FVA reduction provided by the possibility for a bank to post economic capital, on top of reserve capital already included in the above, as variation margin.
- Note that, in principle, uninvested capital (UC) could be used for VM as well, but since UC is not known and could as well be zero in the future, capital is conservatively taken here as (RC+EC).
- Accounting for the use of EC as VM, the VM funding needs are reduced from $(Q - \mathrm{CA})^+$ to $(Q - \mathrm{EC}(L) - \mathrm{CA})^+$.

As a consequence, instead of an exogenous CA value process feeding the dynamics (66) for L, one obtains the following FBSDE system of a forward SDE for L coupled with a backward SDE for the CA value process (assuming n counterparties with survival indicator processes J^i, hence $Q = \sum J^i Q^i$):

$$
\begin{aligned}
& L_0 = z \text{ and, for } t \in (0, \bar{\tau}], \\
& dL_t = d\mathrm{CA}_t + \sum_i (1 - R_i)(Q^i_{\tau_i})^+(-dJ^i_t) \\
& \quad + \Big(\lambda_t\big(\sum_i J^i_t Q^i_t - \mathrm{EC}_t(L) - \mathrm{CA}_t\big)^+ - r_t \mathrm{CA}_t\Big) dt,
\end{aligned}
\tag{73}
$$

where

$$
\begin{aligned}
\mathrm{CA}_t = & \underbrace{\mathbb{E}_t \sum_{t<\tau_i<T} \beta_t^{-1}\beta_{\tau_i}(1 - R_i)(Q^i_{\tau_i})^+}_{\mathrm{CVA}_t} \\
& + \underbrace{\mathbb{E}_t \int_t^{\bar{\tau}} \beta_t^{-1}\beta_s \lambda_s \Big(\sum_i J^i_s Q^i_s - \mathrm{EC}_s(L) - \mathrm{CA}_s\Big)^+ ds}_{\mathrm{FVA}_t}, \quad 0 \le t \le \bar{\tau}.
\end{aligned}
\tag{74}
$$

- Unless $\lambda = 0$, nonstandard coupling between L and CA through the term $\mathrm{EC}_t(L)$, which entails the conditional law of the one-year-ahead increments of L.
- Crépey, Élie, and Sabbagh (2017) show that:
  - This FBSDE for L and CA can be decoupled into an anticipated BSDE (ABSDE) for the underlying FVA process;
  - The previous results are still valid provided one replaces $(Q - \mathrm{CA})^+$ by $(Q - \mathrm{EC}(L) - \mathrm{CA})^+$ everywhere.

Outline

- Our XVA approach can be implemented by means of nested Monte Carlo simulations for approximating the loss process L required as input data in the KVA computations. Contra-assets (and contra-liabilities if wished) are computed at the same time.
- Practical trade-off: unilateral CVA vs. bilateral FVA and KVA.

- Since one of our goals in the numerics is to emphasize the impact on the FVA of the funding sources provided by reserve capital and economic capital, we consider the FBSDE (73)–(74) which accounts for the use of EC (on top of RC) as VM.
- Let

$$\mathrm{FVA}_t^{(0)} = \mathbb{E}_t \int_t^{\bar{\tau}} \beta_t^{-1} \beta_s \lambda_s \big(\sum_i J_s^i Q_s^i\big)^+ ds,$$

which corresponds to the FVA accounting only for the re-hypothecation of the variation margin received on hedges, but ignores the FVA deductions reflecting the possible use of reserve and economical capital as VM.

Picard iteration

$L^{(0)} = z$, $\mathrm{FVA}^{(0)}$ as above, $\mathrm{CA}^{(0)} = \mathrm{CVA} + \mathrm{FVA}^{(0)}$ and, for $k \geq 1$,

$$L_0^{(k)} = z \text{ and, for } t \in (0, \bar{\tau}],$$

$$dL_t^{(k)} = d\mathrm{CA}_t^{(k-1)} - r_t \mathrm{CA}_t^{(k-1)} dt + \sum_i (1 - R_i)(Q^i_{\tau_i})^+(-dJ^i_t)$$

$$+ \lambda_t \Big(\sum_i J^i_t Q^i_t - \max\big(\mathrm{ES}_t(L^{(k-1)}), \mathrm{KVA}_t^{(k-1)}\big) - \mathrm{CA}_t^{(k-1)} \Big)^+ dt,$$

$$\mathrm{CA}_t^{(k)} = \mathrm{CVA}_t + \mathrm{FVA}_t^{(k)} \text{ where } \mathrm{FVA}_t^{(k)} =$$

$$\mathbb{E}_t \int_t^{\bar{\tau}} \beta_t^{-1} \beta_s \lambda_s \Big(\sum_i J^i_s Q^i_s - \max\big(\mathrm{ES}_s(L^{(k)}), \mathrm{KVA}_s^{(k-1)}\big) - \mathrm{CA}_s^{(k-1)} \Big)^{+}$$

$$\mathrm{KVA}_t^{(k)} = h \mathbb{E}_t \int_t^{\bar{\tau}} e^{-\int_t^s (r_u + h) du} \max\big(\mathrm{ES}_s(L^{(k)}), \mathrm{KVA}_s^{(k-1)}\big) ds. \tag{75}$$

- Numerically, one iterates (75) as many times as is required to reach a fixed point within a preset accuracy.
- In the case studies we considered, one iteration ($k = 1$) was found sufficient.
- A second iteration did not bring significant change as
  - In (73)-(74) the FVA feeds into economic capital only through FVA volatility and the economic capital feeds into FVA through a capital term which is typically not FVA dominated
  - In (68), in most cases we have that $\mathrm{EC} = \mathrm{ES}$. The inequality only stops holding when the hurdle rate h is very high and the term structure of EC starts very low and has a sharp peak in a few years, which is quite unusual for a portfolio held on a run-off basis, as considered in XVA computations, which tends to amortize in time.

- However, going even once through (75) necessitates the conditional risk measure simulation of $\mathrm{EC}_t(L)$. On realistically large portfolios, some approximation is required for the sake of tractability.
- The simulated paths of $L^{(1)}$ are used for inferring a deterministic term structure

$$\mathrm{ES}^{(1)}(t) \approx \mathrm{ES}_t(L^{(1)}) \tag{76}$$

of economic capital, obtained by projecting in time instead of conditioning with respect to $\mathcal{G}_t$ in ES.

Outline

- We present two XVA case studies on fixed-income and foreign-exchange portfolios. Toward this end we use the market and credit portfolio models of Albanese, Bellaj, Gimonet, and Pietronero (2011) calibrated to the relevant market data.
- We use nested simulation with primary scenarios and secondary scenarios generated under the risk neutral measure calibrated to derivative data using broker datasets for derivative market data.
- All the computations are run using a 4-socket server for Monte Carlo simulations, Nvidia GPUs for algebraic calculations and Global Valuation Esther as simulation software. Using this super-computer and GPU technology the whole calculation takes a few minutes for building the models, followed by a nested simulation time in the order of about an hour for processing a billion scenarios on a real-life bank portfolio.

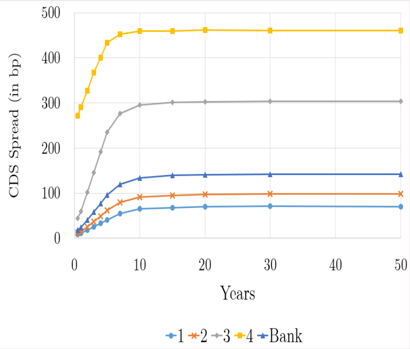
Toy Portfolio

We first consider a portfolio of ten USD currency fixed-income swaps on the date of 11 January 2016 (without initial margins, i.e. for IM $= 0$).

Toy portfolio of swaps (the nominal of each swap is \$$10^4$)

Mat.	Receiver Rate	Payer Rate	
10y	Par 6M	LIBOR 3M	
10y	LIBOR 3M	Par 6M	
5y	Par 6M	LIBOR 3M	
5y	LIBOR 3M	Par 6M	
30y	Par 6M	LIBOR 3M	
30y	LIBOR 3M	Par 6M	
2y	Par 6M	LIBOR 3M	
2y	LIBOR 3M	Par 6M	
15y	Par 6M	LIBOR 3M	1
15y	LIBOR 3M	Par 6M	4

Credit curves of the bank and its four conterparties



Introducing financial contracts one after the other in one or the reverse order in a portfolio at time 0 results in the same aggregated incremental FTP amounts for the bank, equal to the "time 0 portfolio FTP", but in different FTPs for each given contract and counterparty.

Toy portfolio. *Left*: XVA values and standard relative errors (SE). *Right*: Respective impacts when Swaps 5 and 9 are added last in the portfolio.

	\$Value	SE
UCVA_0	471.23	0.46%
$\mathrm{FVA}_0^{(0)}$	73.87	1.06%
FVA_0	3.87	4.3%
KVA_0	668.83	N/A
FTDCVA_0	372.22	0.46%
FTDDVA_0	335.94	0.51%

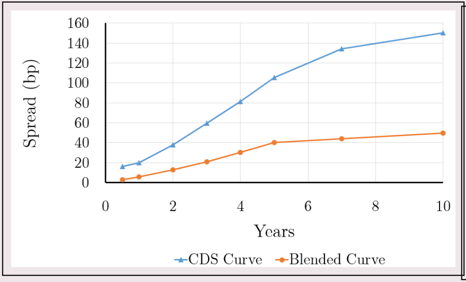
	Swap 5	Swap 9
$\Delta\mathrm{UCVA}_0$	155.46	-27.17
$\Delta\mathrm{FVA}_0^{(0)}$	-85.28	-8.81
$\Delta\mathrm{FVA}_0$	-80.13	-5.80
$\Delta\mathrm{KVA}_0$	127.54	-52.85
$\Delta\mathrm{FTDCVA}_0$	98.49	-23.83
$\Delta\mathrm{FTDDVA}_0$	122.91	-80.13

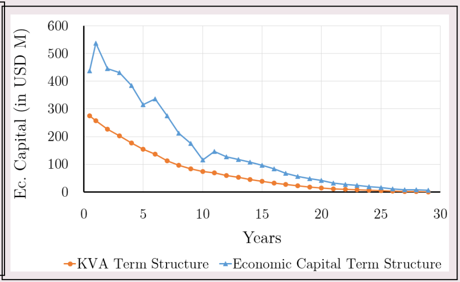
Representative Portfolio

We now consider a representative portfolio with about 2,000 counterparties, 100,000 fixed income trades including swaps, swaptions, FX options, inflation swaps and CDS trades (IM $= 0$).

XVA	\$Value
UCVA_0	242 M
$\mathrm{FVA}_0^{(0)}$	126 M
FVA_0	62 M
KVA_0	275 M
FTDCVA	194 M
FTDDVA	166 M

Left: FVA blended funding curve computed from the ground up based on capital projections.
Right: Term structure of economic capital compared with the term structure of KVA.





Return to FVA

Return to KVA

Return to SCR

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