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Part I : Enlargement of filtration in discrete time

The general problem of enlargement of filtration is the following one : let X be an $\mathbb{F}$-martingale and $\mathbb{G}$ a filtration larger that $\mathbb{F}$. Find conditions such that X is a $\mathbb{G}$ semimartingale and then, give the $\mathbb{G}$ semimartingale decomposition of the process X, i.e. write X as the sum of a $\mathbb{G}$ martingale and a predictable bounded variation process.

Some results are known from the 70's in continuous time, however, in that setting the proofs are not trivial, and one needs to assume specific hypotheses to give a positive answer. Furthermore, many cases are still not solved.

This is important in finance to exclude arbitrages (for example while studying insider trading) and to study the impact of new information when solving an optimization problem on consumption/ terminal wealth.

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Many references and books are available in continuous time (Jeulin, Jacod, Mansuy and Yor, Yor, Protter)

Recent theses with applications to finance: Amendinger, Ankirchner, Aksamit, Deng, Kreher, Falafala. See also papers by these authors and by Acciaio et al., Coculescu et al., Herdegen and Hermann, Kchia and Protter (all in continuous time).

Many results in a discrete time setting can be found in Deng's thesis and the related paper *Tahir Choulli and Jun Deng : Non-arbitrage for Informational Discrete Time Market Models*.

This presentation, where we study enlargement of filtration in discrete time is based on work in progress with Ankirchner, Blanchet-Scaillet and part of the thesis of Romo-Romero.

Our goal is to compute more explicitly the semimartingale decomposition, and to show, with elementary computation, that we recover the classical general formula established in the literature in continuous time.

The interest is mainly from a pedagogical point of view. We shall also present some results in a credit risk framework.

We are working in a discrete time setting: $X = (X_n, n \geq 0)$ is a process and $\mathbb{H} = (\mathcal{H}_n, n \geq 0)$ is a filtration, i.e., a family of σ-algebra such that $\mathcal{H}_n \subset \mathcal{H}_{n+1} \subset \mathcal{G}$. We note $\Delta X_n := X_n - X_{n-1}, n \geq 1$ the increment of X at time n and we set $\Delta X_0 = X_0$.
A process X is $\mathbb{H}$-**adapted** if, for any $n \geq 1$, the random variable X_n is $\mathcal{H}_n$-measurable.
A process X is $\mathbb{H}$-**predictable** if, for any $n \geq 1$, the random variable X_n is $\mathcal{H}_{n-1}$-measurable and X_0 is a constant.
A process X is **integrable** if $E(|X_n|) < \infty$ (resp. $\mathbb{E}(X_n^2) < \infty$) for all $n \geq 0$.
The process X_- is defined as the process equal to X_{n-1} at time n and to 0 for $n = 0$, this process is predictable.

A random variable ζ is said to be **positive** if $\zeta > 0$ a.s., a process X is positive if the r.v. X_n is positive for any $n \geq 0$ and a process A is **increasing** (resp. decreasing) if $A_n \geq A_{n-1}$ (resp. $A_n \leq A_{n-1}$) a.s. , for all $n \geq 1$.

An integrable $\mathbb{H}$-adapted process X is an $\mathbb{H}$-martingale (resp. an $\mathbb{H}$-supermartingale) if $\mathbb{E}(X_n|\mathcal{H}_{n-1}) = X_{n-1}$, or equivalently $\mathbb{E}(\Delta X_n|\mathcal{H}_{n-1}) = 0$ (resp. $\mathbb{E}(X_n|\mathcal{H}_{n-1}) \leq X_{n-1}$).

Basic Facts

Set of martingales

(a) The set of processes of the form $\left(\psi_0 + \sum_{k=1}^{n} \left(\psi_k - \mathbb{E}(\psi_k|\mathcal{H}_{k-1})\right), \, n \geq 0\right)$ where ψ is an $\mathbb{H}$-adapted integrable process is equal to the set of all $\mathbb{H}$-martingales (here, $\sum_{k=1}^{0} \centerdot = 0$)

(b) The set of processes of the form $\left(\psi_0 \prod_{k=1}^{n} \frac{\psi_k}{\mathbb{E}(\psi_k|\mathcal{H}_{k-1})}, n \geq 0\right)$ where ψ is a positive integrable $\mathbb{H}$-adapted process is the set of all positive $\mathbb{H}$-martingales (here, $\prod_{k=1}^{0} \centerdot = 1$).

Basic Facts

Doob's decomposition: Any discrete time process is a semimartingale in any filtration for which it is adapted: $X = M^{\mathbb{F}} + V^{\mathbb{F}}$ where $M^{\mathbb{F}}$ is an $\mathbb{F}$-martingale and $V^{\mathbb{F}}$ is $\mathbb{F}$-predictable, $V_0^{\mathbb{F}} = 0$ and

$$\Delta V_n^{\mathbb{F}} := V_n^{\mathbb{F}} - V_{n-1}^{\mathbb{F}} = \mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}(\Delta X_n|\mathcal{F}_{n-1})\,.$$

Proof: Setting $V^{\mathbb{F}}$ as above, it remains to check that $M^{\mathbb{F}}$ is a martingale. Note that

$$\Delta M_n^{\mathbb{F}} := M_n^{\mathbb{F}} - M_{n-1}^{\mathbb{F}} = X_n - \mathbb{E}(X_n|\mathcal{F}_{n-1})\,.$$

If X is an $\mathbb{F}$-martingale, and $\mathbb{G}$ any filtration such that $\mathbb{F} \subset \mathbb{G}$, it is a $\mathbb{G}$-semimartingale with decomposition $X = M^{\mathbb{G}} + V^{\mathbb{G}}$ where $M^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $V^{\mathbb{G}}$ is $\mathbb{G}$-predictable, and

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Predictable bracket of two martingales

If X and Y are two $\mathbb{F}$-martingales, there exists an $\mathbb{F}$-predictable process K such that $XY - K$ is a martingale and

$$\Delta K_n = \mathbb{E}(Y_n \Delta X_n | \mathcal{F}_{n-1}) = \mathbb{E}(\Delta Y_n \Delta X_n | \mathcal{F}_{n-1})$$

We denote $\langle X, Y \rangle_n := K_n$.

Indeed, from Doob's decomposition the predictable part of the semimartingale XY is $\Delta K_n = \mathbb{E}(X_n Y_n - X_{n-1} Y_{n-1} | \mathcal{F}_{n-1}$.

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The covariation process of two processes $[X, Y]$ is defined by $\Delta [X, Y]_n = \Delta Y_n \Delta X_n$.

Predictable bracket of two semi martingales

The predictable bracket of two semimartingales X, Y is defined as the dual predictable projection of the covariation process $\langle X, Y\rangle = [X, Y]^p$.

For discrete time semimartingales, $[X, Y]_n = \sum_{k=1}^n \Delta X_k \, \Delta Y_k$, and $[X, Y]^p$ is the only predictable (bounded variation) process such that $[X, Y] - [X, Y]^p$ is a martingale, i.e. $[X, Y]^p$ is the predictable part of $[X, Y]$.

From Doob's decomposition

$$(\Delta[X, Y]^p)_n = \mathbb{E}([X, Y]_n - [X, Y]_{n-1} | \mathcal{F}_{n-1}) = \mathbb{E}(\Delta X_n \, \Delta Y_n | \mathcal{F}_{n-1})$$

Then,

$$\Delta\langle X, Y\rangle_n^{\mathbb{F}} = \mathbb{E}(\Delta X_n \, \Delta Y_n | \mathcal{F}_{n-1}) \, .$$

Stochastic integral

The **stochastic integral** of a process Y w.r.t. a process X is the process $Y \centerdot X$ defined as

$$(Y \centerdot X)_n := \sum_{k=1}^{n} Y_k \Delta X_k, \; n \geq 0.$$

For two processes X and Y

$$XY = X_0 Y_0 + X_- \centerdot Y + Y_- \centerdot X + [X,Y] = X_0 Y_0 + X_- \centerdot Y + Y \centerdot X\,.$$

This equality is based on

$$\Delta(XY)_n = X_{n-1}\Delta Y_n + Y_{n-1}\Delta X_n + \Delta X_n \, \Delta Y_n = X_{n-1}\Delta Y_n + Y_n \Delta X_n.$$

If X is a square integrable $\mathbb{H}$-martingale and H an $\mathbb{H}$-predictable square integrable process, then the process $H \centerdot X$ is an $\mathbb{H}$-martingale.

Proof: For H predictable,

$$\mathbb{E}(H_n \Delta X_n | \mathcal{H}_{n-1}) = H_n \mathbb{E}(\Delta X_n | \mathcal{H}_{n-1}) = \mathbb{E}(\Delta M_n^H \Delta X_n | \mathcal{H}_{n-1}) = 0$$

and the result is obvious.

If X and Y are two square integrable $\mathbb{H}$-martingales then $XY - [X, Y]$ is an $\mathbb{H}$-martingale.

Proof: This is a direct consequence of integration by parts formula and the fact that X_- and Y_- are predictable.

Two square integrable martingales X and Y are said to be orthogonal if their product is a martingale, i.e. if $\mathbb{E}(\Delta(XY)_n|\mathcal{H}_{n-1}) = 0$.

This condition is equivalent to any of the following assertions

(a) $\mathbb{E}(\Delta Y_n \Delta X_n|\mathcal{H}_{n-1}) = 0$

(b) $\mathbb{E}(Y_n \Delta X_n|\mathcal{H}_{n-1}) = 0$

(c) $[X, Y]$ is a martingale

(d) $\langle X, Y\rangle = 0$.

Proof: From integration by parts formula, the orthogonality is equivalent to $[X, Y]$ is a martingale, which is equivalent to the two other conditions, due to $\Delta(XY)_n = X_{n-1}\Delta Y_n + Y_n \Delta X_n$, and the fact that $X_- \centerdot Y$ and $Y_- \centerdot X$ are martingales.

Enlargement of filtrations

There are, in continuous time, mainly two kinds of enlargement

Initial enlargement, where L is a random variable and

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(L)\,.$$

Progressive enlargement, where τ is a positive random variable and

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(t \wedge \tau)\,.$$

Initial Enlargement

Initial enlargement: an example (Bridge)

Let $X_n = \sum_{k=1}^{n} Y_k$, where $(Y_k, k \geq 1)$ are i.i.d. centered, be a martingale (an $\mathbb{F}^X$-martingale) and let $\mathcal{G}_n := \mathcal{F}_n^X \vee \sigma(X_N)$, for $n \leq N$.

We need to compute $\Delta A_n = \mathbb{E}(\Delta X_n | \mathcal{F}_{n-1} \vee \sigma(X_N))$. Using the fact that $(Y_k, k \leq N)$ are i.i.d, we have that for any $j \geq n$

$$(Y_j, X_1, \cdots, X_{n-1}, X_N) \overset{\text{loi}}{=} (Y_n, X_1, \cdots, X_{n-1}, X_N)$$

hence

$$\begin{aligned}
\mathbb{E}(Y_n | \mathcal{F}_{n-1} \vee \sigma(X_N)) &= \mathbb{E}(Y_j | \mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{1}{N-(n-1)} \mathbb{E}(Y_n + \cdots + Y_j + \cdots + Y_N | \mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{1}{N-(n-1)} \mathbb{E}(X_N - X_{n-1} | \mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{X_N - X_{n-1}}{N-(n-1)}
\end{aligned}$$

Hence,

$$X_n - \sum_{k=1}^{n} \frac{X_N - X_{k-1}}{N - (k-1)}$$

is a $\mathbb{G}$-martingale.

In continuous time: Brownian bridge. For $\mathbb{G} = \mathbb{F} \vee \sigma(B_1)$,

$$B_t = B_t^{\mathbb{G}} + \int_0^t \frac{B_1 - B_s}{1 - s} ds, \; t \leq 1$$

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Initial enlargement: another example

Let X be a martingale, L be a r.v. taking values in $\mathbb{Z}$ and $p_n(j) = \mathbb{P}(L = j|\mathcal{F}_n^X)$. Define $\mathcal{G}_n = \mathcal{F}_n^X \vee \sigma(L)$.

Then $\Delta V_n^{\mathbb{G}} = \mathbb{E}(\Delta X_n|\mathcal{F}_n \vee \sigma(L))$ and

$$\begin{aligned}
\Delta V_n^{\mathbb{G}} \mathbb{1}_{\{L=j\}} &= \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(\Delta X_n \mathbb{1}_{\{L=j\}}|\mathcal{F}_{n-1})}{\mathbb{P}(L = j|\mathcal{F}_{n-1})} \\
&= \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(p_n(j)\Delta X_n|\mathcal{F}_n)}{p_{n-1}(j)} = \mathbb{1}_{\{L=j\}} \frac{\mathbb{E}(\Delta\langle X, p(j)\rangle_n|\mathcal{F}_{n-1})}{p_{n-1}(j)} .
\end{aligned}$$

On the set $\{L = j\}$, one has $p_n(j) \neq 0, \forall n \geq 0$. Indeed,

$$\mathbb{E}(\mathbb{1}_{\{p_n(j)=0\}} \mathbb{1}_{\{L=j\}}) = \mathbb{E}(\mathbb{1}_{\{p_n(j)=0\}} \mathbb{E}(\mathbb{1}_{\{L=j\}}|\mathcal{F}_n)) = \mathbb{E}(\mathbb{1}_{\{p_n(j)=0\}} p_n(j)) = 0 .$$

Initial enlargement: another example

Let X be an $\mathbb{F}$-martingale, L be a r.v. taking values in $\mathbb{Z}$ and $p_n(j) = \mathbb{P}(L = j|\mathcal{F}_n^X)$ and let $\mathcal{G}_n = \mathcal{F}_n^X \vee \sigma(L)$.

Then $X = X^{\mathbb{G}} + V^{\mathbb{G}}$ where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $\Delta V_n^{\mathbb{G}} = \mathbb{E}(\Delta X_n|\mathcal{F}_n \vee \sigma(L))$ and

$$\begin{aligned}
\Delta V_n^{\mathbb{G}} 1\!\!1_{\{L=j\}} &= 1\!\!1_{\{L=j\}} \frac{\mathbb{E}(\Delta X_n 1\!\!1_{\{L=j\}}|\mathcal{F}_{n-1})}{\mathbb{P}(L = j|\mathcal{F}_{n-1})} \\
&= 1\!\!1_{\{L=j\}} \frac{\mathbb{E}(p_n(j)\Delta X_n|\mathcal{F}_n)}{p_{n-1}(j)} = 1\!\!1_{\{L=j\}} \frac{\mathbb{E}(\Delta\langle X, p(j)\rangle_n|\mathcal{F}_{n-1})}{p_{n-1}(j)}\,.
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On the set $\{L = j\}$, one has $p_n(j) \neq 0, \forall n \geq 0$. Indeed,

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$$X_n = X_n^{\mathbb{G}} + \sum_{k=1}^{n} \frac{\mathbb{E}(\Delta\langle X, p(j)\rangle_n | \mathcal{F}_{n-1})|_{j=L}}{p_{n-1}(L)}$$

In continuous time, under Jacod's hypothesis $\mathbb{P}(L \in du|\mathcal{F}_t) = p_t(u)\mathbb{P}(L \in du)$

$$X_t = X_t^{\mathbb{G}} + \int_0^t \frac{d\langle X, p(u)\rangle_s|_{u=L}}{p_{s-}(L)}$$

Arbitrages

Two simplifications in discrete time are that any non negative local martingale is a martingale and that all kind of arbitrages are equivalent.

If $p(k) > 0$ for any k, then, there exists a $\mathbb{G}$-martingale η such that, for any $\mathbb{F}$ martingale S, $S\eta$ is a $\mathbb{F}^{\sigma(L)}$-local martingale.

Proof: Set $\eta = 1/p(L)$. For an $\mathbb{F}$-martingale X, if $p(k) > 0$, the one has

$$\begin{aligned}\mathbb{E}(\frac{X_n}{p_n(L)}|\mathcal{F}_{n-1} \vee \sigma(L)) &= \sum_{k=-\infty}^{\infty} 1\!\!1_{L=k} \frac{\mathbb{E}(1\!\!1_{\{L=k\}} \frac{X_n}{p_n(k)} |\mathcal{F}_{n-1})}{\mathbb{E}(1\!\!1_{\{L=k\}}|\mathcal{F}_{n-1})} \\ &= \sum_{k=-\infty}^{\infty} 1\!\!1_{\{L=k\}} \frac{\mathbb{E}(X_n|\mathcal{F}_{n-1})}{p_{n-1}(k)} = \frac{X_{n-1}}{p_{n-1}(L)} \,.\end{aligned}$$

A necessary and sufficient condition can be found in Choulli-Deng :

For any $\mathbb{Z}$-valued random variable L, the following are equivalent.

(a) The set $\{p_n(k) = 0 < p_{n-1}^k\}$ is negligible, for all k and n.

(b) For any $\mathbb{F}$-adapted integrable process X satisfying NA($\mathbb{F}$), X satisfies NA($\mathbb{F}^{\sigma()}$.

Progressive Enlargement

We assume here that τ is a random variable valued in $\mathbb{N} \cup \{+\infty\}$, and introduce $\mathcal{G}_n := \mathcal{F}_n \vee \sigma(\tau \wedge n)$.

If $Y_n \in \mathcal{G}_n$, then there exists $y_n \in \mathcal{F}_n$ such that $Y_n 1\!\!1_{\{n<\tau\}} = y_n 1\!\!1_{\{n<\tau\}}$. Any $\mathbb{G}$-predictable process can be written as

$$V_n = V_n^b 1\!\!1_{\{n \leq \tau\}} + V_n^a(\tau) 1\!\!1_{\{\tau<n\}}$$

where, V^b is $\mathbb{F}$ predictable and for any u, $V^a(u)$ is $\mathbb{F}$-predictable.

We introduce two supermartingales

$$Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n), \quad \widetilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_n)$$

and the Doob-Meyer decomposition of $\widetilde{Z} = \widetilde{M} - \widetilde{A}$ where M is an $\mathbb{F}$-martingale and $\widetilde{A}$ an $\mathbb{F}$-predictable increasing process.

We shall use the trivial equalities $\widetilde{Z}_n = \mathbb{P}(\tau > n-1|\mathcal{F}_n), Z_n = \mathbb{P}(\tau \geq n+1|\mathcal{F}_n)$.

On the set $\{n \leq \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are (strictly) positive.

The proof follows from simple arguments

$$\begin{aligned}\mathbb{E}(1\!\!1_{\{n\leq\tau\}}1\!\!1_{\{Z_{n-1}=0\}}) &= \mathbb{E}(\mathbb{P}(n\leq\tau|\mathcal{F}_{n-1})1\!\!1_{\{Z_{n-1}=0\}}) \\ &= \mathbb{E}(\mathbb{P}(n-1<\tau|\mathcal{F}_{n-1})1\!\!1_{\{Z_{n-1}=0\}}) = \mathbb{E}(Z_{n-1}1\!\!1_{\{Z_{n-1}=0\}}) = 0\end{aligned}$$

On the set $\{n > \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are strictly smaller than 1.

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&= \mathbb{E}(\mathbb{P}(n-1<\tau|\mathcal{F}_{n-1})1\!\!1_{\{Z_{n-1}=0\}})\mathbb{E}(Z_{n-1}1\!\!1_{\{Z_{n-1}=0\}}) = 0
\end{aligned}$$

On the set $\{n \leq \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are (strictly) positive. On the set $\{n > \tau\}$, $\widetilde{Z}_n$ and Z_{n-1} are strictly smaller than 1.

For any random time τ, if Y is integrable

$$\mathbb{E}(Y|\mathcal{G}_n)\mathbb{1}_{\{n<\tau\}} = \mathbb{1}_{\{n<\tau\}}\frac{\mathbb{E}(Y\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n)}{Z_n}$$

Indeed, on $\{n < \tau\}$, any $\mathcal{G}_n$ measurable random variable Y_n is equal to an $\mathcal{F}_n$ measurable random variable so that, for $Y_n\mathbb{1}_{\{n<\tau\}} = \mathbb{1}_{\{n<\tau\}}y_n$ which leads to

$$\mathbb{E}(Y_n\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n) = \mathbb{E}(\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n))y_n$$

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$$\mathbb{E}(Y_n\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n) = \mathbb{E}(\mathbb{1}_{\{n<\tau\}}|\mathcal{F}_n)y_n$$

For $Y_n \in \mathcal{F}_n$

$$\begin{aligned}
\mathbb{E}(Y_n \mathbb{1}_{\{\tau \geq n\}} | \mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \widetilde{Z}_n | \mathcal{F}_{n-1}) \\
\mathbb{E}(\frac{Y_n}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} | \mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \mathbb{1}_{\{\widetilde{Z}_n > 0\}} | \mathcal{F}_{n-1}) \,.
\end{aligned}$$

Only the second equality requires a proof

$$\begin{aligned}
\mathbb{E}(\frac{Y_n}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} | \mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \frac{1}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} | \mathcal{F}_{n-1}) \\
&= \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \frac{1}{\widetilde{Z}_n} \mathbb{1}_{\{\tau \geq n\}} \mathbb{1} \widetilde{Z}_n > 0 | \mathcal{F}_{n-1}) = \mathbb{1}_{\{\tau \geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n \mathbb{1}_{\{\widetilde{Z}_n
\end{aligned}$$

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$$\begin{aligned}
\mathbb{E}(Y_n|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau\geq n\}} &= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\widetilde{Z}_n|\mathcal{F}_{n-1}) \\
\mathbb{E}(\frac{Y_n}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\mathbb{1}_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1})\,.
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\mathbb{E}(\frac{Y_n}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\frac{1}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_{n-1}) \\
&= \frac{1}{Z_{n-1}}\mathbb{E}(Y_n\frac{1}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}\mathbb{1}_{\widetilde{Z}_n>0}|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\mathbb{1}_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1})\,.
\end{aligned}$$

Arbitrages

If $Z > 0$, there are no arbitrages before τ.

We prove that if S is an $\mathbb{F}$ martingale, then, there exists a positive $\mathbb{G}$-martingale such that $S^\tau L$ is a local martingale.

The process

$$L_n = \prod_{k=1}^{n}(1+\Delta N_k) = L_{n-1}(1+\Delta N_n)$$

where $\Delta N_k = 1\!\!1_{\tau \geq k}(\frac{Z_{k-1}}{\widetilde{Z}_k} - 1)$, is a positive $\mathbb{G}$-martingale.

Proof : Indeed,

$$\begin{aligned}
\mathbb{E}(1+\Delta N_n|\mathcal{G}_{n-1}) &= 1+\mathbb{E}(1\!\!1_{\tau\geq n}(\frac{Z_{n-1}}{\widetilde{Z}_n}-1)|\mathcal{G}_{n-1}) \\
&= 1+1\!\!1_{\tau>n-1}\frac{1}{Z_{n-1}}\mathbb{E}(1\!\!1_{\tau\geq n}(\frac{Z_{n-1}}{\widetilde{Z}_n}-1)|\mathcal{F}_{n-1}) \\
&= 1+1\!\!1_{\tau>n-1}\frac{1}{Z_{n-1}}\left(\mathbb{E}(\widetilde{Z}_n\frac{Z_{n-1}}{\widetilde{Z}_n}|\mathcal{F}_{n-1}) - Z_{n-1}\right) = 1
\end{aligned}$$

The process $S^\tau L$ is a $(\mathbb{G}, \mathbb{P})$ martingale.

Indeed

$$\begin{aligned}
&\mathbb{E}(S_{(n+1)\wedge\tau}(1 + \mathbb{1}_{\tau\geq n+1}(\frac{Z_n}{\widetilde{Z}_{n+1}} - 1))|\mathcal{G}_n) \\
&= \mathbb{E}(S_{n+1}\mathbb{1}_{\tau\geq n+1}(1 + \frac{Z_n}{\widetilde{Z}_{n+1}} - 1)|\mathcal{G}_n) + \mathbb{E}(S_\tau \mathbb{1}_{\tau<n+1}|\mathcal{G}_n) \\
&= \mathbb{E}(S_{n+1}\mathbb{1}_{\tau\geq n+1}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{G}_n) + S_\tau \mathbb{1}_{\tau<n+1} \\
&= \mathbb{1}_{\tau>n}\frac{1}{Z_n}\mathbb{E}(S_{n+1}\widetilde{Z}_{n+1}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{F}_n) + S_\tau \mathbb{1}_{\tau\leq n} = S_{n\wedge\tau}
\end{aligned}$$

A necessary and sufficient condition can be found in Choulli-Deng : for any S satisfying NA($\mathbb{F}$), S satisfies NA($\mathbb{G}$) is equivalent to $\{\tilde{Z}_n = 0\} = \{Z_{n-1} = 0\}$

Immersion in progressive enlargement

$\mathbb{F}$ is immersed in $\mathbb{G}$ iff any $\mathbb{F}$ martingale is a $\mathbb{G}$-martingale, or equivalently if $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_\infty) = \mathbb{P}(\tau > n|\mathcal{F}_k)$ for any $k \geq n$.

This is the case in the Cox Model.

$\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_\infty)$.

Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$. Then,

$$\begin{aligned} \widetilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_n) &=: \mathbb{P}(\tau > n-1 | \mathcal{F}_n) = \mathbb{P}(\tau > n-1 | \mathcal{F}_{n-1}) = \mathbb{P}(\tau > n-1 | \mathcal{F}_\infty) \\ &= \mathbb{P}(\tau \geq n | \mathcal{F}_\infty) \end{aligned}$$

where the third and the next to last equality follow from immersion assumption. The third equality establishes the predictability of $\widetilde{Z}$. Note that one has $\widetilde{Z}_n = Z_{n-1}$.

Assume now that $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_\infty)$. Then, $\tilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_{n-1})$ and

$$\begin{aligned} \mathbb{P}(\tau > n | \mathcal{F}_n) &= \mathbb{P}(\tau \geq n+1 | \mathcal{F}_n) = \widetilde{Z}_{n+1} \\ &= \mathbb{P}(\tau > n | \mathcal{F}_\infty) \end{aligned}$$

$\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)$.

Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$. Then,

$$\begin{aligned}\widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n) &=: \mathbb{P}(\tau > n-1|\mathcal{F}_n) = \mathbb{P}(\tau > n-1|\mathcal{F}_{n-1}) = \mathbb{P}(\tau > n-1|\mathcal{F}_\infty) \\ &= \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)\end{aligned}$$

where the third and the next to last equality follow from immersion assumption. The third equality establishes the predictability of $\widetilde{Z}$. Note that one has $\widetilde{Z}_n = Z_{n-1}$.

Assume now that $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)$. Then, $\tilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_{n-1})$ and

$$\begin{aligned}\mathbb{P}(\tau > n|\mathcal{F}_n) &= \mathbb{P}(\tau \geq n+1|\mathcal{F}_n) = \widetilde{Z}_{n+1} \\ &= \mathbb{P}(\tau > n|\mathcal{F}_\infty)\end{aligned}$$

$\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_\infty)$.

Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$. Then,

$$\begin{aligned}\widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n) \quad &=: \quad \mathbb{P}(\tau > n-1|\mathcal{F}_n) = \mathbb{P}(\tau > n-1|\mathcal{F}_{n-1}) = \mathbb{P}(\tau > n-1|\mathcal{F}_\infty) \\ &= \quad \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)\end{aligned}$$

where the third and the next to last equality follow from immersion assumption. The third equality establishes the predictability of $\widetilde{Z}$. Note that one has $\widetilde{Z}_n = Z_{n-1}$.

Assume now that $\widetilde{Z}$ is predictable and $\tilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)$. Then, $\tilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_{n-1})$ and

$$\begin{aligned}\mathbb{P}(\tau > n|\mathcal{F}_n) \quad &= \quad \mathbb{P}(\tau \geq n+1|\mathcal{F}_n) = \widetilde{Z}_{n+1} \\ &= \quad \mathbb{P}(\tau > n|\mathcal{F}_\infty)\end{aligned}$$

Semi martingale decomposition, Before τ

Any $\mathbb{F}$-martingale X stopped at τ is a $\mathbb{G}$-semimartingale with decomposition

$$X^\tau = X^{\mathbb{G}} + \sum_{k=0}^{\cdot\wedge\tau} \frac{1}{Z_{k-1}} \Delta \langle \widetilde{M}, X \rangle_k$$

where $\widetilde{Z} = \widetilde{M} - \widetilde{A}$.

Proof: The $\mathbb{G}$-predictable part of the $\mathbb{G}$-semimartingale X is $V^{\mathbb{G}}$ with $\Delta V_n^{\mathbb{G}} = \mathbb{E}(X_n - X_{n-1}|\mathcal{G}_{n-1})$. We apply previous results on the set $n-1 < \tau$,

$$\begin{aligned}
\mathbb{1}_{\{\tau>n-1\}}(V_n^{\mathbb{G}} - V_{n-1}^{\mathbb{G}}) &= \mathbb{1}_{\{\tau>n-1\}}\mathbb{E}(X_n - X_{n-1}|\mathcal{G}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{\{\tau>n-1\}}(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>n-1\}}|\mathcal{F}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\widetilde{Z}_n(X_n - X_{n-1})|\mathcal{F}_{n-1})\,.
\end{aligned}$$

Using now the Doob-Meyer decomposition of $\widetilde{Z}$, and the martingale property of X, we obtain

$$\begin{aligned}\mathbb{E}(\widetilde{Z}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) &= \mathbb{E}((\widetilde{M}_n - \widetilde{A}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\ &= \mathbb{E}(\widetilde{M}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) = \Delta\langle \widetilde{M}, X\rangle_n\end{aligned}$$

and finally

$$1\!\!1_{\{\tau>n-1\}}(V_n^{\mathbb{G}} - V_{n-1}^{\mathbb{G}}) = 1\!\!1_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\Delta\langle \widetilde{M}, X\rangle_n\,.$$

In continuous time

$$X^\tau = X^{\mathbb{G}} + \int_0^{\cdot\wedge\tau} \frac{1}{Z_{s-}} d\langle X, \widetilde{M}\rangle$$

where $\widetilde{M} = \widetilde{Z} - A^0_-$, and A^0 is the dual optional projection of $1\!\!1_{\tau\leq t}$.

Using now the Doob-Meyer decomposition of $\widetilde{Z}$, and the martingale property of X, we obtain

$$\begin{aligned}\mathbb{E}(\widetilde{Z}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) &= \mathbb{E}((\widetilde{M}_n - \widetilde{A}_n)(X_n - X_{n-1})|\mathcal{F}_{n-1}) \\ &= \mathbb{E}(\widetilde{M}_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) = \Delta\langle \widetilde{M}, X\rangle_n\end{aligned}$$

and finally

$$1\!\!1_{\{\tau > n-1\}}(A_n^{\mathbb{G}} - A_{n-1}^{\mathbb{G}}) = 1\!\!1_{\{\tau > n-1\}} \frac{1}{Z_{n-1}} \Delta\langle \widetilde{M}, X\rangle_n \,.$$

In continuous time

$$X^\tau = X^{\mathbb{G}} + \int_0^{\cdot\wedge\tau} \frac{1}{Z_{s-}} d\langle X, \tilde{M}\rangle_s$$

where $\widetilde{M} = \widetilde{Z} - A_-^0$, *with* A^0 *being the* $\mathbb{F}$*-dual optional projection of* $1\!\!1_{\tau \leq t}$ *and* $\widetilde{Z}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t)$.

After τ, Honest times

We now consider the case where τ is honest (and valued in $\mathbb{N}$). We recall the definition and some of the main properties.

A random time is honest, if, for any $n \in \mathbb{N}$, there exists an $\mathcal{F}_n$-measurable random variable $\tau(n)$ such that

$$1\!\!1_{\{\tau \le n\}}\tau = 1\!\!1_{\{\tau \le n\}}\tau(n)$$

or equivalently if there exists $\widehat{\tau}(n)$ such that

$$1\!\!1_{\{\tau < n\}}\tau = 1\!\!1_{\{\tau < n\}}\widehat{\tau}(n)$$

It follows that any $\mathbb{G}$-predictable process V can be written as $V_n = V_n^b 1\!\!1_{\{n \le \tau\}} + V_n^a 1\!\!1_{\{\tau < n\}}$ where V^a, V^b are $\mathbb{F}$-predictable processes.

If τ is honest, $Z_n = \tilde{Z}_n$ on the set $n > \tau$. Furthermore, τ is honest if and only if $\widetilde{Z}_\tau = 1$

For any n,

$$\begin{aligned}\mathbb{P}(\tau = n|\mathcal{F}_n)1\!\!1_{\{n>\tau\}} &= \mathbb{P}(\tau = n|\mathcal{F}_n)1\!\!1_{\{n>\tau;n>\tau(n)\}} = \mathbb{E}(1\!\!1_{\{\tau=n\}}1\!\!1_{\{n>\tau(n)\}}|\mathcal{F}_n)1\!\!1_{\{n>\tau\}} \\ &= \mathbb{E}(1\!\!1_{\{\tau=n\}}1\!\!1_{\{n>\tau(n)\}}1\!\!1_{\{n>\tau\}}|\mathcal{F}_n)1\!\!1_{\{n>\tau\}} = 0\end{aligned}$$

It follows that $Z_n 1\!\!1_{\{\tau<n\}} = \widetilde{Z}_n 1\!\!1_{\{\tau<n\}}$.

Furthermore,

$$\begin{aligned}\widetilde{Z}_n 1\!\!1_{\{\tau=n\}} &= 1\!\!1_{\{\tau=n\}}\mathbb{P}(\tau \geq n|\mathcal{F}_n) = 1\!\!1_{\{\tau=n\}}1\!\!1_{\{\tau(n)=n\}}\mathbb{P}(\tau \geq n|\mathcal{F}_n) \\ &= 1\!\!1_{\{\tau=n\}}\mathbb{E}(1\!\!1_{\{\tau(n)=n\}}1\!\!1_{\{\tau\geq n\}}|\mathcal{F}_n) = 1\!\!1_{\{\tau=n\}}\end{aligned}$$

which implies $\widetilde{Z}_\tau = 1$.

Let $\ell(n) = \sup\{k \leq n : \tilde{Z}_k = 1\}$. Then, on $\tau \leq n$ one has $\tau = \ell(n)$.

Decomposition in the enlarged filtration.

Let X be an $\mathbb{F}$-martingale. Then,

$$X = \widehat{X} + \sum_{k=0}^{\cdot\wedge\tau} \frac{1}{Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k - \sum_{k=\tau}^{\cdot} \frac{1}{1-Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k$$

where $\widehat{X}$ is a $\mathbb{G}$-martingale.

One has

$$1\!\!1_{\tau\leq n}(V^{\mathbb{G}}_{n+1} - V^{\mathbb{G}}_n) \quad = \quad \mathbb{E}(1\!\!1_{\tau\leq n}(X_{n+1} - X_n)|\mathcal{G}_n)\,.$$

We now take the conditional expectation w.r.t. $\mathcal{F}_n$. From the property of honest times, there exists $V^{\mathbb{F}}$, an $\mathbb{F}$-predictable process, such that $V^{\mathbb{G}}_n 1\!\!1_{\tau\leq n} = V^{\mathbb{F}}_n 1\!\!1_{\tau\leq n}$. Taking into account that $V^{\mathbb{F}}$ is predictable, one has

$$\begin{aligned}
\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_n)(V^{\mathbb{G}}_{n+1} - V^{\mathbb{G}}_n) &= \mathbb{E}(1\!\!1_{\tau\leq n}(X_{n+1} - X_n)|\mathcal{F}_n) \\
&= \mathbb{E}(\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_{n+1})(X_{n+1} - X_n)|\mathcal{F}_n)
\end{aligned}$$

$$\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_n)(V^{\mathbb{F}}_{n+1} - V^{\mathbb{F}}_n) \quad = \quad \mathbb{E}(\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_{n+1})(X_{n+1} - X_n)|\mathcal{F}_n)$$

Now, using the fact that

$$\begin{aligned}
\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_n) \quad &= \quad 1 - \mathbb{E}(1\!\!1_{\tau>n}|\mathcal{F}_n) = 1 - Z_n \\
\mathbb{E}(1\!\!1_{\tau\leq n}|\mathcal{F}_{n+1}) \quad &= \quad 1 - \mathbb{E}(1\!\!1_{\tau>n}|\mathcal{F}_{n+1}) = 1 - \mathbb{E}(1\!\!1_{\tau\geq n+1}|\mathcal{F}_{n+1}) = 1 - \widetilde{Z}_{n+1}
\end{aligned}$$

and that X is an $\mathbb{F}$-martingale, we obtain

$$(1 - Z_n)(V^{\mathbb{G}}_{n+1} - V^{\mathbb{G}}_n) = -\mathbb{E}(\tilde{Z}_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n) = -\Delta\langle \widetilde{M}, X\rangle_n \,.$$

It seems important to note that the Doob-Meyer decomposition of Z is not needed.

Brackets in $\mathbb{F}$ and $\mathbb{G}$.

Let X and Y be $\mathbb{F}$ adapted processes (hence, semi martingales)

$$\Delta\langle X, Y\rangle_n^{\mathbb{G}} \mathbb{1}_{n \le \tau} = \mathbb{1}_{n \le \tau} \frac{1}{Z_{n-1}} \mathbb{E}(\widetilde{Z}_n \Delta X_n \Delta Y_n | \mathcal{F}_{n-1})$$

Let τ be an honest time. Then

$$\Delta\langle X, Y\rangle_n^{\mathbb{G}} \mathbb{1}_{\tau < n} = \mathbb{1}_{\tau < n} \frac{1}{1 - Z_{n-1}} \mathbb{E}((1 - \widetilde{Z}_n) \Delta X_n \Delta Y_n | \mathcal{F}_{n-1})$$

Enlargement with a process

For $n \geq 0$ let $U_n(dy)$ be a regular conditional distribution of the random vector $\widehat{Y}_{n-1} = (Y_0, \ldots, Y_{n-1})$ with respect to $\mathcal{F}_n$. Moreover, for $n \geq 1$ let $V_n(dy)$ be a regular conditional distribution of $\widehat{Y}_{n-1}$ with respect to $\mathcal{F}_{n-1}$.

Assume that $U_n(dy)$ is absolutely continuous wrt $V_n(dy)$ for all $n \geq 1$ and $d_n(y) := \frac{U_n(dy)}{V_n(dy)}$. Then, the information drift of X wrt to $(\mathcal{G}_n)$ is given by

$$A_n = \sum_{k=1}^{n} \Delta \langle X, d(z) \rangle \rangle_k \big|_{z=(Y_0,\ldots,Y_{k-1})}.$$

Credit Risk

Let $Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n)$ and $Z_n = M_n - A_n$ its $\mathbb{F}$-supermartingale decomposition. Let $H_n = 1\!\!1_{\{\tau \leq n\}}$.

The process $H_n - \Lambda_{n \wedge \tau}$ is a $\mathbb{G}$-martingale, where $\Delta \Lambda_n = -\frac{\Delta A_n}{Z_{n-1}}$.

Assume now that $\tau = \inf\{n : \Gamma_n \geq \Theta\}$ where Γ is increasing. Then $Z_n = e^{-\Gamma_n}$. If Γ is predictable, $Z_n = 1 - A_n = 1 - e^{-\Gamma_n}$, and $\Delta \Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta \Gamma_n}$.

If Γ is not predictable,

$$\Delta \Lambda_n = 1 - \mathbb{E}(e^{-\Delta \Gamma_n} | \mathcal{F}_{n-1})$$

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If Γ is not predictable,

$$\Delta \Lambda_n = e^{-\Gamma_{n-1}} \frac{1 - \mathbb{E}(e^{-\Delta \Gamma_n} | \mathcal{F}_{n-1})}{Z_{n-1}}$$

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If Γ is predictable, $Z_n = 1 - A_n = 1 - e^{-\Gamma_n}$, and $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta\Gamma_n}$

If Γ is not predictable,

$$\Delta\Lambda_n = e^{-\Gamma_{n-1}} \frac{1 - \mathbb{E}(e^{-\Delta\Gamma_n} | \mathcal{F}_{n-1})}{Z_{n-1}}$$

Credit Risk

Let $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_n)$ and $Z_n = M_n - A_n$ its $\mathbb{F}$-supermartingale decomposition. Let $H_n = 1\!\!1_{\{\tau \leq n\}}$.

The process $H_n - \Lambda_{n\wedge\tau}$ is a $\mathbb{G}$-martingale, where $\Delta\Lambda_n = -\frac{\Delta A_n}{Z_{n-1}}$.

Case of Cox Model: $\tau = \inf\{n :\ \Gamma_n \geq \Theta\}$ where Γ is increasing. Then $Z_n = e^{-\Gamma_n}$.

If Γ is predictable, $Z_n = 1 - A_n = 1 - e^{-\Gamma_n}$, and $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta\Gamma_n}$

If Γ is not predictable,

$$\Delta\Lambda_n = 1 - \mathbb{E}(e^{-\Delta\Gamma_n}|\mathcal{F}_{n-1})$$