BIMRC Summer School on Mathematical Finance

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Part III: Enlargement of filtration in continuous time. Progressive Enlargement.

Let τ be a finite random time, i.e., a finite non-negative random variable constructed on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$, and denote by $\mathbb{G}$ the right-continuous filtration

$$\mathcal{G}_t := \cap_{\epsilon>0} \{\mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon))\} \ .$$

We write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ where $\mathbb{H}$ is the natural filtration of the process $H_t = 1\!\!1_{\{\tau \le t\}}$. Note that τ is an $\mathbb{H}$-stopping time, hence a $\mathbb{G}$-stopping time. We assume in a first part that

(C) all $\mathbb{F}$ martingales are continuous

(A) τ avoids $\mathbb{F}$ stopping times, i.e., for any $\mathbb{F}$-stopping time ϑ, one has $\mathbb{P}(\tau = \vartheta) = 0$.

(all the results admit extension, with serious technical difficulties)

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$$d(XY) = X_- dY + Y_- dX + d[X, Y]$$

For $H_t = 1\!\!1_{\{\tau \le t\}}$, one has $\int_0^t \theta_s dH_s = 1\!\!1_{\{\tau \le t\}} \theta_\tau$.

1 Azéma supermartingale

We introduce the Azéma supermartingale

$$Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$$

which (under (CA)) is a continuous process which admits a Doob-Meyer decomposition

$$Z_t = \mu_t - A_t \,,$$

where μ is a martingale and A a predictable (in fact continuous) increasing process. The process Z is obviously valued in $[0, 1]$.

2 Key lemma

Lemma 2.1. ***Key lemma 1:*** *Let* $X \in \mathcal{F}_T$ *be an integrable r.v. Then, for any* $t \leq T$*,*

$$\mathbb{E}(X \mathbb{1}_{\{\tau < T\}} | \mathcal{G}_t) = \mathbb{1}_{\{t < \tau\}} \frac{\mathbb{E}(X Z_T | \mathcal{F}_t)}{Z_t}$$

PROOF: On the set $\{t < \tau\}$, any $\mathcal{G}_t$ measurable random variable is equal to an $\mathcal{F}_t$-measurable random variable, therefore

$$\mathbb{E}(X \mathbb{1}_{\{\tau < T\}} | \mathcal{G}_t) = \mathbb{1}_{\{t < \tau\}} y_t$$

where y_t is $\mathcal{F}_t$-measurable. Taking conditional expectation w.r.t. $\mathcal{F}_t$, we get $y_t = \frac{\mathbb{E}(Y_t \mathbb{1}_{\{t < \tau\}} | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)}$.

Note that $\mathbb{P}(t < \tau | \mathcal{F}_t)$ does not vanish on the set $\{t < \tau\}$. Indeed, let $C = \{\mathbb{P}(t < \tau | \mathcal{F}_t) > 0\}$. Then

$$\mathbb{P}(C^c \cap \{t < \tau\}) = \mathbb{E}(\mathbb{1}_{C^c} \mathbb{P}(t < \tau | \mathcal{F}_t)) = 0$$

$\triangle$

Key lemma 1: *Let $X \in \mathcal{F}_T$ be an integrable r.v. Then, for any $t \leq T$,*

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where y_t is $\mathcal{F}_t$-measurable. Taking conditional expectation w.r.t. $\mathcal{F}_t$, we get

$$y_t = \frac{\mathbb{E}(X \mathbb{1}_{\{T < \tau\}} | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)} = \frac{\mathbb{E}(X Z_T | \mathcal{F}_t)}{Z_t} \, .$$

Note that $\mathbb{P}(t < \tau | \mathcal{F}_t)$ does not vanish on the set $\{t < \tau\}$. Indeed, let $A = \{\mathbb{P}(t < \tau | \mathcal{F}_t) > 0\}$. Then

$$\mathbb{P}(A^c \cap \{t < \tau\}) = \mathbb{E}(\mathbb{1}_{A^c} \mathbb{P}(t < \tau | \mathcal{F}_t)) = 0$$

$\triangle$

Lemma 2.2. ***Key lemma 2.*** *Let h be an $\mathbb{F}$-predictable process. Then, for $t < T$,*

$$\mathbb{E}(h_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = h_\tau 1\!\!1_{\{\tau\leq t\}} - 1\!\!1_{\{\tau>t\}} \frac{1}{Z_t}\mathbb{E}(\int_t^T h_u dZ_u|\mathcal{F}_t)$$

PROOF: In a first step, the result is established for processes h of the form $h_t = 1\!\!1_{]u,v]}(t)K_u$ where $K_u \in \mathcal{F}_u$. In that case, for $t < u < v < T$, applying the key lemma

$$\mathbb{E}(h_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = \mathbb{E}(K_u 1\!\!1_{\{u<\tau\leq v\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\frac{1}{Z_t}\mathbb{E}(K_u 1\!\!1_{\{u<\tau\leq v\}}|\mathcal{F}_t)$$

It remains to note that

$$\begin{aligned}\mathbb{E}(K_u 1\!\!1_{\{u<\tau\leq v\}}|\mathcal{F}_t) &= \mathbb{E}(K_u 1\!\!1_{\{\tau\leq v\}}|\mathcal{F}_t) - \mathbb{E}(K_u 1\!\!1_{\{\tau\leq u\}}|\mathcal{F}_t)\\ &= \mathbb{E}(K_u(1-Z_v)|\mathcal{F}_t) - \mathbb{E}(K_u(1-Z_u)|\mathcal{F}_t)\\ &= -\mathbb{E}(K_u Z_v|\mathcal{F}_t) + \mathbb{E}(K_u Z_u|\mathcal{F}_t) = -\mathbb{E}(\int_t^T h_r dZ_r|\mathcal{F}_t)\end{aligned}$$

The other cases are done in the same way. The result follows by approximation. $\triangle$

Key lemma 2. *Let h be an $\mathbb{F}$-predictable process. Then, for $t < T$,*

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PROOF: In a first step, the result is established for processes h of the form $h_t = 1\!\!1_{]u,v]}(t)K_u$ where $K_u \in \mathcal{F}_u$. In that case, for $t < u < v < T$, applying the key lemma

$$\mathbb{E}(h_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) \quad = \quad \mathbb{E}(K_u 1\!\!1_{u<\tau<v}|\mathcal{G}_t) = 1\!\!1_{t<\tau}\frac{1}{Z_t}\mathbb{E}(K_u 1\!\!1_{u<\tau<v}|\mathcal{F}_t)$$

It remains to note that

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&= \mathbb{E}(K_u(1-Z_v)|\mathcal{F}_t) - \mathbb{E}(K_u(1-Z_u)|\mathcal{F}_t) \\
&= -\mathbb{E}(K_u Z_v|\mathcal{F}_t) + \mathbb{E}(K_u Z_u|\mathcal{F}_t) = -\mathbb{E}(\int_t^T h_r dZ_r|\mathcal{F}_t)
\end{aligned}$$

The other cases are done in the same way. The result follows by approximation. $\triangle$

3 A Fundamental Martingale

Proposition 3.1. ***The process*** $M_t = H_t - \int_0^{t\wedge\tau} \frac{dA_s}{Z_s}$ ***is a*** $\mathbb{G}$***-martingale.***

PROOF: In a first step, we prove that, for $s < t$

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + 1\!\!1_{\{s<\tau\}'} \frac{1}{Z_s} \mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed,

$$\begin{aligned}
\mathbb{E}(H_t|\mathcal{G}_s) &= 1 - \mathbb{P}(t<\tau|\mathcal{G}_s) = 1 - 1\!\!1_{\{s<\tau\}} \frac{1}{Z_s}\mathbb{E}(Z_t|\mathcal{F}_s) = 1 - 1\!\!1_{\{s<\tau\}} \frac{1}{Z_s}\mathbb{E}(\mu_t - A_t|\mathcal{F}_s) \\
&= 1 - 1\!\!1_{\{s<\tau\}} \frac{1}{Z_s}(\mu_s - A_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\
&= 1 - 1\!\!1_{\{s<\tau\}} \frac{1}{Z_s}(Z_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\
&= 1\!\!1_{\{\tau\le s\}} + 1\!\!1_{\{s<\tau\}} \frac{1}{Z_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)\,.
\end{aligned}$$

The process $M_t = H_t - \int_0^{t\wedge\tau} \frac{dA_s}{Z_s}$ is a $\mathbb{G}$-martingale.

PROOF: In a first step, we prove that, for $s < t$

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed,

$$\begin{aligned}
\mathbb{E}(H_t|\mathcal{G}_s) &= 1 - \mathbb{P}(t<\tau|\mathcal{G}_s) = 1 - 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}(Z_t|\mathcal{F}_s) = 1 - 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}(\mu_t - A_t|\mathcal{F}_s) \\
&= 1 - 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}(\mu_s - A_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\
&= 1 - 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}(Z_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) \\
&= 1\!\!1_{\{\tau\le s\}} + 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)\,.
\end{aligned}$$

In a second step, we prove that, setting, for any v, $K_v = \int_0^v (1-H_s)\frac{dA_s}{Z_s}$,

$$\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) = K_{s\wedge\tau} + 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed, from the key formula, for fixed t and $h_u = K_{t\wedge u}$

$$\begin{aligned}
\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) &= K_{t\wedge\tau}1\!\!1_{\{\tau\le s\}} + 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}\left(-\int_s^\infty K_{t\wedge u}dZ_u|\mathcal{F}_s\right)\\
&= K_\tau 1\!\!1_{\{\tau\le s\}} + 1\!\!1_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}\left(-\int_s^t K_u dZ_u + \int_t^\infty K_t dZ_u|\mathcal{F}_s\right)\\
&= K_{s\wedge\tau}1\!\!1_{\{\tau\le s\}} + 1\!\!1_{\{s<\tau\}}\frac{1}{G_s}\mathbb{E}\left(-\int_s^t K_u dZ_u + K_t Z_t|\mathcal{F}_s\right)
\end{aligned}$$

We now use IP formula, using the fact that K has finite variation and is continuous

$$d(K_t Z_t)) = K_t dZ_t + Z_t dK_t = K_t dZ_t + dA_t$$

hence

$$-\int_s^t K_u dZ_u + K_t Z_t = -K_t Z_t + K_s Z_s + A_t - A_s + K_t Z_t = K_s Z_s + A_t - A_s\,.$$

It follows that

$$
\begin{aligned}
\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) &= K_{s\wedge\tau}\mathbb{1}_{\{\tau\leq s\}} + \mathbb{1}_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}\left(K_s Z_s + A_t - A_s|\mathcal{F}_s\right) \\
&= K_{s\wedge\tau} + \mathbb{1}_{\{s<\tau\}}\frac{1}{Z_s}\mathbb{E}\left(A_t - A_s|\mathcal{F}_s\right) .
\end{aligned}
$$

Assuming that A is absolutely continuous w.r.t. the Lebesgue measure and denoting by a its derivative, we have proved the existence of a $\mathbb{F}$-adapted process λ, called the intensity rate such that the process

$$
H_t - \int_0^{t\wedge\tau} \lambda_u du = H_t - \int_0^t (1-H_u)\lambda_u du
$$

is a $\mathbb{G}$-martingale. More precisely, $\lambda_s = \frac{a_s}{Z_s}$. $\triangle$

4 Martingales

(i) If $Z > 0$, the process $L_t = (1 - H_t)/Z_t$ is a $\mathbb{G}$ -martingale.

(ii) If X is an $\mathbb{F}$-martingale, XL is a $\mathbb{G}$-martingale.

(iii) If the process Z is decreasing and continuous, the process $M_t = H_t - \Gamma(t \wedge \tau)$ is a $\mathbb{G}$ -martingale where $\Gamma = -\ln Z$.

PROOF: (i) From the key lemma, for $t > s$

$$\begin{aligned}
\mathbb{E}(L_t|\mathcal{G}_s) &= \mathbb{E}(\mathbb{1}_{\{\tau>t\}}\frac{1}{Z_t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}\frac{1}{Z_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}\frac{1}{Z_t}|\mathcal{F}_s) \\
&= \mathbb{1}_{\{\tau>s\}}\frac{1}{Z_s}\mathbb{E}(\frac{1}{Z_t}Z_t|\mathcal{F}_s) = \mathbb{1}_{\{\tau>s\}}\frac{1}{Z_s} = L_s
\end{aligned}$$

(ii) From the key lemma,

$$\begin{aligned}
\mathbb{E}(L_t X_t|\mathcal{G}_s) &= \mathbb{E}(\mathbb{1}_{\{\tau>t\}}L_t X_t|\mathcal{G}_s) \\
&= \mathbb{1}_{\{\tau>s\}}\frac{1}{Z_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}\frac{1}{Z_t}X_t|\mathcal{F}_s) \\
&= \mathbb{1}_{\{\tau>s\}}\frac{1}{Z_s}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)\frac{1}{Z_t}X_t|\mathcal{F}_s) = L_s\mathbb{E}(X_t|\mathcal{F}_s) = L_s X_s.
\end{aligned}$$

(iii) From integration by parts formula (H is a finite variation process, and Γ an increasing continuous process):

$$dL_t = (1 - H_t)e^{\Gamma_t}d\Gamma_t - e^{\Gamma_t}dH_t$$

and the process $M_t = H_t - \Gamma(t \wedge \tau)$ can be written

$$M_t \equiv \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u) d\Gamma_u = - \int_{]0,t]} e^{-\Gamma_u} dL_u$$

and is a $\mathbb{G}$-local martingale since L is $\mathbb{G}$-martingale. (It can be noted that, if Γ is not increasing, the differential of e^{Γ} is more complicated.) $\triangle$

5 Credit Risk

One starts with a given filtration $\mathbb{F}$ and a random time (i.e., a positive finite random variable) which represents the default time.

In the first models (Merton, Black and Cox), the default time is the first time when an observable continuous process is hitting a fixed bankruptcy level. Then, the default time is predictable, and this is not fully compatible with financial data.

A stopping time τ is predictable if there exists an increasing sequence (τ_n) of stopping times such that almost surely

$\lim_n \tau_n = \tau\,,$

$\tau_n < \tau$ *for every* n *on the set* $\{\tau > 0\}$.

The second type of model (Duffie, Lando, Schönbucher) is the case where the default arrives "by surprise" as the first jump time of a Poisson process. In the basic model (or Cox model) one defines the default time as the first time where a continuous increasing process hits a random non observable barrier. More precisely, a non negative process λ and a random variable Θ (with exponential law to simplify the computation) independent of λ being given, the default time is defined as

$$\tau = \inf\{t \,:\, \Lambda_t := \int_0^t \lambda_s ds \geq \Theta\}$$

If λ is an $\mathbb{F}$-adapted process and Θ independent of $\mathbb{F}$, the random time τ is not an $\mathbb{F}$-stopping time, and it is convenient to enlarge the filtration, considering the smallest filtration containing $\mathbb{F}$ and making τ a stopping time, or the smallest filtration containing $\mathbb{F}$ and making the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ measurable.

For an inhomogeneous Poisson process with deterministic intensity $(\lambda(s), s \geq 0)$ stopped at its first jump time $T_1 =: \tau$, the compensated martingale

$$M_t = N_{t\wedge\tau} - \int_0^{t\wedge\tau} \lambda(s)ds$$

is a useful tool.

In the basic example of default time, one can prove that

$$\mathbb{1}_{\{\tau\leq t\}} - \int_0^{t\wedge\tau} \lambda_s ds$$

is a martingale in the enlarged filtration. The random time τ is a totally inaccessible $\mathbb{G}$-stopping time and avoids $\mathbb{F}$-stopping times. This model became popular under the name **intensity based model** or the reduced form model.

*An $\mathbb{F}$-stopping time τ is $\mathbb{F}$-**totally inaccessible** if, for any $\mathbb{F}$-**predictable** stopping time ϑ, $\mathbb{P}(\tau = \vartheta < \infty) = 0$. A stopping time τ is $\mathbb{F}$- **accessible** if there exists a sequence T_n of $\mathbb{F}$-**predictable** stopping times such that $\sum \mathbb{P}(\tau = T_n) = 1$.*

Importance on continuity of Λ. Example: if

$$\tau = \inf\{t \,:\, N_t \geq \Theta\}$$

where N is a Poisson process with jumps time (T_n), independent of Θ, the random time τ is such that $\sum_n \mathbb{P}(\tau = T_n) = 1$

$$1\!\!1_{\tau \leq t} - \int_0^{t\wedge\tau} \lambda(1 - e^{-1})\, ds$$

is a $\mathbb{G}$-martingale, τ is a $\mathbb{G}$ stopping time totally inaccessible.

5.1 Dynamics of Prices in a Default Setting in a Cox Model

Here, we are working in a Cox model under the probability measure $\mathbb{Q}$, i.e., $\tau = \inf\{t\ : \Lambda_t := \int_0^t \lambda_s ds \geq \Theta\}$ where Θ is independent of $\mathbb{F}$ under $\mathbb{Q}$. The probability measure $\mathbb{Q}$ is the pricing measure, i.e., such that (discounted) prices are $\mathbb{Q}$-martingales. Our goal is to give the dynamics of prices of some important contingent claims. We recall that we are working under the assumption that all $\mathbb{F}$-martingales are continuous.

5.2 Defaultable Zero-Coupon Bond

A defaultable zero-coupon bond of maturity T pays one monetary unit at time T, if the default has not occurred before T. Let $B_t(T)$ be the price at time t of a default-free bond paying 1 at maturity T given by

$$B_t(T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s\, ds\right) \,\Big|\, \mathcal{F}_t\right].$$

The price $D_t(T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D_t(T) &= \mathbb{E}\left[\mathbb{1}_{\{T<\tau\}} \exp\left(-\int_t^T r_s\, ds\right) \,\Big|\, \mathcal{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T [r_s+\lambda_s]\, ds\right) \,\Big|\, \mathcal{F}_t\right] = m_t^\Lambda \Upsilon_t e^{\int_0 t r_s ds} \end{aligned}$$

where $m_t^\Lambda = \mathbb{E}(e^{-\int_0^T (r_s+\lambda_s)ds}|\mathcal{F}_t)$ and $\Upsilon_t = \mathbb{1}_{\{t<\tau\}} e^{\Lambda_t}$. Then, if $r=0$,

$$dD_t(T) = m_t^\Lambda d\Upsilon_t + \Upsilon_{t-} dm_t^\Lambda = -m_t^\Lambda \Upsilon_{t-} dM_t + \Upsilon_{t-} dm_t^\Lambda = -D_{t-}(T) dM_t + \Upsilon_{t-} dm_t^\Lambda .$$

In the particular case where λ and r are deterministic, $m_t^\Lambda = e^{-\int_0^T (\lambda(s)+r(s))ds}$ and $dm_t^\Lambda = 0$. Hence

$$dD_t(T) = r_t D_t(T)dt - D_{t-}(T)dM_t \, .$$

5.3 Recovery with Payment at Maturity

We assume here that $r = 0$. We consider a contract which pays K_τ at date T, if $\tau \leq T$ and no payment in the case $\tau > T$, where K is a given $\mathbb{F}$-predictable process. The price at time t of this contract is

$$\begin{aligned} S_t &:= \mathbb{E}(K_\tau 1\!\!1_{\{\tau<T\}}|\mathcal{G}_t) = K_\tau 1\!\!1_{\{\tau\leq t\}} + 1\!\!1_{\{t<\tau\}}\mathbb{E}(K_\tau 1\!\!1_{\{t<\tau<T\}}|\mathcal{G}_t) \\ &= K_\tau 1\!\!1_{\{\tau\leq t\}} + 1\!\!1_{\{t<\tau\}}e^{\Lambda_t}\mathbb{E}(\int_t^T K_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t) \\ &= \int_0^t K_u dH_u + \Upsilon_t\left(-\int_0^t K_u e^{-\Lambda_u}\lambda_u du + m_t^K\right) \end{aligned}$$

where $m_t^K := \mathbb{E}(\int_0^T K_u e^{-\Lambda_u}\lambda_u du|\mathcal{F}_t)$.

From $d\Upsilon_t = -\Upsilon_{t-} dM_t$ and

$$d(\Upsilon m^K)_t = \Upsilon_{t-} dm^K_t + m^K_{t-} d\Upsilon_t + d[m^K, \Upsilon]_t = \Upsilon_{t-} dm^K_t + m^K_{t-} d\Upsilon_t$$

we deduce that

$$dS_t = K_t(dH_t - \lambda_t(1 - H_t)dt) - S_{t-} dM_t + \Upsilon_t dm^K_t = (K_t - S_{t-})dM_t + \Upsilon_t dm^K_t .$$

Note that, since m^K is continuous, its covariation process with Υ is null and that one can write $\Upsilon_t dm^K_t$ instead of $\Upsilon_{t-} dm^K_t$. Note also that, from the definition, the process S is a $\mathbb{G}$-martingale. This can be checked looking at the dynamics, since m^K is an $\mathbb{F}$, hence a $\mathbb{G}$, martingale.

5.4 Recovery with Payment at Default Time

Let K be a given $\mathbb{F}$-predictable process. The payment K_τ is done at time τ. Then, in the case $r = 0$, the price of this payment is

$$S_t = 1\!\!1_{\{t<\tau\}}\mathbb{E}(K_\tau 1\!\!1_{\{t<\tau<T\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}e^{\Lambda_t}\mathbb{E}\left[\int_t^T K_u\lambda_u e^{-\Lambda_u}du\Big|\mathcal{F}_t\right].$$

The dynamics of S is

$$dS_t = -S_{t-}dM_t + \Upsilon_t(dm_t^K - K_t e^{-\Lambda_t}\lambda_t dt) = -S_{t-}dM_t + (1-H_t)(e^{\Lambda_t}dm_t^K - K_t\lambda_t dt)$$

and the process $S_t + K_\tau 1\!\!1_{\{\tau<t\}} = S_t + \int_0^t K_s dH_s = \mathbb{E}(K_\tau|\mathcal{G}_t)$ is a $\mathbb{G}$-martingale, as well as the process $S_t + \int_0^{t\wedge\tau} K_s\lambda_s ds$. The quantity $K_t\lambda_t$ which appears in the dynamics of S can be interpreted as a dividend K_t paid at rate λ_t (or with probability $\lambda_t dt = \mathbb{P}(t<\tau<t+dt|\mathcal{F}_t)/\mathbb{P}(t<\tau|\mathcal{F}_t)$).

5.5 Pricing and Hedging a Defaultable Call

We assume that the interest rate is null, that a risky asset with risk-neutral dynamics

$$dY_t = Y_t \sigma dB_t \, ,$$

where B is a Brownian motion and σ is a constant, is traded as well as a defaultable zero-coupon of maturity T with price $D_t(T)$. The reference filtration is that of the Brownian motion B. The price of a defaultable call with payoff $\mathbb{1}_{\{T<\tau\}}(Y_T - K)^+$ is

$$C_t \;=\; \mathbb{E}(\mathbb{1}_{\{T<\tau\}}(Y_T - K)^+ | \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+ | \mathcal{F}_t) = \Upsilon_t m_t^Y$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda_T}(Y_T - K)^+ | \mathcal{F}_t)$. Hence

$$dC_t = \Upsilon_t dm_t^Y - m_t^Y \Upsilon_{t-} dM_t \, .$$

In the particular case where λ is deterministic,

$$m_t^Y = e^{-\Lambda_T} E((Y_T - K)^+ | \mathcal{F}_t) = e^{-\Lambda_T} C_t^Y$$

where C^Y is the price of a call in the Black and Scholes model, and satisfies $C_t^Y = C^Y(t, Y_t)$ and $dC_t^Y = \Delta_t dY_t$ where $\Delta_t$is the Delta-hedge ($\Delta_t = \partial_y C^Y(t, Y_t)$), hence

$$C_t = \Upsilon_t e^{-\Lambda_T} C_t^Y = D_t(T) C_t^Y .$$

From $C_t = D_t(T) C_t^Y$, we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(\Upsilon_t dC_t^Y + C_t^Y d\Upsilon_t) = e^{-\Lambda_T}(\Upsilon_t \Delta_t dY_t - C_t^Y \Upsilon_t dM_t) \\ &= e^{-\Lambda_T}(\Upsilon_t \Delta_t dY_t - C_t^Y \Upsilon_t dM_t) . \end{aligned}$$

Therefore, using the fact that $dD_t(T) = m_t dM_t = -e^{-\Lambda_T}\Upsilon_t dM_t$, we get

$$dC_t = e^{-\Lambda_T}\Upsilon_t \Delta_t dY_t - C_t^Y dD_t(T) = e^{-\Lambda_T}\Upsilon_t \Delta_t dY_t + \frac{C_t}{D_t(T)} dD_t(T)$$

hence, an hedging strategy consists of holding $\frac{C_t}{D_t(T)}$ DZCs, and the sum of the amount of wealth invested in the savings account and the one invested in risky asset is null.

For a general intensity rate $(\lambda_s, s \geq 0)$, one obtains

$$dC_t = \frac{C_{t-}}{D_t(T)} dD_t(T) + \Upsilon_t \frac{m_t^Y}{m_t} dm_t + \Upsilon_t dm_t^Y = \frac{C_{t-}}{D_t(T)} dD_t(T)) + \vartheta_t dY_t .$$

An hedging strategy consists of holding $\frac{C_{t-}}{D_t(T)}$ DZCs.

5.6 Toy model

One default Let τ be a random time with cumulative distribution function $F(t) = \mathbb{P}(\tau \leq t)$ and $G(t) = \mathbb{P}(\tau > t)$.

Then, the process $M_t = H_t - \int_0^{t \wedge \tau} \frac{dF(s)}{G(s)}$ is an $\mathbb{H}$-martingale.

Two defaults

We present here a toy model with two random times τ_1 and τ_2, to underline the role of the filtration and the form of the compensator.

We denote by $\mathbb{H}^1$ the natural filtration of the process $(H_t^1 := 1\!\!1_{\{\tau_1 \leq t\}})$, by $\mathbb{H}^2$ the natural filtration of the process $(H_t^2 := 1\!\!1_{\{\tau_2 \leq t\}})$ and by $\mathbb{G}$ the filtration $\mathbb{G} := \mathbb{H}^1 \vee \mathbb{H}^2$.

We denote by $G(t,s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)$ the survival probability of the pair (τ_1, τ_2) assumed to be strictly positive and continuously differentiable in both variables. Note that $G(t,0) = \mathbb{P}(\tau_1 > t)$ is the survival probability of τ_1. Here, we assume that $G(t,0) = e^{-\lambda t}$, with $\lambda > 0$.

Proposition 5.1. ***The compensator of τ_1 can be computed in two filtrations:***

$$\begin{aligned} M_t^1 &:= H_t^1 - \lambda(t \wedge \tau_1) = H_t^1 - \int_0^t (1 - H_s^1)\lambda ds, \text{ is an } (\mathbb{H}^1, \mathbb{P})\text{-martingale,} \\ M_t^2 &:= H_t^1 - \int_0^t (1 - H_s^1)\lambda_s^2 ds, \text{ is a } (\mathbb{G}, \mathbb{P})\text{-martingale} \end{aligned}$$

where

$$\lambda_t^2 = 1\!\!1_{\{t \le \tau_2\}} \frac{-\partial_1 G(t,t)}{G(t,t)} + 1\!\!1_{\{\tau_2 \le t\}} \frac{\partial_{12} G(t,\tau_2)}{-\partial_1 G(t,\tau_2)}.$$

PROOF: The fact that $\lambda^2(1 - H^1)$ is the $\mathbb{G}$-intensity rate of τ_1 is obtained computing the Doob-Meyer decomposition of the supermartingale $Z_t^1 := \mathbb{P}(\tau_1 > t | \mathcal{H}_t^2)$, obtained as follows.

$$Z_t^1 = H_t^2 \mathbb{P}(\tau_1 > t | \tau_2) + (1 - H_t^2) \frac{\mathbb{P}(\tau_1 > t, \tau_2 > t)}{\mathbb{P}(\tau_2 > t)} = H_t^2 h(t, \tau_2) + (1 - H_t^2)\psi(t)$$

where

$$h(t, v) = \frac{\partial_2 G(t, v)}{\partial_2 G(0, v)}; \quad \psi(t) = G(t, t)/G(0, t).$$

Using the integration by parts formula, one gets

$$dZ_t^1 = (\frac{\partial_2 G(t, t)}{\partial_2 G(0, t)} - \frac{G(t, t)}{G(0, t)}) dH_t^2 + \big(H_t^2 \partial_1 h(t, \tau_2) + (1 - H_t^2)\psi'(t)\big) dt$$

and the result follows. △

5.7 Recent models

The basic model was standard till recently. However many authors note that the fact that τ is totally inaccessible is not realistic.

Coculescu (2009) presents a model where $\tau = \vartheta \wedge \xi$, where ξ is constructed as in the basic example and the graph of ϑ in included in the union of graphs of $\mathbb{F}$-stopping times, and assumes that immersion hypothesis between $\mathbb{F}$ and $\mathbb{G}$ (i.e. $\mathbb{F}$-martingales are $\mathbb{G}$-martingales).

Jiao and Li (2015) present a case where $\tau = \vartheta \wedge \xi$ where ξ is constructed as in the basic example, and the graph of ϑ in included in the union of graphs of $\mathbb{F}$-predictable stopping times. In their model, immersion property between $\mathbb{F}$ and $\mathbb{G}$ is established.

Gehmlich and Schmidt (2015) are interested in the predictable process Λ such that $A_t - \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-martingale. They assume that

$$\Lambda_t = \int_0^{t\wedge\tau} \lambda_s ds + \sum_i \mathbb{1}_{U_i < t\wedge\tau} \Gamma_i$$

where S_i are $\mathbb{G}$ stopping times, $U_i > S_i$ and Γ_i, U_i are $\mathcal{G}_{S_i}$ measurable (in particular, U_i are predictable $\mathbb{G}$ stopping times). This leads to a random time τ which can coincide with U_i. Note that, to construct τ (or to prove that such a τ exists) one needs to enlarge the filtration.

6 Decomposition

A random time τ is called

(a) a **thick** random time if $[\![\tau]\!] \cap [\![T]\!] = \emptyset$ for any $\mathbb{F}$-stopping time T, i.e., if it avoids all finite $\mathbb{F}$-stopping times.

(b) a **thin** random time if its graph $[\![\tau]\!]$ is contained in a thin set, i.e., if there exists a sequence of $\mathbb{F}$-stopping times $(T_n)_{n=1}^{\infty}$ with disjoint graphs such that $[\![\tau]\!] \subset \bigcup_n [\![T_n]\!]$. We say that such a sequence $(T_n)_n$ exhausts the thin random time τ or that $(T_n)_n$ is an exhausting sequence of the thin random time τ.

If τ is a thin random time,and $(T_n)_n$ is an exhausting sequence for τ,

$$\tau = \infty 1\!\!1_{C_0} + \sum_n T_n 1\!\!1_{C_n}$$

where

$$C_0 = \{\tau = \infty\} \quad \text{and} \quad C_n = \{\tau = T_n < \infty\} \quad \text{for} \quad n \geq 1.$$

Any random time τ admits a unique decomposition $\tau = \tau_1 \wedge \tau_2$, where τ_1 is a thick time, τ_2 is a thin time and $\tau_1 \vee \tau_2 = \infty$.

For that purpose, the efficient tool is the optional dual projection of H, denoted H^o

It is enough to take τ_1 and τ_2 of the following form

$$\tau_1 = \tau_{\{\Delta H^o_\tau = 0\}} \quad \text{and} \quad \tau_2 = \tau_{\{\Delta H^o_\tau > 0\}},$$

The $\mathbb{F}$-dual optional projection of H is the increasing $\mathbb{F}$-optional process H^o such that for any $\mathbb{F}$-optional bounded process X,

$$\mathbb{E}(X_\tau) = \mathbb{E}(\int_0^\infty X_s dH_s) = \mathbb{E}(\int_0^\infty X_s dH^o_s)$$

Let τ be a random time and (τ_1, τ_2) its decomposition. Then, the hypothesis $(\mathcal{H}')$ is satisfied for $(\mathbb{F}, \mathbb{F}^{\tau})$ if and only if the hypothesis $(\mathcal{H}')$ is satisfied for $(\mathbb{F}, \mathbb{F}^{\tau_1})$.

7 Restriction of information

Let Z be the $\mathbb{F}$-Azéma supermartinngale of τ. Computing the $\widetilde{\mathbb{F}}$ compensator of τ, for $\widetilde{\mathbb{F}} \subset \mathbb{F}$ is now easy.

Let $Z_t = \mu_t - A_t$ be the $\mathbb{F}$-Doob-Meyer decomposition of Z and assume that $A_t = \int_0^t a_s ds$. The process $\widehat{A}$ defined as $\widehat{A}_t := \mathbb{E}(A_t | \widehat{\mathcal{F}}_t)$ is an $\widehat{\mathbb{F}}$-submartingale and its $\widehat{\mathbb{F}}$-Doob-Meyer decomposition is denoted

$$\widehat{A}_t = \widehat{n}_t + \widehat{\alpha}_t \, .$$

where $\widehat{n}$ is the $\widehat{\mathbb{F}}$-martingale part and $\widehat{\alpha}_t = \int_0^t \mathbb{E}(a_s | \widehat{\mathcal{F}}_s) ds$. Hence, setting $\widehat{\mu}_t = \mathbb{E}(\mu_t | \widehat{\mathcal{F}}_t)$, the $\widehat{\mathbb{F}}$-super-martingale $\widehat{Z}$ admits a $\widehat{\mathbb{F}}$-Doob-Meyer decomposition as

$$\widehat{Z}_t = \widehat{\mu}_t - \widehat{n}_t - \widehat{\alpha}_t$$

where $\widehat{\mu} - \widehat{n}$ is its $\widehat{\mathbb{F}}$-martingale part. It follows that

$$H_t - \int_0^{t \wedge \tau} \frac{d\widehat{\alpha}_s}{\widehat{Z}_s} ds = H_t - \int_0^{t \wedge \tau} \frac{\mathbb{E}(a_s | \widehat{\mathcal{F}}_s)}{\widehat{Z}_s} ds, \quad t \geq 0$$

is a $\widehat{\mathbb{G}}$-martingale and that the $\widehat{\mathbb{F}}$-intensity of τ is equal to $\mathbb{E}(a_s | \widehat{\mathcal{F}}_s) / \widehat{Z}_s$.

8 Before τ, under (CA)

Proposition 8.1. *Under (CA), every $\mathbb{F}$-martingale X stopped at time τ is a $\mathbb{G}$-semi-martingale with canonical decomposition*

$$\boxed{X_t^\tau = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle X, \mu\rangle_s}{Z_s}}$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale.

PROOF: Let Y_s be a $\mathcal{G}_s$-measurable random variable. There exists an $\mathcal{F}_s$-measurable random variable y_s such that $Y_s 1\!\!1_{\{s<\tau\}} = y_s 1\!\!1_{\{s<\tau\}}$, hence, if X is an $\mathbb{F}$-martingale, for $s < t$,

$$
\begin{aligned}
\mathbb{E}(Y_s(X_{t\wedge\tau} - X_{s\wedge\tau})) &= \mathbb{E}(Y_s 1\!\!1_{\{s<\tau\}}(X_{t\wedge\tau} - X_{s\wedge\tau})) \\
&= \mathbb{E}\left(y_s(1\!\!1_{\{s<\tau\le t\}}(X_\tau - X_s) + 1\!\!1_{\{t<\tau\}}(X_t - X_s))\right)
\end{aligned}
$$

From the key lemma

$$
\mathbb{E}\left(y_s 1\!\!1_{\{s<\tau\le t\}} X_\tau\right) = -\mathbb{E}\left(y_s \int_s^t X_u dZ_u\right) .
$$

From integration by parts formula (taking into account the continuity of Z and X)

$$
\int_s^t X_u dZ_u = -X_s Z_s + Z_t X_t - \int_s^t Z_u dX_u - \langle X, Z\rangle_t + \langle X, Z\rangle_s
$$

We have also

$$
\begin{aligned}
\mathbb{E}\left(y_s 1\!\!1_{\{s<\tau\le t\}} X_s\right) &= \mathbb{E}\left(y_s X_s (Z_s - Z_t)\right) \\
\mathbb{E}\left(y_s 1\!\!1_{\{t<\tau\}}(X_t - X_s)\right) &= \mathbb{E}\left(y_s Z_t (X_t - X_s)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(Y_s(X_{t\wedge\tau} - X_{s\wedge\tau})) &= \mathbb{E}\left(y_s(\mathbb{1}_{\{s<\tau\le t\}}(X_\tau - X_s) + \mathbb{1}_{\{t<\tau\}}(X_t - X_s))\right) \\
&= -\mathbb{E}\left(y_s(-X_sZ_s + Z_tX_t - \int_s^t Z_u dX_u - \langle X, Z\rangle_t + \langle X, Z\rangle_s)\right) \\
&\quad -E\left(y_sX_s(Z_s - Z_t)\right) + \mathbb{E}\left(y_sZ_t(X_t - X_s)\right)
\end{aligned}
$$

from the martingale property of X, the blue term has zero expectation, and after simplifications

$$
\mathbb{E}(Y_s(X_{t\wedge\tau} - X_{s\wedge\tau})) = \mathbb{E}\left(y_s(\langle X, Z\rangle_t - \langle X, Z\rangle_s)\right)
$$

From the Doob-Meyer decomposition of Z

$$
\begin{aligned}
\mathbb{E}(Y_s(X_{t\wedge\tau} - X_{s\wedge\tau})) &= \mathbb{E}(y_s(\langle X, \mu\rangle_t - \langle X, \mu\rangle_s)) \\
&= \mathbb{E}\left(y_s \int_s^t \frac{d\langle X, \mu\rangle_u}{Z_u} Z_u\right) = \mathbb{E}\left(y_s \int_s^t \frac{d\langle X, \mu\rangle_u}{Z_u}\, \mathbb{E}(\mathbb{1}_{\{u<\tau\}}|\mathcal{F}_u)\right) \\
&= \mathbb{E}\left(y_s \int_s^t \frac{d\langle X, \mu\rangle_u}{Z_u} \mathbb{1}_{\{u<\tau\}}\right) = \mathbb{E}\left(y_s \int_{s\wedge\tau}^{t\wedge\tau} \frac{d\langle X, \mu\rangle_u}{Z_u}\right) .
\end{aligned}
$$

The result follows. $\triangle$

9 Arbitrages

Let S be a price process in a model with zero interest rate.

The fundamental theorem of asset pricing states that

• No free lunch with vanishing risk (NFLVR) holds if there exists a positive **martingale** L such that SL is a local martingale.

• No upper bounded profit with bounded risk (NUPBR) or No arbitrages of the first kind (NA1) holds if there exists a positive **local martingale** L called a **deflator** such that SL is a local martingale.

If S is continuous and satisfies NA1$(\mathbb{F})$, then S^τ satisfies NA1$(\mathbb{G})$.

Proof: There is no loss of generality in assuming that S is a continuous $\mathbb{F}$-local martingale. We recall that $Z = \mu - A$ and that (C) holds.

Let $\widehat{\mu}$ be the $\mathbb{G}$-martingale associated to μ given by

$$\widehat{\mu}_t := \mu_{t\wedge\tau} - \int_0^{t\wedge\tau} Z_s^{-1} d\langle \mu, \mu\rangle_s$$

Then, $L := \mathcal{E}(\int_0^{\cdot\wedge\tau} Z_s^{-1} d\widehat{\mu_s})$ and $S^\tau L$ are $\mathbb{G}$-local martingales (Use integration by parts), hence L is a local martingale deflator.

We recall that $\mathcal{E}(Y)$ is the solution of $dX = XdY$.

10 Before τ, general case

We define the right-continuous with left limits $\mathbb{F}$-supermartingale

$$Z_t := \mathbb{P}\left(\tau > t \,\middle|\, \mathcal{F}_t\right).$$

One can write

$$Z = m - H^o$$

where m is an $\mathbb{F}$-martingale and H^o is the $\mathbb{F}$-dual optional projection (an increasing process) of $H = 1\!\!1_{[\![\tau,\infty[\![}$. Note that this is NOT the Doob-Meyer decomposition.

Recall the following definition: Let $(A_t,\, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-optional increasing process $(A_t^o, t \geq 0)$, called the dual optional projection of A such that

$$\mathbb{E}\left(\int_{[0,\infty[} Y_s dA_s\right) = \mathbb{E}\left(\int_{[0,\infty[} Y_s dA_s^o\right)$$

for any positive $\mathbb{F}$-optional process Y.

Example: Let $\tau = \inf\{t \,:\, N_t \geq \Theta\}$ where N is a Poisson process with intensity λ, and Θ an exponential random variable independent of $\mathbb{F}^N$. The decreasing process $Z = e^{-N}$ admits a Doob-Meyer decomposition $Z_t = \mu_t - A_t^p$ which can be computed using standard (Stieljes) integration leading to

$$\begin{aligned} d\mu_t &= -e^{-N_{t-}}\gamma dM_t \\ dA_t^p &= e^{-N_{t-}}\gamma\lambda dt \end{aligned}$$

where $\gamma = 1 - \frac{1}{e} > 0$.

The optional decomposition of Z is $Z_t = m_t - (1 - e^{-N_t})$ where $m = 1$.

Note that m is non-negative: indeed $m_t = \mathbb{E}(A^o_\infty | \mathcal{F}_t)$.

Naive remark: if $m_\tau \geq 1$ with $\mathbb{P}(m_\tau > 1) > 0$, and if m is the value of a portfolio, then, there exists a classical arbitrage.

Let $\widetilde{R} := R_{\{\widetilde{Z}_R=0<Z_{R-}\}}$, where R is the first time when Z vanishes and $\widetilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t =$.

Theorem 10.1. *Let $\mathbb{F}$ be a quasi-left continuous filtration. The following conditions are equivalent.*

(i) The $\mathbb{F}$-stopping time $\widetilde{R}$ is infinite ($\widetilde{R} = \infty$).

(ii) For any $\mathbb{F}$-local martingale X, the process X^τ admits a $\mathbb{G}$-local martingale deflator (hence, satisfies NUPBR$(\mathbb{G})$).

A filtration $\mathbb{F}$ is quasi-left continuous if for each $\mathbb{F}$-predictable stopping time T, $\mathcal{F}_T = \mathcal{F}_{T-}$.

11 After τ Honest times

There exists an interesting class of random times τ such that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales, called honest times, introduced by Meyer and studied by Barlow and Jeulin among others.

Definition 11.1. *A random time τ is honest if τ is equal to an $\mathcal{F}_t$-measurable random variable on $\tau < t$.*

Examples 11.2. (i) Let B a Brownian motion and set $\tau = g_1$ where $g_t = \sup\{s < t \,:\, B_s = 0\}$. Then, for $t < 1$, $g_1 = g_t$ on $\{g_1 < t\}$, and g_t is $\mathcal{F}_t$-measurable, hence g_1 is honest.
(ii) Let X be an adapted continuous process and $X^* = \sup X_s$, $X^*_t = \sup_{s \leq t} X_s$. The random time

$$\tau = \sup\{s \,:\, X_s = X^*\}$$

is honest. Indeed, on the set $\{\tau < t\}$, one has $\tau = \sup\{s \,:\, X_s = X^*_t\}$.
(iii) An $\mathbb{F}$-stopping time is honest: indeed $\tau = \tau \wedge t$ on $\tau < t$.

Let Y be a $\mathbb{G}$ predictable process. There exists y and $\widetilde{y}$, two $\mathbb{F}$ predictable processes such that

$$Y_t = y_t \mathbb{1}_{t \leq \tau} + \widetilde{y}_t \mathbb{1}_{\tau < t}$$

11.1 Decomposition

Proposition 11.3. *Let τ be honest. We assume (CA). Then, any $\mathbb{F}$- martingale X is a $\mathbb{G}$ semi-martingale with decomposition*

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X,\mu\rangle_s}{Z_s} - \int_\tau^{\tau\vee t} \frac{d\langle X,\mu\rangle_s}{1-Z_s}\,,$$

where $\widetilde{X}$ is a $\mathbb{G}$-local martingale.

PROOF: Let X be an $\mathbb{F}$-martingale and $K_s \in \mathcal{G}_s$. We define a $\mathbb{G}$-predictable process Y as $Y_u = 1\!\!1_{K_s} 1\!\!1_{]s,t]}(u)$. For $s < t$, one has, using the decomposition of $\mathbb{G}$-predictable processes:

$$\begin{aligned}\mathbb{E}(1\!\!1_{K_s}(X_t - X_s)) &= \mathbb{E}\left(\int_0^\infty Y_u dX_u\right)\\ &= \mathbb{E}\left(\int_0^\tau y_u dX_u\right) + \mathbb{E}\left(\int_\tau^\infty \widetilde{y}_u dX_u\right).\end{aligned}$$

Proposition 11.4. *Let τ be honest. We assume (CA). Then, any $\mathbb{F}$- martingale X is a $\mathbb{G}$ semi-martingale with decomposition*

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, \mu\rangle_s}{Z_s} - \int_\tau^{\tau\vee t} \frac{d\langle X, \mu\rangle_s}{1 - Z_s} \,,$$

where $\widetilde{X}$ is a $\mathbb{G}$-local martingale.

PROOF: Let X be an $\mathbb{F}$-martingale and $K_s \in \mathcal{G}_s$. We define a $\mathbb{G}$-predictable process Y as $Y_u = 1\!\!1_{K_s} 1\!\!1_{]s,t]}(u)$. For $s < t$, one has, using the decomposition of $\mathbb{G}$-predictable processes:

$$\begin{aligned}
\mathbb{E}(1\!\!1_{K_s}(X_t - X_s)) &= \mathbb{E}\left(\int_0^\infty Y_u dX_u\right) \\
&= \mathbb{E}\left(\int_0^\tau y_u dX_u\right) + \mathbb{E}\left(\int_\tau^\infty \widetilde{y}_u dX_u\right) .
\end{aligned}$$

Noting that $\int_0^t \widetilde{y}_u dX_u$ is a martingale yields $\mathbb{E}\left(\int_0^\infty \widetilde{y}_u dX_u\right) = 0$, hence

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{K_s}(X_t - X_s)) &= \mathbb{E}\left(\int_0^\tau (y_u - \widetilde{y}_u) dX_u\right) \\
&= \mathbb{E}\left(\int_0^\infty dA_v \int_0^v (y_u - \widetilde{y}_u) dX_u\right) .
\end{aligned}$$

By integration by parts, setting $N_t = \int_0^t (y_u - \widetilde{y}_u) dX_u$, we get

$$\mathbb{E}(\mathbb{1}_{K_s}(X_t - X_s)) = \mathbb{E}(N_\infty A^p_\infty) = \mathbb{E}(N_\infty \mu_\infty) = \mathbb{E}\left(\int_0^\infty (y_u - \widetilde{y}_u) d\langle X, \mu\rangle_u\right) .$$

Now, it remains to note that

$$\begin{aligned}
&\mathbb{E}\left(\int_0^\infty Y_u\left(\frac{d\langle X,\mu\rangle_u}{Z_{u-}}\mathbb{1}_{\{u\le\tau\}}-\frac{d\langle X,\mu\rangle_u}{1-Z_{u-}}\mathbb{1}_{\{u>\tau\}}\right)\right)\\
&= \mathbb{E}\left(\int_0^\infty\left(y_u\frac{d\langle X,\mu\rangle_u}{Z_{u-}}\mathbb{1}_{\{u\le\tau\}}-\widetilde{y}_u\frac{d\langle X,\mu\rangle_u}{1-Z_{u-}}\mathbb{1}_{\{u>\tau\}}\right)\right)\\
&= \mathbb{E}\left(\int_0^\infty\left(y_u d\langle X,\mu\rangle_u-\widetilde{y}_u d\langle X,\mu\rangle_u\right)\right)\\
&= \mathbb{E}\left(\int_0^\infty\left(y_u-\widetilde{y}_u\right)d\langle X,\mu\rangle_u\right)
\end{aligned}$$

to conclude. □

11.2 Multiplicative decomposition

Assume that Z does not vanish. Under (CA), it can be proved that

$$Z_t = \frac{N_t}{N_t^*}$$

where N is a local (continuous) martingale and $N_t^* = \sup\{N_s, s \leq t\}$ and that $\tau = \sup\{t : N_t = N_t^*\}$. Il follows that $Z_\tau = 1$.

11.3 Arbitrages

Assume that $dS_t = S_t \sigma_t dW_t$ and that τ avoids stopping times. Then, there are arbitrages on the interval $[0, \tau]$ and on the time interval $[\tau, \infty[$

From the multiplicative decomposition of Z, $Z = N/N^*$ and $Z_\tau = 1$, we obtain $N_\tau \geq 1 = N_0$.

$N - 1$ being the value of a portfolio with null initial value, we obtain an arbitrage.

This result does not extend to the case with jumps.

Other arbitrages can be detected using the additive decomposition $Z_t = \mu_t - A_t$ which leads to $\mu_\tau \geq 1 = \mu_0$. It is proved in Jeulin that

$$\tau \textbf{ is honest if and only if } \widetilde{Z}_\tau = 1$$

where $\widetilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t)$. Under (CA), $Z = \widetilde{Z}$.

Let τ be a finite honest time and assume that the market (S^0, S) is complete. Then, if τ is not an $\mathbb{F}$-stopping time, there are classical arbitrages before and after τ.

Other arbitrages can be detected using the additive decomposition $Z_t = \mu_t - A_t$ which leads to $\mu_\tau \geq 1 = \mu_0$.It is proved in Jeulin that

$$\tau \text{ is honest if and only if } \widetilde{Z}_\tau = 1$$

where $\widetilde{Z}_t = \mathbb{P}(\tau \geq t|\mathcal{F}_t)$. Under (CA), $Z = \widetilde{Z}$.

Let τ be a finite honest time and assume that the market (S^0, S) is complete. Then, if τ is not an $\mathbb{F}$-stopping time, there are classical arbitrages before and after τ.

This result extends to the case with jumps.

Before τ

From $\mu = Z + A$ and $Z_\tau = 1$, we deduce that $\mu_\tau \geq 1$.

Since τ is not a stopping time, $\mathbb{P}(A_\tau > 0) > 0$.

The market being complete, the martingale m is the value of a self financing portfolio, with initial value 1, and $\mu_\tau = 1 + \int_0^\tau \varphi_s dS_s$ for an $\mathbb{F}$-predictable φ. Since $\mu_t \geq 0$, the strategy φ is admissible.

After τ: Here, $t > \tau$

Using $\mu = Z + A$, one obtains that $\mu_t - \mu_\tau = Z_t - 1$.

Consider the (finite) $\mathbb{G}$-stopping time

$$\nu := \inf\{t > \tau : Z_t \leq \frac{1}{2}\}.$$

Then,

$$\mu_\nu - \mu_\tau = Z_\nu - 1 \leq \frac{-1}{2} \leq 0,$$

and

$$\mathbb{P}(\mu_\nu - \mu_\tau < 0) > 0.$$

Hence $-\int_\tau^{t\wedge\nu} \varphi_s dS_s = \mu_{t\wedge\tau} - \mu_{t\wedge\nu}$ is the value of a self-financing strategy with initial value 0 and terminal value $\mu_\tau - \mu_\nu \geq 0$ satisfying $\mathbb{P}(\mu_\tau - \mu_\nu > 0) > 0$.

From $\mu = Z + A$ and the fact that $A_t = A_{t\wedge\tau}$, one obtains that $\mu_t - \mu_\tau = Z_t - Z_\tau \geq -2$, hence the strategy is admissible.

The completeness of the $\mathbb{F}$ market seems to be an essential hypothesis to have classical arbitrages:

Let W^1, W^2 be a standard 2-dimensional Brownian motion and

$$dS_t = S_t f(W_t^2) dW_t^1$$

Under regularity assumptions $\mathbb{F}^S = \mathbb{F}^1 \vee \mathbb{F}^2$. Let τ be an $\mathbb{F}^2$ honest time (hence an $\mathbb{F}^S$ honest time). Since W^1 is an $\mathbb{F}^1 \vee \sigma(\tau \wedge \cdot)$ martingale, there are no arbitrages in the enlarged filtration.

11.4 Examples in a Brownian filtration

In this section, we assume that

$$S_t = \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), \quad \sigma > 0 \, .$$

• Consider the following finite random time

$$g := \sup\{t \, : \, S_t = a\},$$

where $0 < a < 1$.

Then $Z_t = 1 - (1 - \frac{S_t}{a})^+$, and

$$dZ_t = 1\!\!1_{\{S_t < a\}} \frac{1}{a} dS_t - \frac{1}{2a} d\ell_t^a$$

Therefore,

$$\varphi := \frac{1}{a} 1\!\!1_{\{S < a\}}$$

$$\begin{aligned}
W_t &= W_t^{\mathbb{G}} + \int_0^{t\wedge g} \frac{d\langle W, m\rangle_s}{Z_s} - \int_{t\wedge g}^{t} \frac{d\langle W, m\rangle_s}{1 - Z_s} \\
&= W_t^{\mathbb{G}} + \int_0^{t\wedge g} \sigma \mathbb{1}_{\{S_s < a\}} ds - \int_{t\wedge g}^{t} \mathbb{1}_{\{S_s < a\}} \frac{\sigma S_s}{a - S_s} ds \\
&= W_t^{\mathbb{G}} + \int_0^{t\wedge g} \sigma \mathbb{1}_{\{S_s < a\}} ds - \int_{t\wedge g}^{t} \frac{\sigma S_s}{a - S_s} ds
\end{aligned}$$

- Let, $S^*_t = \sup\{S_s, s \le t\}$ and

$$\tau = \sup\{t \,:\, S_t = S^*_\infty\} = \sup\{t \,:\, S_t = S^*_t\}$$

Then, $Z_t = \frac{S_t}{S^*_t}$ and $dm_t = \frac{1}{S^*_t} dS_t$, therefore $\varphi_t = \frac{1}{S^*_t}$.

$$\begin{aligned} W_t &= W^{\mathbb{G}}_t + \int_0^{t\wedge\tau} \frac{d\langle W, m\rangle_s}{Z_s} - \int_{t\wedge\tau}^t \frac{d\langle W, m\rangle_s}{1 - Z_s} \\ &= W^{\mathbb{G}}_t + \int_0^{t\wedge\tau} \sigma ds - \int_{t\wedge\tau}^t \frac{\sigma S_s}{S^*_s - S_s} ds \end{aligned}$$

11.5 Example in a Poissonnian filtration

Let $dS_t = S_{t-}\psi dM_t,\ S_0 = 1$ with $\psi > 0$, where M is the compensated martingale of a Poisson process, i.e., $S_t = e^{-\ln(1+\psi)Y_t}$ where $Y_t := \dfrac{\lambda\psi}{\ln(1+\psi)}t - N_t$. Let τ be given by

$$\tau := \sup\{t :\ S_t \geq b\} = \sup\{t : Y_t \leq a\}.$$

where $0 < b < 1$. Then, the process

$$\varphi := \frac{\Psi(Y_- - a - 1)\mathbb{1}_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)\mathbb{1}_{\{Y_- \geq a\}} + \mathbb{1}_{\{Y_- < a+1\}} - \mathbb{1}_{\{Y_- < a\}}}{\psi S_-},$$

where

$$\Psi(x) = \mathbb{P}(T^x < \infty), \quad \text{with} \quad T^x = \inf\{t : x + Y_t < 0\}$$

is an arbitrage opportunity.

Proof On the one hand

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \Psi(Y_t - a)1\!\!1_{\{Y_t \geq a\}} + 1\!\!1_{\{Y_t < a\}} = 1 + 1\!\!1_{\{Y_t \geq a\}}\left(\Psi(Y_t - a) - 1\right).$$

On the other hand, setting $\theta = \dfrac{\mu}{\lambda} - 1$, one shows that the dual optional projection A^o of the process $1\!\!1_{[\tau,\infty)}$ equals

$$A^o = \frac{\theta}{1+\theta} \sum_n 1\!\!1_{[\vartheta_n,\infty)},$$

where ϑ_n is the sequence of $\mathbb{F}$-stopping times defined by $\vartheta_1 = \inf\{t > 0 : Y_t = a\}$ and $\vartheta_n = \inf\{t > \vartheta_{n-1} : Y_t = a\}$.

For any optional increasing process

$$\mathbb{E}(K_\tau) = \mathbb{E}(\sum \mathbb{1}_{\tau=\vartheta_n} K_{\vartheta_n}) = \mathbb{E}(\sum \mathbb{E}(\mathbb{1}_{\tau=\vartheta_n} | \mathcal{F}_{\vartheta_n}) K_{\vartheta_n})$$

and $\mathbb{E}(\mathbb{1}_{\tau=\vartheta_n} | \mathcal{F}_{\vartheta_n}) = \mathbb{P}(T^0 = \infty) = 1 - \Psi(0) = 1 - \frac{1}{1+\theta}$.

11.6 NA1 after τ

Let τ be honest.

Assume that $Z_\tau < 1$ and $\mathbb{F}$ is quasi-left-continuous. Then the following assertions are equivalent.
(a) The set $\{\widetilde{Z} = 1 > Z_-\}$ is evanescent.
(b) For every (bounded) X satisfying NA1$(\mathbb{F})$, $X - X^\tau$ satisfies NA1$(\mathbb{G})$.

12 Under equivalence Jacod's Hypothesis

Under equivalence Jacod's hypothesis, ($\mathcal{H}'$) hypothesis holds for $\mathbb{F}$ and $\mathbb{G}$: any $\mathbb{F}$ martingale, being a $\mathbb{G}$ adapted $\mathbb{F}^{\sigma(\tau)}$-semimartingale is a $\mathbb{G}$-semimartingale.

12.1 Canonical Decomposition in $\mathbb{G}$

Proposition 12.1. *Under (CA), any $(\mathbb{P}, \mathbb{F})$-local martingale X is a $(\mathbb{P}, \mathbb{G})$ semi-martingale with canonical decomposition*

$$X_t = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle X, Z\rangle_s}{Z_{s-}} + \int_{t\wedge\tau}^{t} \frac{d\langle X, p_{.}(\tau)\rangle_s}{p_{s-}(\tau)},$$

where $X^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-local martingale.

12.2 Predictable Representation Property

If $\mathbb{F}$ is a Brownian filtration, and (A) holds, every $X \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{G})$ can be represented as

$$X_t = X_0 + \int_0^t \Phi_s dW_s^{\mathbb{G}} + \int_0^t \Psi_s dM_s,$$

where $W^{\mathbb{G}}$ is the martingale part in the $\mathbb{G}$-canonical decomposition of W, M is the $(\mathbb{P}, \mathbb{G})$-compensated martingale associated with H and Φ, Ψ are $\mathbb{G}$-predictable.

12.3 Characterization of $(\mathbb{P}, \mathbb{G}$ martingales in terms of $(\mathbb{P}, \mathbb{F})$-martingales

Proposition 12.2. *A $\mathbb{G}$-adapted process $Y_t := \tilde{y}_t \mathbb{1}_{t<\tau} + \hat{y}_t(\tau)\mathbb{1}_{\tau \le t}, t \ge 0$, is a $(\mathbb{P}, \mathbb{G})$-martingale if and only if the following two conditions are satisfied*

(i) for ν-a.e u, $\big(\hat{y}_t(u)p_t(u), t \ge u\big)$ is a $(\mathbb{P}, \mathbb{F})$-martingale;

(ii) the process $y = (y_t, t \ge 0)$, given by

$$y_t := \mathbb{E}(Y_t|\mathcal{F}_t) = \tilde{y}_t G_t + \int_0^t \hat{y}_t(u)p_t(u)\nu(du) \ ,$$

is a $(\mathbb{P}, \mathbb{F})$-martingale.

PROOF: Assume, w.l.g., that $Y_t = \mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_t)$ for some $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale $Y^{(\tau)}$. Then $Y_t^{(\tau)} = y_t(\tau)$, where for ν-almost every $u \ge 0$ the process $\big(y_t(u)p_t(u), t \ge 0\big)$ is a $(\mathbb{P}, \mathbb{F})$-martingale. One then has

$$\mathbb{1}_{\tau \le t}\hat{y}_t(\tau) = \mathbb{1}_{\tau \le t}Y_t = \mathbb{1}_{\tau \le t}\mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_t) = \mathbb{E}(\mathbb{1}_{\tau \le t}Y_t^{(\tau)}|\mathcal{G}_t) = \mathbb{1}_{\tau \le t}y_t(\tau) \ ,$$

which implies, that for ν-almost every $u \le t$, the identity $y_t(u) = \hat{y}_t(u)$ holds $\mathbb{P}$-almost surely. So, *(i)* is proved.

Moreover, Y being a $(\mathbb{P}, \mathbb{G})$-martingale, its projection on the smaller filtration $\mathbb{F}$, namely the process y in (ii) is a $(\mathbb{P}, \mathbb{F})$-martingale.

Conversely, assuming *(i)* and *(ii)*, we verify that $\mathbb{E}(Y_t|\mathcal{G}_s) = Y_s$ for $s \leq t$. We start by noting that

$$\mathbb{E}(Y_t|\mathcal{G}_s) = 1\!\!1_{s<\tau} \frac{1}{G_s} \mathbb{E}(Y_t 1\!\!1_{s<\tau}|\mathcal{F}_s) + 1\!\!1_{\tau \leq s} \mathbb{E}(Y_t 1\!\!1_{\tau \leq s}|\mathcal{G}_s) \ . \tag{12.1}$$

We then compute the two conditional expectations the right-hand side:

$$\begin{aligned}
\mathbb{E}(Y_t 1\!\!1_{s<\tau}|\mathcal{F}_s) &= \mathbb{E}(Y_t|\mathcal{F}_s) - \mathbb{E}(Y_t 1\!\!1_{\tau \leq s}|\mathcal{F}_s) \\
&= \mathbb{E}(y_t|\mathcal{F}_s) - \mathbb{E}\big(\mathbb{E}(\hat{y}_t(\tau) 1\!\!1_{\tau \leq s}|\mathcal{F}_t)|\mathcal{F}_s\big) \\
&= y_s - \mathbb{E}\Big(\int_0^s \hat{y}_t(u) p_t(u) \nu(du) |\mathcal{F}_s\Big) \\
&= \tilde{y}_s G_s + \int_0^s \hat{y}_s(u) p_s(u) \nu(du) - \int_0^s \hat{y}_s(u) p_s(u) \nu(du) = \tilde{y}_s G_s \ ,
\end{aligned}$$

where we used Fubini's theorem and the condition *(i)* to obtain the next-to-last identity.

Also

$$\begin{aligned}
\mathbb{E}(Y_t \mathbb{1}_{\tau \leq s} | \mathcal{G}_s) &= \mathbb{E}(\hat{y}_t(\tau) \mathbb{1}_{\tau \leq s} | \mathcal{G}_s) = \mathbb{1}_{\tau \leq s} \frac{1}{p_s(\tau)} \mathbb{E}\big(\hat{y}_t(u) p_t(u) | \mathcal{F}_s\big)_{|u=\tau} \\
&= \mathbb{1}_{\tau \leq s} \frac{1}{p_s(\tau)} \hat{y}_s(\tau) p_s(\tau) = \mathbb{1}_{\tau \leq s} \hat{y}_s(\tau)
\end{aligned}$$

where the next-to-last identity holds in view of the condition *(ii)*. $\triangle$

13 Some References

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Thank you for your attention