Exercises

In this preliminary version of exercises, there are some changes of notation. The indicator process $1\!\!1_{\{\tau\le t\}}$ is denoted A or H, its dual (resp. predictable) projection is A^o (resp. A^p), its dual optional (resp. predictable projection is ${}^oA$ (resp. ${}^pA$). The filtration of A (resp. H) is $\mathbb{A}$ (resp. $\mathbb{H}$). The numbering may be different from the one in the notes.

1.1 Chapter 1

Exercise 1.1.1 Starting from a non continuous on right filtration $\mathbb{F}^0$, define the smallest right-continuous filtration $\mathbb{F}$ which contains $\mathbb{F}^0$.

Exercise 1.1.2 Prove that, for $A \in \mathcal{F}_\tau$, τ_A is a stopping time.

Exercise 1.1.3 Show that for an $\mathbb{F}$-stopping time τ, one has $\tau \in \mathcal{F}_{\tau-}$ and $\mathcal{F}_{\tau-} \subset \mathcal{F}_\tau$. Find an example where $\mathcal{F}_{\tau-} \neq \mathcal{F}_\tau$.

Exercise 1.1.4 Check that if $\mathbb{F} \subset \mathbb{G}$ and τ is an $\mathbb{F}$-stopping time, (resp. an $\mathbb{F}$-predictable stopping time) it is a $\mathbb{G}$-stopping time, (resp. $\mathbb{G}$-predictable stopping time). Give an example where τ is a $\mathbb{G}$-stopping time but not an $\mathbb{F}$-stopping time. Give an example where τ is a $\mathbb{G}$-predictable stopping time, and an $\mathbb{F}$-stopping time, but not a predictable $\mathbb{F}$-stopping time.

Exercise 1.1.5 Let B be a Brownian motion. Prove that $\exp(\lambda B_t - \frac{\lambda^2}{2}t)$ belongs to $(\mathcal{C}_0)$.

Exercise 1.1.6 Prove that a positive local martingale is a super-martingale.

Exercise 1.1.7 Let B be a Brownian motion. Prove that $W_t := \int_0^t (\mathrm{sgn} B_s) dB_s$ defines an $\mathbb{F}^B$ and an $\mathbb{F}^W$-Brownian motion.
Prove that $\beta_t := B_t - \int_0^t \frac{B_s}{s} ds$ defines a Brownian motion (in its own filtration) which is not a Brownian motion in $\mathbb{F}^B$.

Exercise 1.1.8 Let N be a Poisson process. Prove that for any $\theta \in [0,1]$,

$$N_t = \theta(N_t - \lambda t) + (1-\theta)N_t + \theta\lambda t = \mu_t + (1-\theta)N_t + \theta\lambda t$$

where μ is a martingale. For which θ is the finite variation process $(1-\theta)N_t + \theta\lambda t$ a predictable process ?

Exercise 1.1.9 Let τ be a random time. Prove that τ is a $\mathbb{H}$-stopping time, where $\mathbb{H}$ is the natural filtration of $H_t = 1\!\!1_{\{\tau\le t\}}$, and that τ is a $\mathbb{G}$ stopping time, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, for any filtration $\mathbb{F}$.

Exercise 1.1.10 Prove that, if M is a $\mathbb{K}$-martingale and $\mathbb{F} \subset \mathbb{K}$, then $\widehat{M}$ defined as $\widehat{M}_t = \mathbb{E}(M_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale.

Exercise 1.1.11 Prove that, if $\mathbb{K} = \mathbb{F} \vee \widetilde{\mathbb{F}}$ where $\widetilde{\mathbb{F}}$ is independent of $\mathbb{F}$, then any $\mathbb{F}$-martingale remains a $\mathbb{K}$-martingale.

Exercise 1.1.12 Let M a càdlàg martingale. Prove that its predictable projection is M_{t-}.

Exercise 1.1.13 Let X be a measurable process (not necessarily $\mathbb{F}$-adapted) such that $\mathbb{E}(\int_0^t |X_s|ds) < \infty$ and define $Y_t = \int_0^t X_s ds$. Prove that $M_t := {}^{o,\mathbb{F}}Y_t - \int_0^t {}^{o,\mathbb{F}}X_s ds$ is an $\mathbb{F}$-martingale. In particular, for any (bounded) process a (not necessarily $\mathbb{F}$-adapted)

$$\mathbb{E}(\int_0^t a_u du|\mathcal{F}_t) - \int_0^t \mathbb{E}(a_u|\mathcal{F}_u)du$$

is an $\mathbb{F}$-martingale.

Exercise 1.1.14 Prove that if X is bounded and Y predictable, then ${}^p(YX) = Y\,{}^pX$

Exercise 1.1.15 Prove that the $\mathbb{K}$-dual predictable projection of $\int_0^t f(B_s^{(\nu)})ds$ where $\mathcal{K}_t = \sigma(|B_s^{(\nu)}|, s \leq t)$ and $B_t^{(\nu)} = B_t + \nu t$ for a Brownian motion B, is $\int_0^t \mathbb{E}(f(B_s^{(\nu)})|\mathcal{K}_s)ds$ and that

$$\mathbb{E}(f(B_s^{(\nu)})|\mathcal{K}_s) = \frac{f(B_s^{(\nu)})e^{\nu B_s^{(\nu)}} + f(-B_s^{(\nu)})e^{-\nu B_s^{(\nu)}}}{2\cosh(\nu B_s^{(\nu)})}\,.$$

Exercise 1.1.16 Prove that, if $(\alpha_s, s \geq 0)$ is an increasing $\mathbb{F}$-predictable process and X a positive measurable process, then

$$\left(\int_0^{\cdot} X_s d\alpha_s\right)_t^p = \int_0^t {}^pX_s d\alpha_s$$

Exercise 1.1.17 Prove that if X and Y are continuous, $\langle X, Y\rangle = [X, Y]$.
Prove that if M is the compensated martingale of a Poisson process with intensity λ, $[M] = N$ and $\langle M\rangle_t = \lambda t$.

Exercise 1.1.18 Give an example of random time τ where A^p and A^o are different.

Exercise 1.1.19 Let B be a Brownian motion, $\mathbb{F}$ its natural filtration and $B_t^* = \sup_{s\leq t} B_s$. Prove that, for $t < 1$,

$$\mathbb{E}(f(B_1^*)|\mathcal{F}_t) = F(1-t, B_t, B_t^*)$$

with

$$F(s,a,b) \quad = \quad \sqrt{\frac{2}{\pi s}}\left(f(b)\int_0^{b-a} e^{-u^2/(2s)}du + \int_b^{\infty} f(u)\exp\left(-\frac{(u-a)^2}{2s}\right)du\right).$$

Exercise 1.1.20 Let φ be a C^1 function, B a Brownian motion and $B_t^* = \sup_{s\leq t} B_s$. Prove that the process

$$\varphi(B_t^*) - (B_t^* - B_t)\varphi'(B_t^*)$$

is a local martingale.

Exercise 1.1.21 A Useful Lemma: Doob's Maximal Identity.
Let X be a positive continuous martingale such that $X_0 = x$ and $\lim_{t\to\infty} X_t = 0$. Prove that

$$\mathbb{P}(\sup_{t\geq 0} X_t > a) = \left(\frac{x}{a}\right) \wedge 1 \tag{1.1.1}$$

and $\sup_{t\geq 0} X_t \overset{\text{law}}{=} \dfrac{x}{U}$ where U is a random variable with a uniform law on $[0, 1]$.

Exercise 1.1.22 Show that if $X_n, n \geq 1$ is an integrable sequence of random variables, viewed as a discrete time process, adapted to some filtration $\mathbb{F}$, then, there exists a martingale M and a predictable process A such that $X_n = M_n + A_n$.

SOLUTIONS

Exercise 1.1.1: Given the non right-continuous filtration $\mathbb{F}^0 = (\mathcal{F}_t^0)_{t\geq 0}$, we define the smallest right-continuous filtration $\mathbb{F}$ containing $\mathbb{F}^0$ as follows: for any $t \geq 0$

$$\mathcal{F}_t := \bigcap_{\epsilon>0} \mathcal{F}^0_{t+\epsilon}.$$

Indeed, it is by definition right continuous and it satisfies $\mathcal{F}_t \supseteq \mathcal{F}_t^0$ for any t.

Exercise 1.1.3: The stopping time τ is a jump of the predictable process $1\!\!1_{\{\tau<t\}}$, hence belongs to $\mathcal{F}_{\tau-}$. If τ is the first jump of a compound Poisson process $X_t = \sum_{n=1}^{N_t} Y_n$, the r.v. Y_1 belongs to $\mathcal{F}_\tau^X$ but not to $\mathcal{F}_{\tau-}^X$

Exercise 1.1.4: The first time τ where a Poisson process X jumps is not $\mathbb{F}^X$ predictable, but it is predictable in $\mathbb{F}^X$ initially enlarged with τ.

Exercise 1.1.7: The process W is a continuous $\mathbb{F}^B$-martingale with predictable bracket t. The process β is a Gaussian process with mean zero and $\mathbb{E}(\beta_t\beta_s) = t \wedge s$. Note that β is NOT an $\mathbb{F}^B$-Brownian motion. The filtration $\mathbb{F}^\beta$ is strictly smaller than $\mathbb{F}^B$. See [9] for details.

Exercise 1.1.8: It is known that$(M_t := N_t - \lambda t)_t$ is an $\mathbb{F}^N$ martingale, where $\mathbb{F}^N$ is the natural filtration of N. The decomposition stated in the exercise is obvious. From $\mu_t = (1-\theta)M_t$, the process μ is a martingale. The decomposition $N = \mu_t + C_t$ where $C_t = (1-\theta)N_t + \theta\lambda t$ is a decomposition of N as a martingale part and a bounded variation part, but since N is not predictable, C is not predictable (TYhe non predictability of N this can be proved by contradiction, see e.g. Exercise 8.2.2.3 in [jyc:3m] or Liptser-Shiryaev II, Section 18.4).

Exercise 1.1.9: By definition of natural filtration, for every $t \geq 0$, H_t is adapted to $\mathcal{H}_t$, where $(\mathcal{H}_t)_t = \mathbb{H}$, so that $\{\tau \leq t\} \in \mathcal{H}_t$ and τ is an $\mathbb{H}$-stopping time.
If $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t, \forall t \geq 0$, then $\{\tau \leq t\} \in \mathcal{G}_t$ and τ is also a $\mathbb{G}$-stopping time, for any filtration $\mathbb{F}$.

Exercise 1.1.10: It follows immediately by using the properties of conditional expectation: for $s \leq t$ and given that $\mathcal{F}_s \subseteq \mathcal{G}_t$, we have

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}\{\mathbb{E}[M_t|\mathcal{G}_s]|\mathcal{F}_s\} = \mathbb{E}[M_s|\mathcal{F}_s] = M_s,$$

if M is $\mathbb{F}$-adapted.

Analogously, we have, for $s \leq t$,

$$\mathbb{E}[\widehat{M_t}|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[M_s|\mathcal{F}_s] = \widehat{M}_s.$$

Exercise 1.1.11: We prove that for $s \leq t$, $\mathbb{E}[X_t 1\!\!1_G] = \mathbb{E}[X_s 1\!\!1_A]$, where X is an $\mathbb{F}$-martingale and $G \in \mathcal{G}_s$ of the form $G = G_1G_2$, where $G_1 \in \mathcal{F}_s$ and $G_2 \in \widetilde{\mathcal{F}_s}$. We then have, from the assumed independence

$$\begin{aligned}\mathbb{E}[X_t 1\!\!1_G] &= \mathbb{E}[1\!\!1_{G_2}\mathbb{E}[X_t 1\!\!1_{G_1}|\widetilde{\mathcal{F}_s}]] = \mathbb{E}[1\!\!1_{G_2}\mathbb{E}[X_t 1\!\!1_{G_1}]] \\ &= \mathbb{E}[1\!\!1_{G_2}1\!\!1_{G_1}\mathbb{E}[X_t|\mathcal{F}_s]] = \mathbb{E}[1\!\!1_G X_s].\end{aligned}$$

We conclude applying the Monotone Class Theorem (MCT).

For the second part of the exercise, we recall that the Martingale Representation Theorem, in the case of Brownian filtration, states that any $(\mathbb{P}, \mathbb{F})$-martingale M can be written as

$$M_t = M_0 + \int_0^t \xi_s dWs,$$

for some $\mathbb{F}-$predictable process ξ (such that the above stochastic integral is well defined). From Girsanov's Theorem, furthermore, we know that if the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$, in the filtration $\mathbb{F}$, is Z, satisfying $Z_0 = 1$ and $dZ_t = Z_t \eta_t dW_t$ (η $\mathbb{F}-$adapted), then

$$W_t^* := W_t - \int_0^t \eta_s ds, \quad t \geq 0,$$

is a $(\mathbb{Q}, \mathbb{F})$-martingale. Because of the independence of $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ under $\mathbb{Q}$, from the first part of the exercise it follows that W^* is a $(\mathbb{Q}, \mathbb{G})$-martingale, too. To conclude, we apply once more Girsanov theorem to pass, from measure $\mathbb{Q}$ to measure $\mathbb{P}$ in the filtration $\mathbb{G}$ and we immediately obtain the $(\mathbb{P}, \mathbb{G})$ semi-martingale decomposition of M.

Exercise 1.1.12: By definition, we are looking for the predictable process ${}^{(p)}M$ that satisfies

$$\mathbb{E}(M_\tau 1\!\!1_{\tau<\infty} | \mathcal{F}_{\tau_-}) = {}^{(p)} M_\tau 1\!\!1_{\tau<\infty},$$

for any $\mathbb{F}-$predictable stopping time. A known result (see e.g. Dellacherie-Meyer, Vol. II, Ch. VI, Th. 32) states that given a càdlàg local martingale and a predictable stopping time τ we have

$$\mathbb{E}(M_\tau 1\!\!1_{\tau<\infty} | \mathcal{F}_{\tau_-}) = M_{\tau_-} 1\!\!1_{\tau<\infty}, \quad \text{a.s.}$$

and this gives us the desired result.

Exercise 1.1.13: Let us compute the conditional expectation $\mathbb{E}(M_t|\mathcal{F}_s)$, for $s \leq t$, where $(M_t)_t := \Big(\mathbb{E}(\int_0^t a_u du|\mathcal{F}_t) - \int_0^t \mathbb{E}(a_u|\mathcal{F}_u)du\Big)_t$.

$$\begin{aligned}
\mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}\left(\mathbb{E}(\int_0^t a_u du|\mathcal{F}_t) - \int_0^t \mathbb{E}(a_u|\mathcal{F}_u)du|\mathcal{F}_s\right) \\
&= \mathbb{E}(\int_0^t a_u du|\mathcal{F}_s) - \mathbb{E}\left(\int_0^s \mathbb{E}(a_u|\mathcal{F}_u)du + \int_s^t \mathbb{E}(a_u|\mathcal{F}_u)du|\mathcal{F}_s\right) \\
&= \mathbb{E}(\int_0^s a_u du|\mathcal{F}_s) + \mathbb{E}(\int_s^t a_u du|\mathcal{F}_s) - \int_0^s \mathbb{E}(a_u|\mathcal{F}_u)du - \mathbb{E}\left(\int_s^t \mathbb{E}(a_u|\mathcal{F}_u)du|\mathcal{F}_s\right) \\
&= M_s + \mathbb{E}(\int_s^t a_u du|\mathcal{F}_s) - \int_s^t \mathbb{E}(a_u|\mathcal{F}_s)du = M_s.
\end{aligned}$$

Other proof: We consider $s \leq t$ and we compute the following conditional expectation

$$\begin{aligned}
&\mathbb{E}\left({}^oY_t - \int_0^t {}^oX_u du\big|\mathcal{F}_s\right) \stackrel{\text{def}}{=} \mathbb{E}\left({}^o\left(\int_0^t X_u du\right) - \int_0^t {}^oX_u du\big|\mathcal{F}_s\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(\int_0^t X_u du|\mathcal{F}_t\right) - \int_0^t \mathbb{E}\left(X_u|\mathcal{F}_u\right)du\big|\mathcal{F}_s\right),
\end{aligned}$$

where we have used Theorem VI.7.10 in Rogers-Williams (1994), that states that if X is a bounded right-continuous process, then Z is indistinguishable from ${}^oX$ if and only if Z is an adapted right-continuous process such that $Z_t = \mathbb{E}[X_t|\mathcal{F}_t]$. We then find (notice that by setting the problem we have implicitly assumed that the above optional projections exist)

$$\begin{aligned}
&\mathbb{E}\left({}^oY_t - \int_0^t {}^oX_u du\big|\mathcal{F}_s\right) = \mathbb{E}\left(\int_0^s X_u du|\mathcal{F}_s\right) - \int_0^s \mathbb{E}\left(X_u|\mathcal{F}_u\right)du \\
&+\mathbb{E}\left(\mathbb{E}\left(\int_s^t X_u du|\mathcal{F}_t\right) - \int_s^t \mathbb{E}\left(X_u|\mathcal{F}_u\right)du\big|\mathcal{F}_s\right) \\
&={}^o Y_s - \int_0^s {}^oX_u du + \mathbb{E}\left(\int_s^t X_u du|\mathcal{F}_s\right) - \int_s^t \mathbb{E}\left(X_u|\mathcal{F}_s\right)du \\
&={}^o Y_s - \int_0^s {}^oX_u du.
\end{aligned}$$

Exercise 1.1.14: By definition, given a predictable stopping time τ, we look for a process ${}^{(p)}(YX)$ such that

$$\mathbb{E}(Y_\tau X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}) = {}^{(p)}(YX)\mathbb{1}_{\{\tau<\infty\}}.$$

By looking carefully at the definition of the $\sigma-$algebra $\mathcal{F}_{\tau-}$, it is clear that, for a predictable stopping time τ, the random variable $Y_\tau \mathbb{1}_{\{\tau<\infty\}}$ is $\mathcal{F}_{\tau-}-$measurable and we have

$$\mathbb{E}(Y_\tau X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}) = {}^{(p)}(X_\tau)Y_\tau \mathbb{1}_{\{\tau<\infty\}},$$

the conclusion follows.

Exercise 1.1.15: By using the results provided in the Example at page 10 of the notes, we have, applying Girsanov's theorem to pass from B_s to $B_s+\nu s$, for any s,

$$\begin{aligned}\mathbb{E}(f(B_s+\nu s)|\mathcal{F}_s^{|B|}) &= \frac{\mathbb{E}(f(B_s)e^{\nu B_s-\frac{\nu^2 s}{2}}|\mathcal{F}_s^{|B|})}{\mathbb{E}(e^{\nu B_s-\frac{\nu^2 s}{2}}|\mathcal{F}_s^{|B|})} = \frac{f(|B_s|)e^{\nu|B_s|}+f(-|B_s|)e^{-\nu|B_s|}}{2\cosh(\nu|B_s|)}\\ &= \ ?\end{aligned}$$

Once proven that the above projection exists, we know that $\left(\int_0^t f(B_s^{(\nu)})ds\right)^{(p)} = \int_0^t {}^{(p)}f(B_s^{(\nu)})ds$ and as predictable projection of the integrand we take the continuous process $\mathbb{E}(f(B_s^{(\nu)})|\mathcal{G}_s^{(\nu)})$.

Exercise 1.1.16: We set $A_t := \int_0^t X_s d\alpha_s$, so that $dA_t = X_t d\alpha_t, t\geq 0$ and, given a positive $\mathbb{F}$-predictable process Y, we consider the following stochastic integral, looking for the integrable increasing $\mathbb{F}$-predictable process A^p such that

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA_s^p\right).$$

By hypothesis, $(\alpha_s, s\geq 0)$ is an increasing predictable process and we have to consider the predictable projection of X and

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s X_s d\alpha_s\right) = \mathbb{E}\left(\int_0^\infty Y_s\, {}^{p}X_s d\alpha_s\right).$$

Exercise 1.1.17: If X and Y are continuous, then $\Delta[X,Y]_t = \Delta X_t \Delta Y_t = 0$ and the covariation process is continuous and equal to $< X, Y >$.

Let us recall a general result: if X is a stochastic process with independent, stationary increments (in French, "P.A.I.S."), satisfying $\mathbb{E}[|X_t|]<\infty, \forall t$ and $\mathbb{E}[|X_t^2|]<\infty,\forall t$, then

$$M_t := X_t - \mathbb{E}(X_t)$$

and

$$M_t^2 - \mathbb{E}(X_t^2)$$

are $\mathbb{F}^X-$martingales, where $\mathbb{F}^X$ denotes the natural filtration associated to X.

In particular, in the case of a Poisson process N with deterministic constant intensity λ, $M_t := N_t - \lambda t$ and $M_t^2 - \lambda t$ are $\mathbb{F}^N$ martingales. From the definition of predictable quadratic variation process of M we then find that $\langle M\rangle_t = \lambda t$ (the deterministic process $(\lambda t)_t$ is predictable). We then find $(\Delta N)^2 = \Delta N$ and $M_t^2 - N_t = M_t^2 - \lambda t + M_t$ is a martingale.

Exercise 1.1.18: Let τ be the first jump time of a Poisson process and M the martingale $M_t = N_t - \lambda t$. Then $Z_t = \mathbb{1}_{t<\tau} = 1 - \mathbb{1}_{\tau\leq t} = 1 - A_t^{(p)}$ whereas the Doob-Meyer decomposition of Z is $Z_t = M_{t\wedge\tau} - \lambda(t\wedge\tau)$

Exercise 1.1.19: First of all we prove that effectively $\sup_{s\leq 1} B_s = \sup_{s\leq t} B_s \vee (\widehat{M}_{1-t} + B_t) = M_t \vee (\widehat{M}_{1-t} + B_t)$, by recalling that, given a Brownian motion B, the process $(B_{t+s} - B_s)_t =: (\hat{B}_t)_t$

denotes another Brownian motion. Then, exploiting the independence property of the increments of a Brownian motion and the measurability of M_t and B_t with respect to $\mathcal{F}_t$, we have

$$\mathbb{E}(f(M_1)|\mathcal{F}_t) = \mathbb{E}\left(f(M_t \vee (\widehat{M}_{1-t} + B_t))|\mathcal{F}_t\right) = \mathbb{E}\left(f(b \vee (\widehat{M}_{1-t} + a))\right)_{|a=B_t, b=M_t}.$$

The result is an immediate consequence of the fact that the random variable $\widehat{M}_{1-t}$ has same law of $|\widehat{B}_{1-t}|$.

Exercise 1.1.20: As a first step we assume that φ is C^2. Then, from integration by parts and using the fact that B^* is increasing

$$(B_t^* - B_t)\varphi'(B_t^*) = \int_0^t \varphi'(B_s^*)\, d(B_s^* - B_s) + \int_0^t (B_s^* - B_s)\varphi''(B_s^*)dB_s^*.$$

Now, we note that $\int_0^t (B_s^* - B_s)\varphi''(B_s^*)dB_s^* = 0$, since dB^* is carried by $\{s : B_s^* = B_s\}$, and that $\int_0^t \varphi'(B_s^*)dB_s^* = \varphi(B_t^*) - \varphi(0)$. The result follows. The general case is obtained using the Monotone Class Theorem.

Exercise 1.1.21: (i) Let us consider the case $x < a$ and introduce $T_a := \inf\{t \geq 0 : M_t \geq a\}$. By applying Doob's optional sampling Theorem to the martingale M and to the finite stopping time $T_a \wedge t$, we find

$$\begin{aligned} x &= \mathbb{E}(M_{T_a \wedge t}) = a\mathbb{P}(T_a \leq t) + \mathbb{E}(M_t \mathbb{1}_{\{T_a > t\}}) \\ &= a\mathbb{P}\left(\sup_{0 \leq s \leq t} M_s \geq a\right) + \mathbb{E}\left(M_t \mathbb{1}_{\{T_a > t\}}\right). \end{aligned} \tag{1.1.2}$$

By letting t go to infinity, recalling that, by hypothesis, $\lim_{t\to\infty} M_t = 0$ and thanks to the dominated convergence theorem, we finally find

$$\mathbb{P}(\sup_{0 \leq s \leq +\infty} M_s \geq a) = \mathbb{P}(\sup_t M_t \geq a) = \frac{x}{a}.$$

In the case when $x \leq a$, evidently $\mathbb{P}(T_a \leq t) = 1$ and the result follows. Furthermore, $\mathbb{P}(\frac{x}{U} \geq a) = \mathbb{P}(U \leq \frac{x}{a}) = \left(\frac{x}{a}\right) \wedge 1$.

(ii) We consider Equation (1.1.2) in the case when $x = 1$, namely

$$1 = \mathbb{E}(M_{T_a \wedge t}) = a\mathbb{P}\left(\sup_{0 \leq s \leq t} M_s \geq a\right) + \mathbb{E}\left(M_t \mathbb{1}_{\{T_a > t\}}\right)$$

and we let t go to infinity, obtaining

$$1 = a\mathbb{P}\left(\sup_t M_t \geq a\right) + \mathbb{E}\left(M_\infty \mathbb{1}_{\{T_a > +\infty\}}\right).$$

If we know that $\sup M_t \overset{law}{=} \frac{x}{U} = \frac{1}{U}$, choosing $a > 1$ we have

$$1 = a\frac{1}{a} + \mathbb{E}\left(M_\infty \mathbb{1}_{\{T_a > +\infty\}}\right),$$

meaning that $\mathbb{E}\left(M_\infty \mathbb{1}_{\{T_a > +\infty\}}\right) = 0$. Now, $\mathbb{P}(T_a > +\infty) = \mathbb{P}(\sup_t M_t < a) = 1 - \left(\frac{1}{a} \wedge 1\right) = 1 - \frac{1}{a}$ and for $a \to +\infty$ we have $\mathbb{P}(T_a > +\infty) \to 1$ and it follows that $\mathbb{E}(M_\infty) = 0$ and, being M positive, $M_\infty = 0$ a.s.

Exercise 1.1.13: Let ϑ a bounded stopping time. Then,

$$\mathbb{E}(M_\vartheta) = \mathbb{E}\int \mathbb{1}_{s<\theta}(X_s - {}^oX_s)ds = \int \mathbb{E}(\mathbb{1}_{s<\theta}X_s - {}^o(\mathbb{1}_{s<\theta}X_s))ds = 0$$

1.2 Chapter 2

In this section, τ is a random time admitting the cumulative distribution function equal to F which is supposed to satisfy $F(t) < 1$ and to be continuous, $\Gamma = -\ln(1-F)$, and $\mathbb{G}$ is the filtration $\mathbb{F}$ progressively enlarged with τ.

Exercise 1.2.1 Assume that Y is $\mathcal{H}_\infty$-measurable, so that $Y = h(\tau)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$. Prove that

$$\mathbb{E}(Y|\mathcal{H}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + \frac{1}{1-F(t)}1\!\!1_{\{t<\tau\}}\int_t^\infty h(u)\,dF(u). \tag{1.2.1}$$

Prove that

$$\mathbb{E}(Y|\mathcal{H}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + 1\!\!1_{\{t<\tau\}}\int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\,d\Gamma(u)\,.$$

Find a predictable process φ so that $dY_t = \varphi_t dM_t$.

Exercise 1.2.2 a) Prove that the process $L_t := 1\!\!1_{\{\tau>t\}}\exp(\Gamma(t))$ is an $\mathbb{H}$-martingale and

$$L_t = 1 - \int_{]0,t]} L_{u-}dM_u \tag{1.2.2}$$

In particular, for $t < T$,

$$\mathbb{E}(1\!\!1_{\{\tau>T\}}|\mathcal{H}_t) = 1\!\!1_{\{\tau>t\}}\,\exp(-\Gamma(t))\,.$$

b) Let $d\mathbb{Q}_{|\mathcal{H}_t} = L_t d\mathbb{P}_{|\mathcal{H}_t}$. Prove that $\mathbb{Q}(\tau \le t) = 0$.

Exercise 1.2.3 a) Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a (bounded) Borel measurable function. Prove that the process

$$Y_t := \exp\left(1\!\!1_{\{\tau\le t\}}h(\tau)\right) - \int_0^{t\wedge\tau}(e^{h(u)}-1)\,d\Gamma(u) \tag{1.2.3}$$

is an $\mathbb{H}$-martingale. Find a predictable process φ such that

$$dY_t = \varphi_t dM_t$$

Exercise 1.2.4 (i) Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a (bounded) Borel measurable function. Prove that the process

$$Y_t := \exp\left(1\!\!1_{\{\tau\le t\}}h(\tau)\right) - \int_0^{t\wedge\tau}(e^{h(u)}-1)\,d\Gamma(u)$$

is an $\mathbb{A}$-martingale. Find an $\mathbb{A}$-predictable process φ such that $dY_t = \varphi_t dM_t$.
(ii) Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Prove that the process

$$Y_t := (1 + 1\!\!1_{\{\tau\le t\}}h(\tau))\exp\Big(-\int_0^{t\wedge\tau}h(u)\,d\Gamma(u)\Big)\,.$$

is an $\mathbb{A}$-martingale. Find an $\mathbb{A}$-predictable process φ such that $dY_t = \varphi_t dM_t$. Give a condition on h so that Y is positive. In that case, find an $\mathbb{A}$-predictable process ψ such that $dY_t = Y_{t-}\psi_t dM_t$.

Exercise 1.2.5 Let B be a Brownian motion and $\tau = \inf\{t : B_t = a\}$. Find the compensator of τ in the progressive enlargement of $\mathbb{F}^B$- with τ and the $\mathbb{A}$-compensator of τ, where $\mathbb{A}$ is the natural filtration of the process A.

Exercise 1.2.6 In this exercise, F is only continuous on right, and $F(t-)$ is the left limit of F at point t. Prove that the process $(M_t, t \geq 0)$ defined as

$$M_t = H_t - \int_0^{\tau \wedge t} \frac{dF(s)}{1 - F(s-)} = H_t - \int_0^t (1 - H_s) \frac{dF(s)}{1 - F(s-)}$$

is an $\mathbb{H}$-martingale.

Exercise 1.2.7 Prove that τ is independent of $\mathcal{F}_\infty$ if and only if λ is a deterministic function.

Exercise 1.2.8 Assume that

$$dS_t = S_t((r - \delta)dt + \sigma dB_t),\ S_0 = 1$$

where B is a Brownian motion and let $\tau = \inf\{t \,:\, S_t \leq \alpha\}$, with $\alpha < 1$. Define $\mathbb{K} = (\mathcal{K}_t, t \geq 0)$ as the filtration generated by the observations of S at given times $t_1, \ldots, t_n$ with $t_n \leq t < t_{n+1}$, that is, $\mathcal{H}_t = \sigma(S_s, s \leq t_n)$ for $t_n \leq t < t_{n+1}$. Compute the $\mathbb{K}$-intensity rate of τ.

Exercise 1.2.9 Prove that, in a Cox model, τ is independent of $\mathcal{F}_\infty$ if and only if λ is a deterministic function. Prove that $\mathbb{A}$ is, in general, not immersed in $\mathbb{G}$. Prove that, if λ is deterministic, $\mathbb{A}$ is immersed in $\mathbb{G}$.

Exercise 1.2.10 Write the risk-neutral dynamics of the price of the recovery for a general interest rate r.

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Exercise 1.2.4: (i) Noting that

$$\exp\left(1\!\!1_{\{\tau \leq t\}} h(\tau)\right) = 1\!\!1_{\{\tau \leq t\}}(e^{h(\tau)} - 1) + 1 = \int_0^t (e^{h(s)} - 1) dA_s + 1$$

the martingale property is obtained from Proposition 2.1.5. It follows that $dY_t = (e^{h(t)} - 1)dM_t$.
(ii) In a first step, we establish that

$$Y_t = \exp\Big(-\int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big) = + \int_{]0,t]} h(u) \exp\Big(-\int_0^{t\wedge\tau} h(s)\, d\Gamma(s)\Big)\, dA_u.$$

Using Itô's formula, we obtain

$$\begin{aligned} dY_t &= \exp\Big(-\int_0^t (1 - A_u) h(u)\, d\Gamma(u)\Big)\big(h(t)\, dA_t - (1 - A_t) h(t)\, d\Gamma(t)\big) \\ &= h(t) \exp\Big(-\int_0^t (1 - A_u) h(u)\, d\Gamma(u)\Big)\, dM_t. \end{aligned}$$

This shows that $\widehat{M}^h$ is a local $\mathbb{H}$-martingale. If $h > 1$ is is a positive local martingale. It can be checked directly that $\mathbb{E}(Y_t) = 1$. Hence the process Y is indeed an $\mathbb{A}$-martingale.

Exercise 1.2.5: The $\mathbb{F}^B$-compensator of τ is the predictable process $1\!\!1_{\{\tau \leq t\}}$, the $\mathbb{A}$-compensator of τ is $\int_0^{t\wedge\tau} \frac{f(s)}{\mathbb{P}(\tau > s)} ds$ where f is the density of τ.

Exercise 1.2.8: It suffices to compute

$$Z_t = \mathbb{P}(\tau > t | \mathcal{K}_t) = 1\!\!1_{\{\tau > t_n\}} \Phi(t - t_n, \alpha - S_{t_n})$$

where $\Phi(t, z) = \mathbb{P}(\inf_{s \leq t} S_s > z)$ and $t_n \leq t < t_{n+1}$, and then apply Itô's formula to find the Doob-Meyer decomposition of Z. See [3] for details.

Exercise 1.2.9 If τ is independent of $\mathcal{F}_\infty$, then $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t) = e^{-\Lambda_t}$ is deterministic. Conversely, if Λ is deterministic, $\mathbb{P}(\tau > t|\mathcal{F}_\infty)$ is deterministic, hence the independence.
In general the $\mathbb{A}$-martingale $A_t - \int_0^{t\wedge\tau} \frac{f(s)}{\mathbb{P}(\tau > s)} ds$ is not a $\mathbb{G}$ martingale. In a Cox process, if λ is deterministic, the independence of τ and $\mathcal{F}_\infty$ implies that $\mathbb{A}$ is immersed in $\mathbb{G}$.

1.3 Chapter 3

Exercise 1.3.1 Prove that $\mathbb{H}$ is, in general, not immersed in $\mathbb{G}$. Prove that, if λ is deterministic, $\mathbb{H}$ is immersed in $\mathbb{G}$.

Exercise 1.3.2 Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$ and that W is an $\mathbb{F}$-Brownian motion. Prove that W is a $\mathbb{G}$-Brownian motion without using the bracket.

Exercise 1.3.3 Prove that, if $\mathbb{F}$ is immersed in $\mathbb{K}$, then, for any t, $\mathcal{F}_t = \mathcal{K}_t \cap \mathcal{F}_\infty$.

Exercise 1.3.4 Show that, if $\tau \in \mathcal{F}_\infty$, immersion holds between $\mathbb{F}$ and $\mathbb{F} \vee \mathbb{H}$ where $\mathbb{H}$ is generated by $H_t = 1\!\!1_{\tau \leq t}$ if and only if τ is an $\mathbb{F}$-stopping time.

Exercise 1.3.5 Prove that, if $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$ and if $\mathbb{Q}$ is a probability equivalent to $\mathbb{P}$, then, any $(\mathbb{Q}, \mathbb{F})$-semi-martingale is a $(\mathbb{Q}, \mathbb{G})$-semi-martingale. Let

$$\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}; \quad \mathbb{Q}|_{\mathcal{F}_t} = \ell_t \mathbb{P}|_{\mathcal{F}_t} .$$

and X be a $(\mathbb{Q}, \mathbb{F})$ martingale. Assuming that $\mathbb{F}$ is a Brownian filtration and that L is continuous, prove that

$$X_t + \int_0^t \left(\frac{1}{\ell_s} d\langle X, \ell \rangle_s - \frac{1}{L_s} d\langle X, L \rangle_s \right)$$

is a $(\mathbb{G}, \mathbb{Q})$ martingale.
In a general case, prove that

$$X_t + \int_0^t \frac{L_{s-}}{L_s} \left(\frac{1}{\ell_{s-}} d[X, \ell]_s - \frac{1}{L_{s-}} d[X, L]_s \right)$$

is a $(\mathbb{G}, \mathbb{Q})$ martingale. See Jeulin and Yor [5].

Exercise 1.3.6 Let $\mathbb{F} \subset \mathbb{K}$ and $\mathbb{P}$ be a probability measure. Let L be a positive $(\mathbb{F}, \mathbb{P})$-martingale with $L_0 = 1$ and define

$$\mathbb{Q}|_{\mathcal{K}_t} = L_t \mathbb{P}|_{\mathcal{K}_t}; \quad \mathbb{Q}|_{\mathcal{F}_t} = \ell_t \mathbb{P}|_{\mathcal{F}_t} .$$

Prove that $\mathbb{F} \hookrightarrow \mathbb{K}$ under $\mathbb{Q}$ if and only if:

$$\forall T, \forall X \geq 0, \; X \in \mathcal{F}_T, \forall t < T, \qquad \frac{\mathbb{E}_{\mathbb{P}}(X L_T | \mathcal{K}_t)}{L_t} = \frac{\mathbb{E}_{\mathbb{P}}(X \ell_T | \mathcal{F}_t)}{\ell_t}$$

Exercise 1.3.7 Assume that $\mathbb{F}$ is immersed in $\widetilde{\mathbb{F}}$ and τ is an $\widetilde{\mathbb{F}}$ stopping time. Prove that any $\mathbb{F}$ is immersed in $\mathbb{G}$.

Exercise 1.3.8 Assume that $\mathcal{F}_t^{(L)} = \mathcal{F}_t \vee \sigma(L)$ where L is a random variable. Find under which conditions on L, immersion property holds.

Exercise 1.3.9 Construct an example where some $\mathbb{F}$-martingales are $\mathbb{G}$-martingales, but not all $\mathbb{F}$ martingales are $\mathbb{G}$-martingales.

Exercise 1.3.10 Assume that $\mathbb{F} \subset \widetilde{\mathbb{G}}$ where $(\mathcal{H})$ holds for $\mathbb{F}$ and $\widetilde{\mathbb{G}}$.
a) Let τ be a $\widetilde{\mathbb{G}}$-stopping time. Prove that $(\mathcal{H})$ holds for $\mathbb{F}$ and $\mathbb{F}^\tau = \mathbb{F} \vee \mathbb{H}$ where $\mathcal{H}_t = \sigma(\tau \wedge t)$.
b) Let $\mathbb{G}$ be such that $\mathbb{F} \subset \mathbb{G} \subset \widetilde{\mathbb{G}}$. Prove that $\mathbb{F}$ be immersed in $\mathbb{G}$.

Exercise 1.3.11 Assume that $\mathcal{F}_t^{(\tau)} = \mathcal{F}_t \vee \sigma(\tau)$ where τ is a positive random variable, and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(\tau \wedge t)$. Find under which conditions on τ the filtration $\mathbb{G}$ is immersed in $\mathbb{F}^{(\tau)}$.

Exercise 1.3.12 Prove that in a Cox model, immersion holds.

Exercise 1.3.13 Prove that if $\mathbb{H}$ and $\mathbb{F}$ are immersed in $\mathbb{G}$, and if any $\mathbb{F}$ martingale is continuous, then τ and $\mathcal{F}_\infty$ are independent.

Exercise 1.3.14 Assume that immersion property holds and let, for every u, $y_t(u)$ be an $\mathbb{F}$-martingale. Prove that, for $t > s$,

$$\mathbb{1}_{\tau\leq s}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) = \mathbb{1}_{\tau\leq s}y_s(\tau)$$

Exercise 1.3.15 Prove that $\mathbb{G}$ is immersed in $\mathbb{F}\vee\sigma(\tau)$ if and only if τ is constant.

Exercise 1.3.16 (A different proof of Norros' result) Suppose that

$$\mathbb{P}(\tau \leq t|\mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbb{F}$-adapted process. Prove, using the inverse of Γ that the random variable Γ_τ is independent of $\mathcal{F}_\infty$, with exponential law of parameter 1.

Exercise 1.3.17 Let $\mathbb{F}\hookrightarrow\mathbb{K}$ and a be a $\mathbb{K}$-adapted process. Prove that $\mathbb{E}(\int_0^t a_s ds|\mathcal{F}_t) = \int_0^t \mathbb{E}(a_s|\mathcal{F}_s)ds$.

Exercise 1.3.18 Show that, if $\tau\in\mathcal{F}_\infty$, immersion holds between $\mathbb{F}$ and the progressive enlargement of $\mathbb{F}$ with τ if and only if τ is an $\mathbb{F}$-stopping time.

Exercise 1.3.19 Let $\tau_i, i = 1,2$ be two random times such that $\mathbb{P}(\tau_1 = \tau_2) = 0$ and $\mathbb{A}^i$ the filtration, associated to τ_i. Prove that $\mathbb{A}^i, i = 1,2$ are immersed in $\mathbb{A} := \mathbb{A}^1\vee\mathbb{A}^2$ if and only if $\tau_i, i = 1,2$ are independent.

Exercise 1.3.20 Let $\mathbb{F}$ be the Brownian filtration generated by B and $X_t = \int_0^t \mathbb{1}_{\{B_s>0\}}dB_s$. Prove that the process X is an $\mathbb{F}$-martingale, however, $\mathbb{F}^X$ is not immersed in $\mathbb{F}$.

Exercise 1.3.21 Let $\mathbb{G}$ be a Brownian filtration generated by B and $\mathbb{F}$ the filtration generated by $\beta_t = \int_0^t \mathrm{sgn}(B_s)dB_s$. Prove that $\mathbb{F}\hookrightarrow\mathbb{G}$ and β enjoys PRP as well in $\mathbb{F}$ and in $\mathbb{G}$.

SOLUTIONS

Exercise 1.3.4 Immersion is equivalent to $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_=\infty)$ and the last quantity is $\mathbb{1}_{\tau>t}$.

Exercise 1.3.2: By definition (see e.g. [4], Definition 1.4.1.1) a continuous process X is said to be a Brownian motion, if, between the others, one of the following equivalent properties is satisfied: either the processes $(X_t)_{t\geq 0}$ and $(X_t^2 - t)_{t\geq 0}$ are $\mathbb{F}^X-$local martingales, or, for any $\lambda\in\mathbb{R}$, $\Big(\exp(\lambda X_t - \frac{\lambda^2}{2}t)\Big)_{t\geq 0}$ is an $\mathbb{F}^X-$local martingale.
Since immersion property holds between $\mathbb{F}$ and $\mathbb{G}$, the result is immediate.

Exercise ??: Because of property $(\mathcal{H}_2)$ in Proposition 2.1.1, given $A\in\mathcal{K}_t\cap\mathcal{F}_\infty$, we have $\mathbb{1}_A = \mathbb{E}(\mathbb{1}_A|\mathcal{F}_\infty) = \mathbb{E}(\mathbb{1}_A|\mathcal{F}_t)$, meaning that $A\in\mathcal{F}_t$. Conversely, if $B\in\mathcal{F}_t$, it also holds $B\in\mathcal{K}_t$ and $B\in\mathcal{F}_\infty$ and we have equality between $\mathcal{F}_t$ and $\mathcal{K}_t\cap\mathcal{F}_\infty$.

Exercise 1.3.5: Any $(\mathbb{F},\mathbb{Q})$ martingale is an $(\mathbb{F},\mathbb{P})$ semimartingale (Girsanov), hence a $(\mathbb{G},\mathbb{P})$ semimartingale (immersion) and a $(\mathbb{G},\mathbb{Q})$ semimartingale (Girsanov).
We reproduce the proof of Jeulin & Yor [5, Theorem 3].

In a first step we assume that $\mathbb{F}$-martingales are continuous. Given an $(\mathbb{F},\mathbb{Q})-$martingale X and using that $(\frac{d\mathbb{P}}{d\mathbb{Q}})_{|\mathcal{F}_t} = \frac{1}{\ell_t}$, we know, thanks to Girsanov's theorem, that

$$\widetilde{X}_t := X_t - \int_0^t \ell_s d\langle X, \frac{1}{\ell}\rangle_s, \quad t\geq 0,$$

is an $(\mathbb{F},\mathbb{P})$-(local) martingale, hence, by immersion, a $(\mathbb{G},\mathbb{P})$-(local) martingale. We then apply once more Girsanov's theorem to pass from $\mathbb{P}$ to $\mathbb{Q}$ under $\mathbb{G}$ by means of L, so that we obtain the following $(\mathbb{G},\mathbb{Q})$-(local) martingale, for any t,

$$\bar{X}_t := \widetilde{X}_t - \int_0^t \frac{1}{L_s} d\langle \widetilde{X}, L\rangle_s = X_t - \int_0^t \ell_s d\langle X, \frac{1}{\ell}\rangle_s - \int_0^t \frac{1}{L_s} d\langle X, L\rangle_s .$$

In the general case,

$$\widetilde{X}_t := X_t - \int_0^t \ell_s d[X, \frac{1}{\ell}]_s, \quad t \geq 0,$$

is a $(\mathbb{F},\mathbb{P})$-(local) martingale, that remains a $(\mathbb{G},\mathbb{P})$-(local) martingale, hence

$$\bar{X}_t := \widetilde{X}_t - \int_0^t \frac{1}{L_s} d[\widetilde{X}, L]_s$$

is a $(\mathbb{G},\mathbb{Q})$ local martingale. One has

$$\begin{aligned}
\bar{X}_t &= X_t - \int_0^t \ell_s d[X, \frac{1}{\ell}]_s - \int_0^t \frac{1}{L_s} d[X, L]_s + \int_0^t \frac{\ell_s}{L_s} d[[X, \frac{1}{\ell}], L]_s \\
&= X_t - \int_0^t \ell_s (\frac{\Delta L_s}{L_s} - 1) d[X, \frac{1}{\ell}]_s - \int_0^t \frac{1}{L_s} d[X, L]_s \\
&= X_t + \int_0^t \ell_s (\frac{\Delta L_s}{L_s} - 1) d[X, \frac{1}{\ell}]_s - \int_0^t \frac{1}{L_s} d[X, L]_s \\
&= X_t - \int_0^t \frac{\ell_s L_{s-}}{L_s} d[X, \frac{1}{\ell}]_s - \int_0^t \frac{1}{L_s} d[X, L]_s
\end{aligned}$$

The result follows from $\ell_s d[X, \frac{1}{\ell}]_s = -\frac{1}{\ell_{s-}} d[X, \ell]_s$.

Exercise 1.3.6: Note that, for $X \in \mathcal{F}_T$,

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) = \frac{1}{L_t}\mathbb{E}_{\mathbb{P}}(XL_T|\mathcal{G}_t) \quad , \quad \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) = \frac{1}{\ell_t}\mathbb{E}_{\mathbb{P}}(X\ell_T|\mathcal{G}_t)$$

and that, from MCT, the hypothesis $(\mathcal{H})$ holds under $\mathbb{Q}$ if and only if, $\forall T, \forall X \in F_T, \forall t \leq T$, one has

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) .$$

Exercise 1.3.8: First of all let us notice that if ζ is independent of $\mathcal{F}_\infty$, then immersion holds between $\mathbb{F}$ and $\mathbb{F}^{\sigma(\zeta)}$. We will show that independence is not only a sufficient, but also a necessary condition for immersion property to hold. We will exploit property (ii) in Proposition **??**, namely the fact that the hypothesis $(\mathcal{H})$ is equivalent, for any $t \geq 0$, to the conditional independence of $\mathcal{F}_t^{\sigma(\zeta)}$ and $\mathcal{F}_\infty$ given $\mathcal{F}_t$. Let us, then, consider an $\mathcal{F}_t^{\sigma(\zeta)}$-measurable random variable of the form $F_t h(\zeta)$, with F_t an $\mathcal{F}_t$-measurable and an $\mathcal{F}_\infty$-measurable random variable F_∞. Immersion property is equivalent, then, to

$$\mathbb{E}(F_t h(\zeta) F_\infty | \mathcal{F}_t) \stackrel{(\mathcal{H})}{=} \mathbb{E}(F_t h(\zeta)|\mathcal{F}_t)\mathbb{E}(F_\infty|\mathcal{F}_t),$$

for any $t \geq 0$.

In particular, taking a constant random variable F_t and $t = 0$ we find

$$\mathbb{E}(h(\zeta)F_\infty) = \mathbb{E}(h(\zeta))\mathbb{E}(F_\infty)$$

and we find that immersion property is equivalent to the independence of random variables of the form $h(\zeta)$ and F_∞. As usual, an application of the Monotone Class Theorem allows us to conclude the proof.

Exercise 1.3.9 To construct an example it suffices to recall Example 1.1.7, namely we define $\mathcal{G}_t := \mathcal{F}_\infty, \forall\, t \geq 0$. In this particular case, only constant $\mathbb{F}-$martingales remain $\mathbb{G}-$martingales.

Exercise 1.3.11: Immersion holds if and only if

$$\mathbb{E}(h(\tau)|\mathcal{G}_t) = \mathbb{E}(h(\tau)|\mathcal{G}_\infty)$$

Since $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma(\tau)$, and τ being positive, $\mathcal{G}_0$ is the trivial sigma-algebra, this condition reduces to $\mathbb{E}(h(\tau)|\mathcal{G}_t) = h(\tau)$. In particular $\mathbb{E}(h(\tau)) = h(\tau)$, hence τ is constant.

Exercise 1.3.13 For h a bounded Borel function and X a bounded $\mathcal{F}_\infty$-measurable martingale, the two processes $\mathbb{E}(h(\tau)|\mathcal{A}_t)$ and $\mathbb{E}(X|\mathcal{F}_t)$ are $\mathbb{G}$-martingales. The continuity hypothesis implies that they are orthogonal. Then

$$\mathbb{E}(h(\tau)X) = \mathbb{E}(h(\tau))\mathbb{E}(X)\,.$$

Note that the assumption (C) can be replaced by assumption (A).

Exercise 1.3.19: Let $M_t^f = \mathbb{E}(f(\tau_1)|\mathcal{A}_t^1)$ and $M_t^g = \mathbb{E}(g(\tau_2)|\mathcal{A}_t^2)$. From immersion property, these processes are $\mathbb{A}$-martingales. The pure jump martingale M^f (resp. M^g) has a single jump at time τ_1 (resp. at τ_2) hence, they are orthogonal, and

$$\mathbb{E}(f(\tau_1)g(\tau_2)) = \mathbb{E}(M_\infty^f M_\infty^g) = M_0^f M_0^g = \mathbb{E}(M_\infty^f)\mathbb{E}M_\infty^g) = \mathbb{E}(f(\tau_1))\mathbb{E}(g(\tau_2))$$

Exercise 1.3.20: There are $\mathbb{F}^M$ martingales which are discontinuous, hence $\mathbb{F}^M$ is not immersed in $\mathbb{F}$ [8, ex 4.25, page 216]. Example of such a discontinuous martingale: let $T_a = \inf\{t : B_t = a\}$ and $\sigma := \inf\{t > T_{-1}, B_t = 0\}$. Then, σ is a totally inaccessible $\mathbb{F}^M$-stopping times, which admits a continuous compensator. The compensated $\mathbb{F}^M$-martingale is discontinuous. See [2, Proposition 9].

1.4 Chapter 4

Exercise 1.4.1 a) Prove that the Riemann integral $\int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$ is absolutely convergent.
b) Prove that, for $0 \leq s < t \leq 1$, $\mathbb{E}(B_t - B_s|B_1 - B_s) = \frac{t-s}{1-s}(B_1 - B_s)$

Exercise 1.4.2 Using the notation of Proposition 4.1.2, prove that B_1 and β are independent. Check that the projection of β on $\mathbb{F}^B$ is equal to B.

Exercise 1.4.3 Consider the SDE

$$\left\{\begin{array}{rcl} dX_t &=& -\dfrac{X_t}{1-t}\,dt + dW_t\,;\; 0 \leq t < 1 \\ X_0 &=& 0 \end{array}\right.$$

1. Prove that
$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s}\,;\; 0 \leq t < 1\,.$$
2. Prove that $(X_t, t \geq 0)$ is a Gaussian process. Compute its expectation and its covariance.
3. Prove that $\lim_{t\to 1} X_t = 0$.

Exercise 1.4.4 (See Jeulin and Yor [6]) Let $X_t = \int_0^t \varphi_s dB_s$ where φ is predictable such that $\int_0^t \varphi_s^2 ds < \infty$. Prove that the following assertions are equivalent

1. X is an $\mathbb{F}^{(B_1)}$-semimartingale with decomposition
$$X_t = \int_0^t \varphi_s d\beta_s + \int_0^{t\wedge 1} \frac{B_1 - B_s}{t-s}\,\varphi_s ds$$

2. $\int_0^1 |\varphi_s| \frac{|B_1 - B_s|}{1-s} ds < \infty$

3. $\int_0^1 \frac{|\varphi_s|}{\sqrt{1-s}} ds < \infty$

Exercise 1.4.5 Prove that, for any enlargement of filtration the compensated martingale M remains a semi-martingale.

Exercise 1.4.6 Prove that any $\mathbb{F}^N$-martingale is a $\mathbb{G}^*$-semimartingale.

Exercise 1.4.7 Prove that

$$\eta_t = N_t - \int_0^{t\wedge T} \frac{N_T - N_s}{T-s} ds - (t-T)^+ ,$$

Prove that

$$\langle \eta \rangle_t = \int_0^{t\wedge T} \frac{N_T - N_s}{T-s} ds + (t-T)^+ \, .$$

Therefore, $(\eta_t, t \leq T)$ is a compensated $\mathbb{G}^*$-Poisson process, time-changed by $\int_0^t \frac{N_T - N_s}{T-s} ds$, i.e., $\eta_t = \widetilde{M}(\int_0^t \frac{N_T - N_s}{T-s} ds)$ where $(\widetilde{M}(t), t \geq 0)$ is a compensated Poisson process.

Exercise 1.4.8 A process X fulfills the **harness property** if

$$\mathbb{E}\left(\frac{X_t - X_s}{t-s} \Big| \mathcal{F}_{s_0],\,[T} \right) = \frac{X_T - X_{s_0}}{T - s_0}$$

for $s_0 \leq s < t \leq T$ where $\mathcal{F}_{s_0],\,[T} = \sigma(X_u, u \leq s_0, u \geq T)$. Prove that a process with the harness property satisfies

$$\mathbb{E}\left(X_t \Big| \mathcal{F}_{s],\,[T} \right) = \frac{T-t}{T-s} X_s + \frac{t-s}{T-s} X_T \, ,$$

and conversely. Prove that, if X satisfies the harness property, then, for any fixed T,

$$M_t^T = X_t - \int_0^t du \frac{X_T - X_u}{T-u} , \; t < T$$

is an $\mathcal{F}_{t],\,[T}$-martingale and conversely. See [3M] for more comments.

Exercise 1.4.9 Assume that $\mathbb{F}$ is a Brownian filtration. Then, check directly that $\mathbb{E}(\int_0^t \frac{d\langle p_\cdot(L), X\rangle_s}{p_{s-}(L)} |\mathcal{F}_t)$ is an $\mathbb{F}$-martingale.

Exercise 1.4.10 Prove that if there exists a probability $\mathbb{Q}^*$ equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}^*$, the r.v. L is independent of $\mathcal{F}_\infty$, then every $(\mathbb{P}, \mathbb{F})$-semi-martingale X is also an $(\mathbb{P}, \mathbb{F}^{(L)})$-semi-martingale. See the last Chapter

Exercise 1.4.11 Prove that, if τ is an $\mathbb{F}$ stopping time, $\mathbb{G} = \mathbb{F}$.

Exercise 1.4.12 Prove that

$$\{\tau > t\} \subset \{Z_t > 0\} \tag{1.4.1}$$

(where the inclusion is up to a negligible set).

Exercise 1.4.13 Let τ be an honest time. Prove that

$$\mathbb{E}(f(\tau)|\mathcal{F}_t) = f(\tau)(1 - Z_t) + \mathbb{E}(\int_t^\infty f(s) dA_s^p |\mathcal{F}_t)$$

Exercise 1.4.14 Prove that $\mathcal{G}_t^* := \{A \in \mathcal{F}_\infty : A = (\widetilde{A}_t \cap \{\tau \le t\}) \cup (\widehat{A}_t \cap \{\tau > t\})$ for some $\widehat{A}_t, \widetilde{A}_t \in \mathcal{F}_t\}$ defines indeed a filtration (i.e., the increasing property holds).

Exercise 1.4.15 Prove that any $\mathbb{F}$-stopping time is honest

Exercise 1.4.16 Prove that, under (CA)

$$\mathbb{E}(\int_0^{t\wedge\tau} \frac{d\langle M, \mu\rangle_s}{Z_{s-}} - \int_\tau^{\tau\vee t} \frac{d\langle M, \mu\rangle_s}{1 - Z_{s-}} |\mathcal{F}_t)$$

is an $\mathbb{F}$-local martingale, without using the semimartingale decomposition.

Exercise 1.4.17 Let B be a Brownian motion and

$$\begin{aligned} T_a^{(\nu)} &= \inf\{t : B_t + \nu t = a\} \\ G_a^{(\nu)} &= \sup\{t : B_t + \nu t = a\} \end{aligned}$$

Prove that

$$(T_a^{(\nu)}, G_a^{(\nu)}) \stackrel{\text{law}}{=} \left(\frac{1}{G_\nu^{(a)}}, \frac{1}{T_\nu^{(a)}}\right)$$

Give the law of the pair $(T_a^{(\nu)}, G_a^{(\nu)})$.

Exercise 1.4.18 Let X be a transient diffusion, such that

$$\begin{aligned} \mathbb{P}_x(T_0 < \infty) &= 0, x > 0 \\ \mathbb{P}_x(\lim_{t\to\infty} X_t = \infty) &= 1, x > 0 \end{aligned}$$

and note s the scale function satisfying $s(0^+) = -\infty, s(\infty) = 0$. Prove that for all $x, t > 0$,

$$\mathbb{P}_x(G_y \in dt) = \frac{-1}{2s(y)} p_t^{(m)}(x, y)dt$$

where $p^{(m)}$ is the density transition w.r.t. the speed measure m.

Exercise 1.4.19 Prove that, if $Z = N/N^*$ is the multiplicative decomposition of Z, then $L = \frac{1}{N}$.

SOLUTIONS

Exercise 1.4.2: : The $\mathbb{F}^{\sigma(B_1)}$-Brownian motion β is independent of $\mathcal{F}_0^{\sigma(B_1)}$.

Exercise ??: The optimal terminal wealth for $\mathbb{F}$-predictable portfolio is of the form $X_T^* = (U')^{-1}(\nu L_T)$ for some L_T Radon-Nikodym density of an e.m.m.. Convexity inequality proves that $\mathbb{E}(U(X_T^{\mathbb{G}}) - U(X_T^*)) \le \mathbb{E}((X_T^{\mathbb{G}} - X_T^*)\nu L_T) = \mathbb{E}(X_T^{\mathbb{G}}\nu L_T) - \nu x$ and the conclusion follows, since, L being also a $\mathbb{G}$ e.m.m. one has $\mathbb{E}(X_T^{\mathbb{G}}\nu L_T) \le x$.

1.5 Chapter 5

Exercise 1.5.1 Solve the optimization problem in $\mathbb{F}$ and $\mathbb{F}^{\sigma(B_1)}$ for power utility function.

Exercise 1.5.2 Assume that $\mathbb{F}$ is a Brownian filtration. Then, check directly that

$$\mathbb{E}\left(\int_0^t \frac{1}{p_{s-}^\zeta} d\langle p^u, X\rangle_s|_{u=\zeta} |\mathcal{F}_t\right)$$

is an $\mathbb{F}$-martingale.

Exercise 1.5.3 Prove that if there exists a probability $\mathbb{Q}^*$ equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}^*$, the r.v. ζ is independent of $\mathcal{F}_\infty$, then every $(\mathbb{F}, \mathbb{P})$-semimartingale X is also an $(\mathbb{F}^{\sigma(\zeta)}, \mathbb{P})$-semimartingale.

Exercise 1.5.4 Find the $\mathbb{F}^{\sigma(\zeta)}$-semimartingale decomposition of $\mathbb{F}$-martingales when $\mathbb{F}$ is a Brownian filtration and $\zeta = 1\!\!1_{\{a \leq B_T \leq b\}}$. Discuss arbitrages opportunities.

Exercise 1.5.5 Let τ be an honest time. Prove that

$$\mathbb{E}(h(\tau)|\mathcal{F}_t) = h(\tau)(1 - Z_t) + \mathbb{E}(\int_t^\infty h(s) dA^p_s | \mathcal{F}_t).$$

Exercise 1.5.6 Prove that

$$\mathcal{G}^*_t := \{A \in \mathcal{F}_\infty \,:\, A = (\widetilde{A}_t \cap \{\tau \leq t\}) \cup (\widehat{A}_t \cap \{\tau > t\}) \text{ for some } \widehat{A}_t, \widetilde{A}_t \in \mathcal{F}_t\}$$

defines indeed a filtration (i.e., the increasing property holds).

Exercise 1.5.7 Assume $\mathbb{F}$ is a Brownian filtration and τ an honest time. Let X be a positive $\mathbb{F}$ martingale. Prove that, if Z is continuous and $Z = N/N^*$ is its multiplicative decomposition, then $X^\tau L$ is a $\mathbb{G}$-local martingale for $L = \frac{1}{N^\tau}$.

Exercise 1.5.8 Let τ and τ^* be two honest times. Show that $\tau \vee \tau^*$ is an honest time.

Exercise 1.5.9 Compute the projection of the martingale L (defined earlier as $L_t = \frac{1}{p^\tau_t}, t \geq 0$) on $\mathbb{G}$.

Exercise 1.5.10 If τ is a pseudo-stopping time and Z is continuous, prove that Z_τ has a uniform law.

SOLUTIONS

Exercise 1.5.2: Note that the result is obvious from the decomposition theorem: indeed taking expectation w.r.t. $\mathcal{F}_t$ of the two sides of

$$\widetilde{X}_t = X_t - \int_0^t \frac{1}{p^\zeta_{s-}} d\langle p^u, X\rangle_s|_{u=\zeta}$$

leads to

$$\mathbb{E}(\widetilde{X}_t|\mathcal{F}_t) = X_t - \mathbb{E}(\int_0^t \frac{1}{p^\zeta_{s-}} d\langle p^u, X\rangle_s|_{u=\zeta}|\mathcal{F}_t),$$

and $\mathbb{E}(\widetilde{X}_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale.

Our aim is to check it directly. Writing $dp^u_t = p^u_t \sigma_t(u) dB_t$ and $dX_t = x_t dB_t$, we note that $\mathbb{P}(\zeta \in \mathbb{R}|\mathcal{F}_s) = \int_\mathbb{R} p^u_s \eta(du) = 1$ implies that

$$\begin{aligned}\int_\mathbb{R} p^u_s \eta(du) &= \int_\mathbb{R} p^u_0 \eta(du) + \int_0^t dB_s \int_\mathbb{R} \sigma_s(u) p^u_s \eta(du) \\ &= 1 + \int_0^t dB_s \int_\mathbb{R} \sigma_s(u) p^u_s \eta(du)\end{aligned}$$

hence $\int_\mathbb{R} \sigma_s(u) p^u_s \eta(du) = 0$. The process $\mathbb{E}(\int_0^t \frac{1}{p^\zeta_{s-}} d\langle p^u, X\rangle_s|_{u=\zeta}|\mathcal{F}_t)$ is equal to a martingale plus $\int_0^t \mathbb{E}(\sigma_s(\zeta) x_s|\mathcal{F}_s) ds = \int_0^t ds\, x_s \int_\mathbb{R} \sigma_s(u) p^u_s \eta(du) = 0$.

Exercise 1.5.7: By Itô's formula and the fact that one can write $dN = \varphi dX$

$$\begin{aligned}\frac{1}{N^\tau} &= 1 - \frac{1}{(N^\tau)^2} \cdot N^\tau + \frac{1}{(N^\tau)^3} \cdot \langle N \rangle^\tau = 1 - \frac{\varphi}{(N^\tau)^2} \cdot X^\tau + \frac{\varphi}{(N^\tau)^3} \cdot \langle X^\tau, N\rangle \\ &= 1 - \frac{\varphi}{(N^\tau)^2} \cdot \widehat{X}^\tau,\end{aligned}$$

where $\widehat{X}$ is a $\mathbb{G}$-martingale. This shows that $1/N^\tau$ is a positive continuous $\mathbb{G}$-local martingale satisfying $1/N_0^\tau = 1$. Let X be an $\mathbb{F}$-martingale. Using integration by parts

$$\frac{X^\tau}{N^\tau} = X_0 + \frac{1}{N^\tau} \cdot X^\tau + X^\tau \cdot \frac{1}{N^\tau} + \left\langle X^\tau, \frac{1}{N^\tau} \right\rangle = X_0 + \frac{1}{N^\tau} \cdot \widehat{X}^\tau + X^\tau \cdot \frac{1}{N^\tau} \,.$$

Exercise 1.5.9: An application of the key Lemma yields to the corresponding Radon-Nikodym density on $\mathbb{G}$:

$$d\mathbb{P}^*{}_{|\mathcal{G}_t} = \ell_t \; d\mathbb{P}_{|\mathcal{G}_t} \,,$$

with

$$\begin{aligned}\ell_t &:= \mathbb{E}(L_t|\mathcal{G}_t) = \mathbb{1}_{t<\tau} \frac{1}{G_t} \int_t^\infty \nu(du) + \mathbb{1}_{\tau \le t} \frac{1}{p_t(\tau)} \\ &= \mathbb{1}_{t<\tau} \frac{G(t)}{G_t} + \mathbb{1}_{\tau\le t} \frac{1}{p_t(\tau)} \,.\end{aligned}$$

1.6 Chapter 6

1.7 Chapter 7

Exercise 1.7.1 Prove that if X is a (square-integrable) $\mathbb{F}$-martingale, XL is a $\mathbb{G}$ -martingale, where $L = (1_H)/Z$.

Exercise 1.7.2 We consider, as in the paper of Biagini et al. [1] a mortality bond, a financial instrument with payoff $Y = \int_0^{\tau\wedge T} Z_s ds$, where $Z_s = \mathbb{P}(\tau > s|\mathcal{F}_s)$ where $\mathbb{F}$ is a continuous filtration. We assume that Z is continuous, admits a Doob-Meyer decomposition as $Z = \mu - A$ and does not vanish.

1. Compute, in the case $r = 0$, the price Y_t of the mortality bond. It will be convenient to introduce $N_t = \mathbb{E}(\int_0^T Z_s^2 ds|\mathcal{F}_t)$. Is the process N a $(\mathbb{P}, \mathbb{F})$ martingale? a $(\mathbb{P}, \mathbb{G})$-martingale?
2. Determine the processes α, β and γ so that

$$dY_t = \alpha_t dM_t + \beta_t (dN_t - \frac{1}{Z_t} d\langle N, Z\rangle_t) + \gamma_t (dZ_t - \frac{1}{Z_t} d\langle Z\rangle_t)$$

3. Determine the price $D(t, T)$ of a defaultable zero-coupon bond with maturity T, i.e., a financial asset with terminal payoff $\mathbb{1}_{T<\tau}$. Give the dynamics of this price.
4. We now assume that $\mathbb{F}$ is a Brownian filtration, and that a risky asset with dynamics

$$dS_t = S_t(bdt + \sigma dW_t)$$

is traded. Explain how one can hedge the mortality bond.

for a more exhaustive study.

Exercise 1.7.3 Prove that, if τ is an $\mathbb{F}$ stopping time, $\mathbb{G} = \mathbb{F}$.

Exercise 1.7.4 Prove that
$$\{\tau > t\} \subset \{Z_t > 0\} \tag{1.7.1}$$
(where the inclusion is up to a negligible set).

Exercise 1.7.5 Prove that if X is a (square-integrable) $\mathbb{F}$-martingale, XL is a $\mathbb{G}$ -martingale, where L is defined in Proposition **??**.

Exercise 1.7.6 We consider, as in the paper of Biagini et al. [1] a mortality bond, a financial instrument with payoff $Y = \int_0^{\tau\wedge T} Z_s ds$, where $Z_s = \mathbb{P}(\tau > s|\mathcal{F}_s)$ where $\mathbb{F}$ is a continuous filtration. We assume that Z is continuous, admits a Doob-Meyer decomposition as $Z = \mu - A$ and does not vanish.

1. Compute, in the case $r = 0$, the price Y_t of the mortality bond. It will be convenient to introduce $N_t = \mathbb{E}(\int_0^T Z_s^2 ds|\mathcal{F}_t)$. Is the process N a $(\mathbb{P},\mathbb{F})$ martingale? a $(\mathbb{P},\mathbb{G})$-martingale?
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$$dY_t = \alpha_t dM_t + \beta_t(dN_t - \frac{1}{Z_t} d\langle N, Z\rangle_t) + \gamma_t(dZ_t - \frac{1}{Z_t} d\langle Z\rangle_t)$$
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4. We now assume that $\mathbb{F}$ is a Brownian filtration, and that a risky asset with dynamics
$$dS_t = S_t(bdt + \sigma dW_t)$$
is traded. Explain how one can hedge the mortality bond.

Exercise 1.7.7 Let τ be an honest time. Prove that
$$\mathbb{E}(f(\tau)|\mathcal{F}_t) = f(\tau)(1 - Z_t) + \mathbb{E}(\int_t^\infty f(s) dA_s^p|\mathcal{F}_t)$$

Exercise 1.7.8 Prove that $\mathcal{G}_t^* := \{A \in \mathcal{F}_\infty \,:\, A = (\widetilde{A}_t \cap \{\tau \le t\}) \cup (\widehat{A}_t \cap \{\tau > t\})$ for some $\widehat{A}_t, \widetilde{A}_t \in \mathcal{F}_t\}$ defines indeed a filtration (i.e., the increasing property holds).

Exercise 1.7.9 Prove that any $\mathbb{F}$-stopping time is honest

Exercise 1.7.10 Prove that, under (CA)
$$\mathbb{E}(\int_0^{t\wedge\tau} \frac{d\langle M, \mu\rangle_s}{Z_{s-}} - \int_\tau^{\tau\vee t} \frac{d\langle M, \mu\rangle_s}{1 - Z_{s-}}|\mathcal{F}_t)$$
is an $\mathbb{F}$-local martingale, without using the semimartingale decomposition.

Exercise 1.7.11 Let X be a drifted Brownian motion with positive drift ν and Λ_y^ν its last passage time at level y. Prove that
$$\mathbb{P}_x(\Lambda_y^{(\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x - y + \nu t)^2\right) dt\,,$$
and
$$\mathbb{P}_x(\Lambda_y^{(\nu)} = 0) = \begin{cases} 1 - e^{-2\nu(x-y)}, & \text{for} \quad x > y \\ 0 & \text{for} \quad x < y\,. \end{cases}$$

Prove, using time inversion that, for $x=0$,

$$\Lambda_y^{(\nu)} \stackrel{\text{law}}{=} \frac{1}{T_\nu^{(y)}}$$

where

$$T_a^{(b)} = \inf\{t \,:\, B_t + bt = a\}$$

See Madan et al. [7].

Exercise 1.7.12 The aim of this exercise is to compute, for $t<T<1$, the quantity $\mathbb{E}(h(W_T)1\!\!1_{\{T<g_1\}}|\mathcal{G}_t)$, which is the price of the claim $h(S_T)$ with barrier condition $1\!\!1_{\{T<g_1\}}$.
Prove that

$$\mathbb{E}(h(W_T)1\!\!1_{\{T<g_1\}}|\mathcal{F}_t) = \mathbb{E}(h(W_T)|\mathcal{F}_t) - \mathbb{E}\Big(h(W_T)\Phi\Big(\frac{|W_T|}{\sqrt{1-T}}\Big)\,\Big|\,\mathcal{F}_t\Big)\,,$$

where

$$\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^x \exp\left(-\frac{u^2}{2}\right) du\,.$$

Define $k(w) = h(w)\Phi(|w|/\sqrt{1-T})$. Prove that $\mathbb{E}\Big(k(W_T)\,\Big|\,\mathcal{F}_t\Big) = \widetilde{k}(t,W_t)$, where

$$\begin{aligned}
\widetilde{k}(t,a) &= \mathbb{E}\Big(k(W_{T-t}+a)\Big)\\
&= \frac{1}{\sqrt{2\pi(T-t)}}\int_{\mathbb{R}} h(u)\Phi\Big(\frac{|u|}{\sqrt{1-T}}\Big)\exp\Big(-\frac{(u-a)^2}{2(T-t)}\Big)\,du.
\end{aligned}$$

Exercise 1.7.13 Let M be a positive martingale, such that $M_0=1$ and $\lim_{t\to\infty} M_t = 0$. Let $a\in[0,1[$ and define $G_a = \sup\{t \,:\, M_t = a\}$. Prove that

$$\mathbb{P}(G_a \le t|\mathcal{F}_t) = \left(1-\frac{M_t}{a}\right)^+$$

Assume that, for every $t>0$, the law of the r.v. M_t admits a density $(m_t(x), x\ge 0)$, and $(t,x)\to m_t(x)$ may be chosen continuous on $(0,\infty)^2$ and that $d\langle M\rangle_t = \sigma_t^2 dt$, and there exists a jointly continuous function $(t,x)\to\theta_t(x) = \mathbb{E}(\sigma_t^2|M_t=x)$ on $(0,\infty)^2$. Prove that

$$\mathbb{P}(G_a\in dt) = (1-\frac{M_0}{a})\delta_0(dt) + 1\!\!1_{\{t>0\}}\frac{1}{2a}\theta_t(a)m_t(a)dt$$

Exercise 1.7.14 Let B be a Brownian motion and

$$\begin{aligned}
T_a^{(\nu)} &= \inf\{t \,:\, B_t+\nu t = a\}\\
G_a^{(\nu)} &= \sup\{t \,:\, B_t+\nu t = a\}
\end{aligned}$$

Prove that

$$(T_a^{(\nu)}, G_a^{(\nu)}) \stackrel{\text{law}}{=} \left(\frac{1}{G_\nu^{(a)}}, \frac{1}{T_\nu^{(a)}}\right)$$

Give the law of the pair $(T_a^{(\nu)}, G_a^{(\nu)})$.

Exercise 1.7.15 Let X be a transient diffusion, such that

$$\begin{aligned}
\mathbb{P}_x(T_0<\infty) &= 0, x>0\\
\mathbb{P}_x(\lim_{t\to\infty} X_t = \infty) &= 1, x>0
\end{aligned}$$

and note s the scale function satisfying $s(0^+)=-\infty, s(\infty)=0$. Prove that for all $x,t>0$,

$$\mathbb{P}_x(G_y\in dt) = \frac{-1}{2s(y)}p_t^{(m)}(x,y)dt$$

where $p^{(m)}$ is the density transition w.r.t. the speed measure m.

1.8 Chapter 8

Exercise 1.8.1 Prove that $(Y_t(\tau), t \geq 0)$ is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale if and only if $Y_t(x)p_t(x)$ is a family of $\mathbb{F}$-martingales.

Exercise 1.8.2 Let $\mathbb{F}$ be a Brownian filtration. Prove that, if X is a square integrable $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale, then, there exists a function h and a process ψ such that

$$X_t = h(\tau) + \int_0^t \psi_s(\tau) dB_s$$

Exercise 1.8.3 Give a direct check of Proposition **??** in a Brownian filtration

Exercise 1.8.4 Prove that the change of probability measure generated by the two processes

$$z_t = (L_t^{\mathbb{F}})^{-1}, \qquad z_t(\theta) = \frac{p_\theta(\theta)}{p_t(\theta)}$$

provides a model where the immersion property holds true, and where the intensity processes does not change

Exercise 1.8.5 Check that

$$\mathbb{E}(\int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_s}{G_{s^-}} - \int_{t\wedge\tau}^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s^-}(\theta)}\Big|_{\theta=\tau} |\mathcal{F}_t)$$

is an $\mathbb{F}$-martingale.
Check that that

$$\mathbb{E}(\int_0^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s^-}(\theta)}\Big|_{\theta=\tau} |\mathcal{G}_t)$$

is a $\mathbb{G}$ martingale.

Exercise 1.8.6 Let λ be a positive $\mathbb{F}$-adapted process and $\Lambda_t = \int_0^t \lambda_s ds$ and Θ be a strictly positive random variable such that there exists a family $\gamma_t(u)$ which satisfies $\mathbb{P}(\Theta > \theta|\mathcal{F}_t) = \int_\theta^\infty \gamma_t(u)du$. Let $\tau = \inf\{t > 0 : \Lambda_t \geq \Theta\}$.Prove that the density of τ is given by

$$p_t(\theta) = \lambda_\theta \gamma_t(\Lambda_\theta) \text{ if } t \geq \theta \qquad \text{and} \qquad p_t(\theta) = \mathbb{E}[\lambda_\theta \gamma_\theta(\Lambda_\theta)|\mathcal{F}_t] \text{ if } t < \theta.$$

Conversely, if we are given a density p, prove that it is possible to construct a threshold Θ such that τ has p as density.

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