BIMRC Summer School on Mathematical Finance

Beijing,7/9 June 2017

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Part II: Enlargement of filtration in continuous time. General facts and initial enlargement.

The problem of enlargement of filtration is the following.

Let X be an $\mathbb{F}$-martingale and $\mathbb{G}$ a filtration larger than $\mathbb{F}$ (that is such that $\mathcal{F}_t \subset \mathcal{G}_t$ for any t).

In finance, its means that one of the agent (the $\mathbb{G}$ one) has more information than the other (the $\mathbb{F}$ one).

What can be said about X in the filtration $\mathbb{G}$? This is important when one would like to define stochastic integrals of the form $\int \vartheta_s dX_s$ when ϑ is $\mathbb{G}$ -adapted (e.g. if an investor has access to some information not contained in the prices). If X is a price process (an $\mathbb{F}$-semimartingale), which is not a $\mathbb{G}$-semimartingale, the informed agent has obviously arbitrage opportunities due to the fundamental theorem of asset pricing.

A semimartingale is a process of the form $X = M + A$ where M is a martingale and A a bounded variation process.

Examples and counter examples:

- Let $\mathcal{G}_t = \mathcal{F}_\infty$ and $\mathcal{F}_0$ trivial. Then, only constant $\mathbb{F}$-martingales are $\mathbb{G}$-semimartingales.

- Let $\mathbb{F}$ be the natural filtration of a Brownian motion and $\mathcal{G}_t = \mathcal{F}_{t+\epsilon}, \forall t$. Then, the $\mathbb{F}$-BM is not a $\mathbb{G}$-semimartingale.

- Let $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$ with $\widetilde{\mathcal{F}}_\infty$ independent of $\mathcal{F}_\infty$ under a probability $\mathbb{Q}$, equivalent to $\mathbb{P}$. Then all $(\mathbb{P}, \mathbb{F})$-martingales are $(\mathbb{P}, \mathbb{G})$-semimartingales

- Initial enlargement : Let ζ be a random variable and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\zeta)$.
Example: Let S be a price process $\mathbb{F}$-adapted and $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(S_T)$. Obviously there are arbitrages.

- Progressive enlargement : Let τ a random time, i.e., a non negative random variable. Then, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ is the smallest filtration which contains $\mathbb{F}$ and makes τ a stopping time.

1 Drift information for an Enlargement in a Brownian setting

We assume in this part that

• $\mathbb{F}$ is the filtration generated by a Brownian motion W and $\mathbb{G}$ is a filtration larger than $\mathbb{F}$

• there exists an integrable $\mathbb{G}$-adapted process $\mu^{\mathbb{G}}$ such that $dW_t = dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt$ where $W^{\mathbb{G}}$ is a $\mathbb{G}$-BM,

• In the financial market, a risky asset with price S (an $\mathbb{F}$-adapted positive process) and a riskless asset $S^0 \equiv 1$ are traded. This market is supposed to be arbitrage free. More precisely, we assume that S is a $(\mathbb{P}, \mathbb{F})$ (local) martingale, $dS_t = S_t \sigma_t dW_t$.

Let X be the wealth process associated with an $\mathbb{F}$ (resp. $\mathbb{G}$) -predictable strategy $\hat{\pi}$: $dX_t = \widehat{\pi}_t dS_t$.

Our goal is to solve $\sup(\mathbb{E}(\ln X_T), \hat{\pi} \in \mathbb{F})$ and $\sup(\mathbb{E}(\ln X_T), \hat{\pi} \in \mathbb{G})$.

● For $\widehat{\pi} \in \mathbb{F}$, restricting attention to positive wealth processes,

$$dX_t = \widehat{\pi}_t S_t \sigma_t dW_t = \pi_t X_t dW_t$$

and

$$X_t = x \exp\left(\int_0^t \pi_s dW_s - \frac{1}{2}\int_0^t \pi_s^2 ds\right) .$$

Then, assuming some regularity on the set of strategies

$$\mathbb{E}(\ln X_T) = \ln x + \mathbb{E}(\int_0^T \pi_s dW_s - \frac{1}{2}\int_0^T \pi_s^2 ds) = \ln x - \frac{1}{2}\int_0^T \mathbb{E}(\pi_s^2) ds \, .$$

The optimal strategy is $\pi = 0$ and $\mathbb{E}(\ln X_T^*) = \ln x$.

• For $\widehat{\pi} \in \mathbb{G}$

$$dX_t = \pi_t X_t dW_t = \pi_t X_t (dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt)$$

so that

$$X_t = x \exp\left(\int_0^t \pi_s dW_s^{\mathbb{G}} - \frac{1}{2}\int_0^t \pi_s^2 ds + \int_0^t \pi_s \mu_s^{\mathbb{G}} ds\right)$$

Therefore

$$\mathbb{E}(\ln X_T) = \ln x + \mathbb{E}\left(-\frac{1}{2}\int_0^T \pi_s^2 ds + \int_0^T \pi_s \mu_s^{\mathbb{G}} ds\right)$$

and the optimal strategy is $\pi^* = \mu$ and

$$\sup_{\mathbb{G}} \mathbb{E}(\ln X_T) = \ln x + \frac{1}{2}\int_0^T \mathbb{E}((\mu_s^{\mathbb{G}})^2) ds$$

Finally

$$\sup_{\pi \in \mathbb{F}} \mathbb{E}(\ln X_T) = \ln x \leq \sup_{\pi \in \mathbb{G}} \mathbb{E}(\ln X_T) = \ln x + \mathbb{E}\left(\frac{1}{2}\int_0^T (\mu_s^{\mathbb{G}})^2 ds\right)$$

which leads to a finite utility if

$$\mathbb{E}\left(\int_0^T (\mu_s^{\mathbb{G}})^2 ds\right) < \infty \,.$$

2 Immersion Hypothesis

2.1 Definition

The filtration $\mathbb{F}$ is said to be immersed in $\mathbb{G}$ if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. This is also referred to as the $(\mathcal{H})$ hypothesis in the literature.

Proposition 2.1. *Immersion holds for $(\mathbb{F}, \mathbb{G})$ is equivalent to any of the following properties:*

$(i)\; \forall\, t \geq 0,$ *the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$, i.e.,*
$\forall\, t \geq 0,\; \forall\, G_t \in L^2(\mathcal{G}_t), \forall\, F \in L^2(\mathcal{F}_\infty), \mathbb{E}(G_t\, F|\mathcal{F}_t) = \mathbb{E}(G_t|\mathcal{F}_t)\mathbb{E}(F|\mathcal{F}_t)$.

$(ii)\; \forall\, t \geq 0,\; \forall\, G_t \in L^1(\mathcal{G}_t),\; \mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$.

$(iii)\; \forall\, t \geq 0,\; \forall\, F \in L^1(\mathcal{F}_\infty),\; \mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$.

Furthermore, if immersion holds for $(\mathbb{F}, \mathbb{G})$, then $\mathcal{G}_t \cap \mathcal{F}_\infty = \mathcal{F}_t$.

PROOF: First we prove that immersion implies (i).

Let $F \in L^2(\mathcal{F}_\infty)$ and assume that immersion holds for $(\mathbb{F}, \mathbb{G})$. This implies that the $\mathbb{F}$-martingale $X_t = \mathbb{E}(F|\mathcal{F}_t)$ is a $\mathbb{G}$-martingale with terminal value $X_\infty = F$, hence $X_t = \mathbb{E}(F|\mathcal{F}_t) = \mathbb{E}(F|\mathcal{G}_t)$.

It follows that for any t and any $G_t \in L^2(\mathcal{G}_t)$:

$$\mathbb{E}(FG_t|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{G}_t)|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)|\mathcal{F}_t) = \mathbb{E}(G_t|\mathcal{F}_t)\mathbb{E}(F|\mathcal{F}_t)$$

which is exactly (i).

• To prove (i) $\Rightarrow$ (ii), let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$.

Under (i),

$$\mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(F|\mathcal{F}_t)\mathbb{E}(G_t|\mathcal{F}_t)) \stackrel{(i)}{=} \mathbb{E}(\mathbb{E}(FG_t|\mathcal{F}_t)) = \mathbb{E}(FG_t)$$

which implies $\mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$ that is (ii).

• Next, we give a proof of (ii) $\Rightarrow$ (iii).

Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$.

If (ii) holds, then, for $F \in L^2(\mathcal{F}_\infty)$,

$$\mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)) = \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) \stackrel{(ii)}{=} \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_\infty)) = \mathbb{E}(FG_t),$$

which implies $\mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$, that is (iii). Finally, obviously (iii) implies $(\mathcal{H})$.

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which implies $\mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$ that is (ii).

• Next, we give a proof of (ii) $\Rightarrow$ (iii).
Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$.

If (ii) holds, then, for $F \in L^2(\mathcal{F}_\infty)$,

$$\mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)) = \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) \stackrel{(ii)}{=} \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_\infty)) = \mathbb{E}(FG_t),$$

which implies which implies $\mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$, that is (iii). • Finally, obviously (iii) implies $(\mathcal{H})$.

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Under (i),

$$\mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(F|\mathcal{F}_t)\mathbb{E}(G_t|\mathcal{F}_t)) \stackrel{(i)}{=} \mathbb{E}(\mathbb{E}(FG_t|\mathcal{F}_t)) = \mathbb{E}(FG_t)$$

which implies $\mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$ that is (ii).

•Next, we give a proof of (ii) $\Rightarrow$ (iii).

Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$.

If (ii) holds, then, for $F \in L^2(\mathcal{F}_\infty)$,

$$\mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)) = \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) \stackrel{(ii)}{=} \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_\infty)) = \mathbb{E}(FG_t),$$

which implies which implies $\mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$, that is (iii).

• Finally, obviously (iii) implies $(\mathcal{H})$.

The proof of $\mathcal{G}_t \cap \mathcal{F}_\infty = \mathcal{F}_t$ is now simple.

We have only to check that $\mathcal{G}_t \cap \mathcal{F}_\infty \subset \mathcal{F}_t$.

For $A \in \mathcal{G}_t \cap \mathcal{F}_\infty$, we have $\mathbb{1}_A = \mathbb{E}(\mathbb{1}_A|\mathcal{F}_\infty) = \mathbb{E}(\mathbb{1}_A|\mathcal{F}_t)$ which implies that $A \in \mathcal{F}_t$. $\triangle$

2.2 Change of probability

In general, immersion is not stable by change of probability. Nevertheless, it is true under specific conditions.

Proposition 2.2. *Assume that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, and let $\mathbb{Q}$ be equivalent to $\mathbb{P}$, with $\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}$ where L is assumed to be $\mathbb{F}$-adapted. Then, $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{Q}$.*

PROOF: Let N be an $(\mathbb{F}, \mathbb{Q})$-martingale, then $(N_t L_t, t \geq 0)$ is a $(\mathbb{F}, \mathbb{P})$-martingale, and since $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, $(N_t L_t, t \geq 0)$ is a $(\mathbb{G}, \mathbb{P})$-martingale which implies that N is a $(\mathbb{G}, \mathbb{Q})$-martingale, L being as well a $(\mathbb{G}, \mathbb{P})$-martingale. $\triangle$

2.3 Example: reduced form in Credit Risk

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$. *A nonnegative* $\mathbb{F}$*-adapted process* λ *is given*. We assume that there exists, on the space $(\Omega, \mathcal{G}, \mathbb{P})$, a random variable Θ, independent of $\mathcal{F}_\infty$, with an exponential law: $\mathbb{P}(\Theta \geq t) = e^{-t}$. We define the default time τ as the first time when the increasing process $\Lambda_t = \int_0^t \lambda_s \, ds$ is above the random level Θ, i.e.,

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq \Theta\}.$$

In particular, using the increasing property of Λ, one gets $\{\tau > s\} = \{\Lambda_s < \Theta\}$. We assume that $\Lambda_t < \infty, \forall t, \ \Lambda_\infty = \infty$, hence τ is a real-valued r.v. We define $H_t = 1\!\!1_{\{\tau \leq t\}}$ and $\mathcal{H}_t = \sigma(H_s \, : \, s \leq t)$. We introduce the smallest right-continuous filtration $\mathbb{G}$ which contains $\mathbb{F}$ and turns τ in a stopping time $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

Lemma 2.3. *The conditional distribution function of τ given the σ-field $\mathcal{F}_t$ is for $t \geq s$*

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp\big(-\Lambda_s\big)\,. \tag{2.1}$$

PROOF: The proof follows from the equality $\{\tau > s\} = \{\Lambda_s < \Theta\}$. From the independence assumption and the $\mathcal{F}_t$-measurability of Λ_s for $s \leq t$, we obtain

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}\Big(\Lambda_s < \Theta \,\Big|\, \mathcal{F}_t\Big) = \exp\big(-\Lambda_s\big).$$

In particular, we have

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty), \tag{2.2}$$

and, for $t \geq s$, $\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}(\tau > s|\mathcal{F}_s)$. Let us notice that the process $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ is here an increasing process, as the right-hand side of (2.4) is. $\triangle$

Lemma 2.4. *The conditional distribution function of τ given the σ-field $\mathcal{F}_t$ is for $t \geq s$*

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp\big(-\Lambda_s\big)\,. \tag{2.3}$$

PROOF: The proof follows from the equality $\{\tau > s\} = \{\Lambda_s < \Theta\}$. From the independence assumption and the $\mathcal{F}_t$-measurability of Λ_s for $s \leq t$, we obtain

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}\Big(\Lambda_s < \Theta \,\Big|\, \mathcal{F}_t\Big) = \exp\big(-\Lambda_s\big).$$

In particular, we have

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty), \tag{2.4}$$

and, for $t \geq s$, $\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}(\tau > s|\mathcal{F}_s)$. Let us notice that the process $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ is here an increasing process, as the right-hand side of (2.4) is. $\triangle$

Immersion holds in that example: Indeed any $\mathbb{F}$ martingale is an $\mathbb{F} \vee \sigma(\Theta)$ martingale. Being $\mathbb{G}$-adapted, it is also a $\mathbb{G}$ martingale.

We have used the fact that, if $\mathbb{F} \subset \mathbb{K}$ and if an $\mathbb{F}$-adapted process X is a $\mathbb{K}$-martingale, then X is an $\mathbb{F}$-martingale.

Proposition 2.5. *Let $Y \in \mathcal{F}_T$. Then*

$$\mathbb{E}(Y\mathbb{1}_{\tau>T}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}e^{\Lambda_t}\mathbb{E}(Ye^{-\Lambda_T}|\mathcal{F}_t)$$

PROOF: From definition of $\mathbb{G}$, for any $Y_t \in \mathcal{G}_t$, there exists $y_t \in \mathcal{F}_t$ such that

$$Y_t\mathbb{1}_{t<\tau} = y_t\mathbb{1}_{t<\tau}$$

Then,

$$\mathbb{E}(Y\mathbb{1}_{\tau>T}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}Y_t = \mathbb{1}_{t<\tau}y_t$$

Taking $\mathcal{F}_t$ conditional expectations, this implies that

$$\mathbb{E}(Y\mathbb{1}_{\tau>T}|\mathcal{F}_t) = e^{-\Lambda_t}y_t$$

therefore

$$\mathbb{E}(Ye^{-\Lambda_T}|\mathcal{F}_t) = e^{-\Lambda_t}y_t\,.$$

Proposition 2.6. *Let $Y \in \mathcal{F}_T$. Then*

$$\mathbb{E}(Y \mathbb{1}_{\tau > T} | \mathcal{G}_t) = \mathbb{1}_{t < \tau} e^{\Lambda_t} \mathbb{E}(Y e^{-\Lambda_T} | \mathcal{F}_t)$$

PROOF: From definition of $\mathbb{G}$, for any $Y_t \in \mathcal{G}_t$, there exists $y_t \in \mathcal{F}_t$ such that

$$Y_t \mathbb{1}_{t<\tau} = y_t \mathbb{1}_{t<\tau}$$

Then,

$$\mathbb{E}(Y \mathbb{1}_{\tau > T} | \mathcal{G}_t) = \mathbb{1}_{t<\tau} Y_t = \mathbb{1}_{t<\tau} y_t$$

Taking $\mathcal{F}_t$ conditional expectations, this implies that

$$\mathbb{E}(Y \mathbb{1}_{\tau > T} | \mathcal{F}_t) = e^{-\Lambda_t} y_t$$

therefore

$$\mathbb{E}(Y e^{-\Lambda_T} | \mathcal{F}_t) = e^{-\Lambda_t} y_t \, .$$

Proposition 2.7. *The process $L_t = e^{\Lambda_t}(1 - H_t)$ is a $\mathbb{G}$ martingale. The process M*

$$M_t = H_t - \Lambda_{t \wedge \tau}$$

is a $\mathbb{G}$-martingale.

PROOF:

$$\begin{aligned} \mathbb{E}(L_t | \mathcal{G}_s) &= \mathbb{1}_{s<\tau} e^{\Lambda_s} \mathbb{E}(L_t | \mathcal{F}_s) \\ &= \mathbb{1}_{s<\tau} \frac{1}{Z_s} \mathbb{E}(e^{\Lambda_t} e^{-\Lambda_t} | \mathcal{F}_s) = L_s \end{aligned}$$

Then, from $dL_t = e^{\Lambda_t}(-dH_t + (1 - H_t)\lambda_t dt)$ we obtain $dM_t = -e^{-\Lambda_t} dL_t$. △

Proposition 2.8. *The process $L_t = e^{\Lambda_t}(1 - H_t)$ is a-$\mathbb{G}$ martingale. The process M*

$$M_t = H_t - \Lambda_{t\wedge\tau}$$

is a $\mathbb{G}$-martingale.

PROOF:

$$\begin{aligned} \mathbb{E}(L_t|\mathcal{G}_s) &= \mathbb{1}_{s<\tau} e^{\Lambda_s} \mathbb{E}(L_t|\mathcal{F}_s) \\ &= \mathbb{1}_{s<\tau} e^{\Lambda_s} \mathbb{E}(e^{\Lambda_t} e^{-\Lambda_t}|\mathcal{F}_s) = L_s \end{aligned}$$

Then, from $dL_t = e^{\Lambda_t}(-dH_t + (1 - H_t)\lambda_t dt)$ we obtain $dM_t = -e^{\Lambda_t} dL_t$. △

If $\mathbb{F}$ is immersed in $\mathbb{G}$, then for any utility function U defined on $\mathbb{R}^+$,

$$\max_{\pi \in \mathcal{A}^{\mathbb{F}}} \mathbb{E}(U(X_T^{\pi,x})) = \max_{\pi \in \mathcal{A}^{\mathbb{G}}} \mathbb{E}(U(X_T^{\pi,x})).$$

PROOF: The solution of the optimisation problem in $\mathbb{F}$ is known to be $X_T^* = (U')^{-1}(\lambda L_T^*)$ where L is the Radon Nikodym density optimal solution of the dual problem and λ is a parameter such that $\mathbb{E}(L_T^* X_T^*) = x$. Being an $\mathbb{F}$-martingale, L^* is a $\mathbb{G}$-martingale by immersion property, and is a Radon-Nikodym density in $\mathbb{G}$, hence for any strategy $\pi \in \mathcal{A}^{\mathbb{G}}$, the process $X^\pi L^*$ is a $\mathbb{G}$-supermartingale (if one restrict attention to strategies such that the wealth is non negative) with initial value x. By concavity of the utility function

$$\mathbb{E}(U(X_T^\pi) - U(X_T^*)) \leq \mathbb{E}((X_T^\pi - X_T^*)U'(X_T^*)) = \lambda \mathbb{E}((X_T^\pi - X_T^*)L_T^*)) \leq 0$$

which proves that X_T^* is optimal for $\mathcal{A}^{\mathbb{G}}$-strategies. $\triangle$

2.4 Pseudo-stopping times

The property that Z is a decreasing process does not imply that immersion holds.

An $\mathbb{F}$-pseudo-stopping time is a random time such that $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$ for any bounded $\mathbb{F}$-martingale M. In a Brownian filtration, random times with a decreasing continuous Azéma supermartingale (i.e., $\mathbb{P}(\tau > t|\mathcal{F}_t)$) are pseudo-stopping times. They enjoy the property that any $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale.

An Example:

Let S be defined through $dS_t = \sigma S_t dW_t$, where W is a Brownian motion and σ a constant. Let $\tau = \sup\{t \leq 1 \,:\, S_1 - 2S_t = 0\}$, that is the last time before 1 at which the price is equal to half of its terminal value at time 1.

Note that

$$\{\tau \leq t\} = \{\inf_{t\leq s\leq 1} 2S_s \geq S_1\} = \{\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}\}$$

Since $\frac{S_s}{S_t}, s \geq t$ and $\frac{S_1}{S_t}$ are independent from $\mathcal{F}_t$,

$$\mathbb{P}(\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}|\mathcal{F}_t) = \mathbb{P}(\inf_{t\leq s\leq 1} 2S_{s-t} \geq S_{1-t}) = \Phi(1-t)$$

where $\Phi(u) = \mathbb{P}(\inf_{s\leq u} 2S_s \geq S_u)$. It follows that the supermartingale Z is a deterministic decreasing function, hence, τ is a pseudo-stopping time and S is a $\mathbb{G}$-martingale up to time τ and there are no arbitrages up to τ.

There are obviously classical arbitrages after τ, since S_1 is known at time τ.

3 Initial Enlargement

We study initial enlargement of a filtration $\mathbb{F}$ with a real valued random variable ζ, where the enlarged filtration is

$$\mathcal{F}_t^{\sigma(\zeta)} = \cap_{\epsilon>0} \{\mathcal{F}_{t+\epsilon} \vee \sigma(\zeta)\} \ .$$

We work in a rather general framework and we study $(\mathcal{H}')$ hypothesis between $\mathbb{F}$ and $\mathbb{F}^{\sigma(\zeta)}$, i.e., conditions such that $\mathbb{F}$-martingales are $\mathbb{F}^{\sigma(\zeta)}$-semimartingales. Note that, under $(\mathcal{H}')$ hypothesis, every $\mathbb{F}$-semimartingale $(X_t)_{t\leq T}$ is also an $\mathbb{F}^{\sigma(\zeta)}$-semimartingale.

3.1 Jacod's equivalence criterion

Let $P_t(\omega, du)$ be the conditional law of ζ given $\mathcal{F}_t$; i.e.,

$$\mathbb{E}(h(\zeta)|\mathcal{F}_t) = \int_{\mathbb{R}} h(x) P_t(\cdot, dx)$$

We say that ζ satisfies **Jacod's equivalence criterion** if for each $t \geq 0$,

$$P_t(\omega, du) \sim \eta(du)$$

where η is the law of ζ.

There exists a family of **positive** martingales $p(x)$ such that

$$\mathbb{P}(\zeta > u|\mathcal{F}_t) = \int_u^\infty p_t(x)\eta(dx)$$

Here, the main difficulty is to prove that one an find a "regular" version of p, in order to take care about negligeable sets and to be able to define the process $p(\zeta)$.

Lemma 3.1. *The process L defined as $L_t = \frac{1}{p_t(\zeta)}, t \geq 0$ is a $(\mathbb{P}, \mathbb{F}^{(\sigma(\zeta)})$-martingale. Let $\mathbb{P}^*$ be the probability measure defined on $\mathbb{F}^{\sigma(\zeta)}$ as*

$$d\mathbb{P}^*{}_{|\mathcal{F}_t^{\sigma(\zeta)}} = L_t \, d\mathbb{P}_{|\mathcal{F}_t^{\sigma(\zeta)}}.$$

Under $\mathbb{P}^*$, the random variable ζ is independent of $\mathcal{F}_t$ for any $t \geq 0$ and, moreover

$$\mathbb{P}^*{}_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t} \text{ for any } t \geq 0, \quad \mathbb{P}^*{}_{|\sigma(\zeta)} = \mathbb{P}_{|\sigma(\zeta)}.$$

PROOF: From the definition of p, setting $L_t^x := \frac{1}{p_t(x)}$, one has, for any (bounded) Borel function h and any $\mathcal{F}_s$-measurable (bounded) random variable K_s

$$\begin{aligned}
\mathbb{E}(L_t h(\zeta) K_s) &= \mathbb{E}(K_s \int_{\mathbb{R}} L_t^x h(x) p_t(x) \eta(dx)) = \mathbb{E}(K_s \int_{\mathbb{R}} h(x) \eta(dx)) \\
&= \int_{\mathbb{R}} h(x) \eta(dx) \mathbb{E}(K_s) = \mathbb{E}(K_s) \, \mathbb{E}(h(\zeta)) \,.
\end{aligned}$$

The particular case $t = s$ leads to $\mathbb{E}(L_s h(\zeta) K_s) = \mathbb{E}(h(\zeta)) \, \mathbb{E}(K_s)$, hence $\mathbb{E}(L_s h(\zeta) K_s) = \mathbb{E}(L_t h(\zeta) K_s)$, and it follows that L is a martingale.

Lemma 3.2. *The process L defined as $L_t = \frac{1}{p_t(\zeta)}$, $t \geq 0$ is a $(\mathbb{P}, \mathbb{F}^{(\sigma(\zeta)})$-martingale. Let $\mathbb{P}^*$ be the probability measure defined on $\mathbb{F}^{\sigma(\zeta)}$ as*

$$d\mathbb{P}^*{}_{|\mathcal{F}_t^{\sigma(\zeta)}} = L_t \; d\mathbb{P}_{|\mathcal{F}_t^{\sigma(\zeta)}}.$$

Under $\mathbb{P}^*$, the random variable ζ is independent of $\mathcal{F}_t$ for any $t \geq 0$ and, moreover

$$\mathbb{P}^*{}_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t} \;\; \text{for any } t \geq 0, \quad \mathbb{P}^*{}_{|\sigma(\zeta)} = \mathbb{P}_{|\sigma(\zeta)}.$$

PROOF: From the definition of p, setting $L_t^x := \frac{1}{p_t(x)}$, one has, for any (bounded) Borel function h and any $\mathcal{F}_s$-measurable (bounded) random variable K_s

$$\begin{aligned} \mathbb{E}(L_t h(\zeta) K_s) &= \mathbb{E}(K_s \int_{\mathbb{R}} L_t^x h(x) p_t(x) \eta(dx)) = \mathbb{E}(K_s \int_{\mathbb{R}} h(x) \eta(dx)) \\ &= \int_{\mathbb{R}} h(x) \eta(dx) \mathbb{E}(K_s) = \mathbb{E}(K_s)\, \mathbb{E}(h(\zeta))\,. \end{aligned}$$

The particular case $t = s$ leads to $\mathbb{E}(L_s h(\zeta) K_s) = \mathbb{E}(h(\zeta))\, \mathbb{E}(K_s)$, hence $\mathbb{E}(L_s h(\zeta) K_s) = \mathbb{E}(L_t h(\zeta) K_s)$, and it follows that L is a martingale.

Note that, since $p_0(x) = 1$, one has $\mathbb{E}(1/p_t(\zeta)|\mathcal{F}_0^{\sigma(\zeta)}) = 1/p_0(\zeta) = 1$.
Now, we prove the required independence. From the above,

$$\mathbb{E}^*(h(\zeta)K_s) = \mathbb{E}(L_s h(\zeta)K_s) = \mathbb{E}(h(\zeta))\,\mathbb{E}(K_s)$$

where $\mathbb{E}^*$ is the expectation under $\mathbb{P}^*$. For $h = 1$ (resp. $K_s = 1$), one obtains $\mathbb{E}^*(K_s) = \mathbb{E}(K_s)$ (resp. $\mathbb{E}^*(h(\zeta)) = \mathbb{E}(h(\zeta))$) and the assertion is proved. $\triangle$

It is now obvious that, under positive density hypothesis, NFLVR holds in the enlarged filtration $\mathbb{F} \vee \sigma(\tau)$. Indeed, the $(\mathbb{F}, \mathbb{P})$ martingale S is an $(\mathbb{F}, \mathbb{P}^*)$ martingale, and - using the independence property - an $(\mathbb{F}^{\sigma(\tau)}, \mathbb{P}^*)$ martingale.

Optimisation Under the equivalence hypothesis on can reduce the study of optimal $\mathbb{F}^{\sigma(\zeta)}$-predictable portfolio using results on $\mathbb{F}$-predictable optimal portfolio. Indeed, for $dX_t = \pi_t(\zeta)dS_t$, one has

$$\mathbb{E}(U(X_T(\zeta))) = \mathbb{E}^*(p_T(\zeta)U(X_T(\zeta))) = \int \mathbb{E}^*(p_T(u)U(X_T(u)))\eta(du)$$

and the problem can be solved finding an optimal $\mathbb{F}$-portfolio of the problem $\mathbb{E}^*(p_T(u)U(X_T(u)))$.

See Hillairet and Jiao for examples.

$\mathbb{F}^{\sigma(\zeta)}$-semimartingale decomposition of $\mathbb{F}$ martingales

Proposition 3.3. *Any $(\mathbb{P},\mathbb{F})$-local martingale X is a $(\mathbb{P},\mathbb{F}^{\sigma(\zeta)})$-semimartingale with canonical decomposition*

$$X_t = X_t^{(\zeta)} + \int_0^t \frac{d\langle X, p_\cdot(\zeta)\rangle_s}{p_{s-}(\zeta)},$$

where $X^{(\zeta)}$ is a $(\mathbb{P},\mathbb{F}^{\sigma(\zeta)})$-local martingale.

PROOF: If X is a $(\mathbb{P},\mathbb{F})$-martingale, it is a $(\mathbb{P}^*,\mathbb{F}^{(\zeta)})$-martingale, too (Indeed, since $\mathbb{P}$ and $\mathbb{P}^*$ are equal on $\mathbb{F}$, X is a $(\mathbb{P}^*,\mathbb{F})$ martingale, hence, using the fact that ζ is $\mathbb{P}^*$ independent of $\mathbb{F}$, it is a $(\mathbb{P}^*,\mathbb{F}^{(\zeta)})$ martingale). Noting that $d\mathbb{P} = p_t(\zeta)d\mathbb{P}^*$ on $\mathcal{F}_t^{(\zeta)}$, Girsanov's theorem tells us that the process $X^{(\zeta)}$, defined by

$$X_t^{(\zeta)} = X_t - \int_0^t \frac{d\langle X, p_\cdot(\zeta)\rangle_s}{p_{s-}(\zeta)}$$

is a $(\mathbb{P},\mathbb{F}^{(\zeta)})$-martingale. $\triangle$

Proposition 3.4. *Any $(\mathbb{P},\mathbb{F})$-local martingale X is a $(\mathbb{P},\mathbb{F}^{\sigma(\zeta)})$-semimartingale with canonical decomposition*

$$X_t = X_t^{(\zeta)} + \int_0^t \frac{d\langle X, p.(\zeta)\rangle_s}{p_{s-}(\zeta)},$$

where $X^{(\zeta)}$ is a $(\mathbb{P},\mathbb{F}^{\sigma(\zeta)})$-local martingale.

PROOF: If X is a $(\mathbb{P},\mathbb{F})$-martingale, it is a $(\mathbb{P}^*,\mathbb{F}^{\sigma(\zeta)})$-martingale, too (Indeed, since $\mathbb{P}$ and $\mathbb{P}^*$ are equal on $\mathbb{F}$, X is a $(\mathbb{P}^*,\mathbb{F})$ martingale, hence, using the fact that ζ is $\mathbb{P}^*$ independent of $\mathbb{F}$, it is a $(\mathbb{P}^*,\mathbb{F}^{\sigma(\zeta)})$ martingale). Noting that $d\mathbb{P} = p_t(\zeta)d\mathbb{P}^*$ on $\mathcal{F}_t^{\sigma(\zeta)}$, Girsanov's theorem tells us that the process $X^{(\zeta)}$, defined by

$$X_t^{(\zeta)} = X_t - \int_0^t \frac{d\langle X, p.(\zeta)\rangle_s}{p_{s-}(\zeta)}$$

is a $(\mathbb{P},\mathbb{F}^{\sigma(\zeta)})$-martingale. $\triangle$

Example Enlargement with $\zeta := \int_0^\infty f(s)dB_s$

Let $\zeta := \int_0^\infty f(s)dB_s$ where f is a deterministic function such that $\int_0^\infty f^2(s)ds < \infty$ and $\int_t^\infty f^2(s) \neq 0$. It is easy to compute $p_t\,(x)$, since conditionally on $\mathcal{F}_t$, ζ is Gaussian, with mean $m_t = \int_0^t f\,(s)\,dB_s$, and variance $\sigma^2(t) = \int_t^\infty f^2\,(s)\,ds$. Therefore, $\mathbb{P}(\zeta \leq x|\mathcal{F}_t) = \Phi(\dfrac{x - m_t}{\sigma(t)})$, where Φ is the cumulative distribution function of a standard Gaussian law, and the absolute continuity requirement is satisfied with:

$$p_t(x)\nu(dx) = \frac{1}{\sigma(t)}\varphi(\frac{x - m_t}{\sigma(t)})dx,$$

where φ is the density of a standard Gaussian law, and ν the law of ζ (a centered Gaussian law with variance $\sigma^2(0)$). Note that, from Itô's calculus,

$$dp_t(x) = p_t(x)\frac{x - m_t}{\sigma^2(t)}dm_t\;,$$

hence, $d\langle p(x), B\rangle_{x=\zeta} = p_t(\zeta)\frac{1}{\sigma^2(t)}(\zeta - m_t)f(t)dt$.

Then B is an $\mathbb{F}^{\sigma(\zeta)}$-semimartingale with canonical decomposition:

$$B_t = \widetilde{B}_t + \int_0^t ds \frac{f(s)}{\sigma^2(s)} \left(\int_s^\infty f(u)\, dB_u \right).$$

Many results extend to the weaker hypothesis

We say that ζ satisfies **Jacod's equivalence criterion** if for each $t \geq 0$,

$$P_t(\omega, du) << \eta(du)$$

3.2 Bridge

Some situations require another criteria: for example the Brownian Bridge

Let B be a Brownian motion. The conditional law of $\zeta := B_1$ is not equivalent (not even absolutely continuous) to the law of B_1.

Proposition 3.5. *Let $\mathcal{F}_t^{\sigma(B_1)} = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon} \vee \sigma(B_1)$. The process*

$$\beta_t := B_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

is an $\mathbb{F}^{\sigma(B_1)}$-martingale, and an $\mathbb{F}^{\sigma(B_1)}$ Brownian motion. In other words,

$$B_t = \beta_t + \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

is the decomposition of B as an $\mathbb{F}^{\sigma(B_1)}$-semi-martingale.

PROOF: One has $\mathcal{F}_t \vee \sigma(B_1) = \mathcal{F}_t \vee \sigma(B_1 - B_t)$. Then, since $\mathcal{F}_s$ is independent of $(B_{s+h} - B_s, h \geq 0)$, one has, for $s < t$:

$$\mathbb{E}(B_t - B_s | \mathcal{F}_s^{\sigma(B_1)}) = \mathbb{E}(B_t - B_s | B_1 - B_s) = \frac{t-s}{1-s}(B_1 - B_s) \, .$$

For $s < t < 1$,

$$\begin{aligned}
\mathbb{E}(\int_s^t \frac{B_1 - B_u}{1-u} du | \mathcal{F}_s^{\sigma(B_1)}) &= \int_s^t \frac{1}{1-u} \mathbb{E}(B_1 - B_u | B_1 - B_s) \, du \\
&= \int_s^t \frac{1}{1-u} \left(B_1 - B_s - \mathbb{E}(B_u - B_s | B_1 - B_s)\right) du \\
&= \int_s^t \frac{1}{1-u} \left(B_1 - B_s - \frac{u-s}{1-s}(B_1 - B_s)\right) du \\
&= \frac{1}{1-s}(B_1 - B_s) \int_s^t du = \frac{t-s}{1-s}(B_1 - B_s)
\end{aligned}$$

It follows that $\mathbb{E}(\beta_t - \beta_s | \mathcal{F}_s^{\sigma(B_1)}) = 0$ hence, β is an $\mathbb{F}^{\sigma(B_1)}$-martingale (and an $\mathbb{F}^{\sigma(B_1)}$-Brownian motion). $\triangle$

Note that there is a "trap" to avoid. One can think that any $\mathbb{F}$-martingale, being a stochastic integral w.r.t. the Brownian motion will be a $\mathbb{F}^{\sigma(B_1)}$-semimartingale.

This is not the case.

Take $\int_0^\cdot \theta_s dB_s$ be a martingale. One needs a condition on θ to insure that $\int_0^{t\wedge 1} \theta_s \frac{B_1 - B_s}{1-s} ds$ is well defined.

Insider trading

Let

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

where μ and σ are constants, be the price of a risky asset. Assume that the riskless asset has an constant interest rate r. We denote by $\theta = \frac{\mu - r}{\sigma}$ the risk premium.

The wealth of an agent holding π_t^0 shares of the savings account and π_t^1 shares of the underlying risky process is $X_t = \pi_t^0 e^{rt} + \pi_t^1 S_t$. The self financing condition is that

$$dX_t = \pi_t^0 de^{rt} + \pi_t^1 dS_t = rX_t dt + \pi_t^1 (dS_t - rS_t dt)$$

With the change of notation $\pi_t = \pi_t^1 S_t / X_t$ (so that the wealth remains non negative) one has

$$dX_t = rX_t dt + \pi_t \sigma X_t (dB_t + \theta dt),\ X_0 = x$$

Here π^1 is the number of shares of the risky asset, and π the proportion of wealth invested in the risky asset. It follows that

$$\ln(X_T^{\pi,x}) = \ln x + \int_0^T (r - \frac{1}{2}\pi_s^2 \sigma^2 + \theta \pi_s \sigma) ds + \int_0^T \sigma \pi_s dB_s$$

Then, assuming that the local martingale represented by the stochastic integral is in fact a martingale,

$$\mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + \int_0^T \mathbb{E}\left(r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma\right) ds$$

The portfolio which maximizes $\mathbb{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\theta}{\sigma}$ and

$$\sup \mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + T\left(r + \frac{1}{2}\theta^2\right)$$

We now enlarge the filtration with S_1 (or equivalently, with B_1). In the enlarged filtration, setting, for $t < 1$, $\alpha_t = \frac{B_1 - B_t}{1-t}$, the dynamics of S are

$$dS_t = S_t((\mu + \sigma\alpha_t)dt + \sigma d\beta_t),$$

where β is defined in Proposition 3.5 and the dynamics of the wealth are

$$dX_t = rX_t dt + \pi_t\sigma X_t(d\beta_t + \widetilde{\theta}_t dt), \; X_0 = x$$

with $\widetilde{\theta}_t = \frac{\mu - r}{\sigma} + \alpha_t = \frac{\mu - r}{\sigma} - \frac{B_1 - B_t}{1-t}$. Assuming again that the stochastic integral which appears is a martingale, the portfolio which maximizes $\mathbb{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\widetilde{\theta}_s}{\sigma}$.

Then, for $T < 1$,

$$\begin{aligned}
\ln(X_T^{\pi,x,*}) &= \ln x + \int_0^T (r + \frac{1}{2}\widetilde{\theta}_s^2)ds + \int_0^T \widetilde{\vartheta}_s d\beta_s \\
\mathbb{E}(\ln(X_T^{\pi,x,*})) &= \ln x + \int_0^T (r + \frac{1}{2}(\theta^2 + \mathbb{E}(\alpha_s^2) + 2\theta\mathbb{E}(\alpha_s))ds \\
&= \ln x + (r + \frac{1}{2}\theta^2)T + \frac{1}{2}\int_0^T \mathbb{E}(\alpha_s^2)ds
\end{aligned}$$

where we have used the fact that $\mathbb{E}(\alpha_t) = 0$. Let

$$\begin{aligned}
V^{\mathbb{F}}(x) &= \max \mathbb{E}(\ln(X_T^{\pi,x}))\,;\ \pi \text{ is } \mathbb{F} \text{ adapted} \\
V^{\mathbb{G}}(x) &= \max \mathbb{E}(\ln(X_T^{\pi,x}))\,;\ \pi \text{ is } \mathbb{G} \text{ adapted}
\end{aligned}$$

Then $V^{\mathbb{G}}(x) = V^{\mathbb{F}}(x) + \frac{1}{2}\mathbb{E}\int_0^T \alpha_s^2 ds = V^{\mathbb{F}}(x) - \frac{1}{2}\ln(1-T)$.

If $T = 1$, the value function is infinite: there is an arbitrage opportunity and there does not exist an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \leq 1)$ is a $\mathbb{G}$-martingale. However, for any $\epsilon \in]0, 1]$, there exists a uniformly integrable $\mathbb{G}$-martingale L defined as

$$dL_t = \frac{\mu - r + \sigma\alpha_t}{\sigma} L_t d\beta_t, \, t \leq 1 - \epsilon, \quad L_0 = 1\,,$$

such that, setting $d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{P}|_{\mathcal{G}_t}$, the process $(e^{-rt}S_t, t \leq 1 - \epsilon)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.

The same computations can be done if μ and σ are $\mathbb{F}$-adapted.

3.3 Poisson bridge

Let N be a Poisson process with constant intensity 1, $\mathcal{F}_t^N = \sigma(N_s, s \le t)$ its natural filtration. The process $M_t = N_t - t$ is a martingale. Let $\mathcal{F}_t^{\sigma(N_T)} = \sigma(N_s, s \le t; N_T)$ be the natural filtration of N enlarged with the terminal value N_T of the process N.

Proposition 3.6. *The process*

$$\eta_t = M_t - \int_0^{t \wedge T} \frac{M_T - M_s}{T - s} ds,$$

is an $\mathbb{F}^{\sigma(N_T)}$-martingale, or

$$\eta_t = N_t - \int_0^t \frac{N_T - N_s}{T - s} ds = N_t - \Lambda_t$$

is an $\mathbb{F}^{\sigma(N_T)}$-martingale.

For $0 < s < t < T$,

$$\mathbb{E}(N_t - N_s | \mathcal{F}_t^{\sigma(N_T)} = \mathbb{E}(N_t - N_s | N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)$$

where the last equality follows from the fact that, if X and Y are independent with Poisson laws with parameters μ and ν respectively, then

$$\mathbb{P}(X = k | X + Y = n) = \frac{n!}{k!(n-k)!}\alpha^k (1-\alpha)^{n-k}$$

where $\alpha = \frac{\mu}{\mu+\nu}$. Hence,

$$\begin{aligned}
\mathbb{E}\left(\int_s^t du \frac{N_T - N_u}{T-u} | \mathcal{F}_s^{\sigma(N_T)}\right) &= \int_s^t \frac{du}{T-u}\left(N_T - N_s - \mathbb{E}(N_u - N_s | \mathcal{F}_s^{\sigma(N_T)})\right) \\
&= \int_s^t \frac{du}{T-u}\left(N_T - N_s - \frac{u-s}{T-s}(N_T - N_s)\right) \\
&= \int_s^t \frac{du}{T-s}(N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)\,.
\end{aligned}$$

Therefore,

$$\mathcal{F}_s^{\sigma(N_T)}) = \frac{t-s}{T-s}(N_T - N_s) - \frac{t-s}{T-s}(N_T - N_s) = 0$$

and the result follows.

Optimisation We suppose that the interest rate is null and that the risky asset has dynamics

$$dS_t = S_{t-}\left(\mu dt + \sigma dW_t + \phi dM_t\right)$$

Let $(X_t, t \geq 0)$ be the wealth of an un-informed agent whose portfolio is described by (π_t). Then

$$dX_t = \pi_t X_{t-}(\mu dt + \sigma dW_t + \phi dM_t) \tag{3.1}$$

Then,

$$X_t = x\exp\left(\int_0^t \pi_s(\mu - \phi\lambda)ds + \int_0^t \sigma\pi_s dW_s + \frac{1}{2}\int_0^t \sigma^2\pi_s^2 ds + \int_0^t \ln(1+\pi_s\phi)dN_s\right)$$

Then

$$\mathbb{E}[\ln(X_T)] = \ln(x) + \int_0^T \mathbb{E}(\mu\pi_s - \frac{1}{2}\sigma^2\pi_s^2 + \lambda(\ln(1+\phi\pi_s) - \phi\pi_s)ds\,.$$

Our aim is to solve

$$V(x) = \sup_\pi \mathbb{E}\left(\ln(X_T^{x,\pi})\right)$$

The maximum attainable wealth for the uninformed agent is obtained using the constant strategy $\tilde{\pi}$

for which

$$\tilde{\pi}\mu + \lambda[\ln(1+\tilde{\pi}\phi) - \tilde{\pi}\phi] - \frac{1}{2}\tilde{\pi}^2\sigma^2 = \sup_{\pi}[\pi\mu + \lambda[\ln(1+\pi\phi) - \pi\phi] - \frac{1}{2}\pi^2\sigma^2] \, .$$

Hence

$$\tilde{\pi} = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda - \sigma^2)^2 + 4\sigma^2\phi\mu}\right) \, .$$

The quantity under the square root is $(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\lambda$ and is non-negative. The sign to be used depends on the sign of quantities related to the parameters. The optimal $\widetilde{\pi}$ is the only one such that $1 + \phi\widetilde{\pi} > 0$.

We assume now that the informed agent knows N_T from time 0. Therefore, his wealth evolves according to the dynamics

$$dX_t^* = \pi_t X_{t-}^* [(\mu + \phi(\gamma_t - \lambda)]dt + \sigma dW_t + \phi dM_t^*]$$

where $\gamma_t = \frac{N_T - N_t}{T-t}$. The optimal portfolio π^* is now such that
$\mu - \lambda\phi + \phi\gamma_s[\frac{1}{1+\pi^*\phi}] - \pi^*\sigma^2 = 0$ and is given by

$$\pi_s^* = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\gamma_s}\right),$$