Chapter 1

Enlargement of filtration in discrete time

Contents

Introduction

In these notes, we present classical results on enlargement of filtration, in a discrete time framework. In such a setting, any $\mathbb{F}$-martingale is a semimartingale for any filtration $\mathbb{G}$ larger than $\mathbb{F}$, and one can think that there are not so many things to do. From our point of view, one interest of our paper is that the proofs of the semimartingale decomposition formula are simple, and give a pedagogical support to understand the general formulae obtained in the literature in continuous time. It can be noted that many results are established in continuous time under the hypothesis that all $\mathbb{F}$-martingales are continuous or, in the progressive enlargement case, that the random time avoids the $\mathbb{F}$-stopping times and the extension to the general case is difficult. In discrete time, one can not make any of such assumptions, since martingales are discontinuous and random times valued in the set of integers do not avoid $\mathbb{F}$-stopping times.
Many books are devoted to discrete time in Finance, among them Shreve [18], Shiryaev [19], and lecture notes are available on line Privault [15], Spreij [21].

In the first section, we recall some well know facts. Section 2 is devoted to the case of initial enlargement. Section 3 presents the case of progressive enlargement with a random time τ. We give a "model-free" definition of arbitrages in the context of enlargement of filtration, we study some examples in initial enlargement and give, in a progressive enlargement setting, necessary and sufficient conditions to avoid arbitrages before τ. We present the particular case of honest times (which are the standard example in continuous time) and we give conditions to obtain immersion property. We also give also various characterizations of pseudo-stopping times. In Section 4, we consider enlargement with a process, and in Section 5, we study credit risk.

1.1 Some well known Results and Definitions

In these notes, we are working in a discrete time setting: $X = (X_n, n \geq 0)$ is a process on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and $\mathbb{H} = (\mathcal{H}_n, n \geq 0)$ is a filtration, i.e., a family of σ-algebra such that $\mathcal{H}_n \subset \mathcal{H}_{n+1} \subset \mathcal{G}$. We note $\Delta X_n := X_n - X_{n-1}, n \geq 1$ the increment of X at time n and we set $\Delta X_0 = X_0$. An inequality (or equality) between two random variables is always $\mathbb{P}$ a.s.
We recall, for the ease of the reader some basic definitions.
A process X is $\mathbb{H}$-**adapted** if, for any $n \geq 1$, the random variable X_n is $\mathcal{H}_n$-measurable.
A process X is $\mathbb{H}$-**predictable** if, for any $n \geq 1$, the random variable X_n is $\mathcal{H}_{n-1}$-measurable and X_0 is a constant.
A process X is **integrable** (resp. square integrable) if $E(|X_n|) < \infty$ (resp. $\mathbb{E}(X_n^2) < \infty$) for all $n \geq 0$.
The process X_- is defined as the process equal to X_{n-1} at time n and to 0 for $n = 0$, this process is predictable.
A random variable ζ is said to be **positive** if $\zeta > 0$ a.s., a process X is positive if the r.v. X_n is positive for any $n \geq 0$ and a process A is **increasing** (resp. decreasing) if $A_n \geq A_{n-1}$ (resp. $A_n \leq A_{n-1}$) a.s. , for all $n \geq 1$. For two r.v. ζ and ξ, we write $\zeta < \xi$ for $\xi - \zeta > 0$, a.s. (resp. $\zeta \leq \xi$ for $\xi - \zeta \geq 0$); for two processes X, Y we write $X < Y$ for $X_n < Y_n$, for any $n \geq 0$.
For a probability measure $\mathbb{P}$, we note $\mathbb{P}_n$ (or $\mathbb{P}_n^{\mathbb{H}}$ in case of ambiguity) the restriction of $\mathbb{P}$ to $\mathcal{H}_n$.
We recall that, if $\zeta > 0$, and $\mathcal{G}$ a sigma-algebra, then $\mathbb{E}(\zeta|\mathcal{G}) > 0$. Indeed, for $A = \{\mathbb{E}(\zeta|\mathcal{G}) = 0\}$, one has $\mathbb{E}(\zeta \mathbb{1}_A) = 0$.
The tower property states that, if $\mathcal{F}$ and $\mathcal{G}$ are sigma-algebra, with $\mathcal{F} \subset \mathcal{G}$, then

$$\mathbb{E}(\mathbb{E}(\zeta|\mathcal{G})|\mathcal{F}) = \mathbb{E}(\mathbb{E}(\zeta|\mathcal{F})|\mathcal{G}) = \mathbb{E}(\zeta|\mathcal{F}) .$$

For a process X, we denote $X_{\infty^-} = \lim_{n\to\infty} X_n$, if the limit exists.

1.1.1 $\mathbb{H}$-martingales

An integrable $\mathbb{H}$-adapted process X is an $\mathbb{H}$-martingale (resp. an $\mathbb{H}$-supermartingale) if $\mathbb{E}(X_n|\mathcal{H}_{n-1}) = X_{n-1}$, or equivalently $\mathbb{E}(\Delta X_n|\mathcal{H}_{n-1}) = 0$ (resp. $\mathbb{E}(X_n|\mathcal{H}_{n-1}) \le X_{n-1}$).

We give some obvious results on the form of $\mathbb{H}$-martingales.

Proposition 1.1.1 *(a) The set of processes of the form* $(\psi_0 + \sum_{k=1}^n \big(\psi_k - \mathbb{E}(\psi_k|\mathcal{H}_{k-1})\big),\, n \ge 0)$ *where* ψ *is an* $\mathbb{H}$*-adapted integrable process is equal to the set of all* $\mathbb{H}$*-martingales (here,* $\sum_{k=1}^0 \centerdot = 0$*)*
(b) The set of processes of the form $(\psi_0 \prod_{k=1}^n \frac{\psi_k}{\mathbb{E}(\psi_k|\mathcal{H}_{k-1})}, n \ge 0)$ *where* ψ *is a positive integrable* $\mathbb{H}$*-adapted process is the set of all positive* $\mathbb{H}$*-martingales (here,* $\prod_{k=1}^0 \centerdot = 1$*).*

Proof. (a) Let X be a process such that $X_n = \psi_0 + \sum_{k=1}^n \big(\psi_k - \mathbb{E}(\psi_k|\mathcal{H}_{k-1})\big)$, $n \ge 0$ where ψ is an $\mathbb{H}$-adapted integrable process. Then, X is integrable, as a difference of integrable processes, and

$$\mathbb{E}(X_n - X_{n-1}|\mathcal{H}_{n-1}) \quad = \quad \mathbb{E}(\psi_n - \mathbb{E}(\psi_n|\mathcal{H}_{n-1})|\mathcal{H}_{n-1}) = 0\,.$$

Therefore X is an $\mathbb{H}$-martingale.
Let X be an $\mathbb{H}$-martingale. Then, $X_n = \Delta X_0 + \sum_{k=1}^n \Delta X_k - \mathbb{E}(\Delta X_k|\mathcal{H}_{k-1})$ with $\Delta X_0 = X_0$ and $\sum_{k=1}^0 \centerdot = 0$. For $\psi = \Delta X$, we obtain the result.
(b) Let $X_n = \psi_0 \prod_{k=1}^n \frac{\psi_k}{\mathbb{E}(\psi_k|\mathcal{H}_{k-1})}, n \ge 0$. Then,

$$\mathbb{E}(X_n|\mathcal{H}_{n-1}) = X_{n-1}\,\mathbb{E}(\frac{\psi_n}{\mathbb{E}(\psi_n|\mathcal{H}_{n-1})}|\mathcal{H}_{n-1}) = X_{n-1}\,.$$

If X is a positive martingale, setting $\psi_k = X_k$ leads to the result. □

Lemma 1.1.2 *A predictable martingale* X *is constant.*

Proof. Indeed, $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = X_n$ and $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = X_{n-1}$ lead to $X_n = X_0$, for any n. □

1.1.2 Doob's Decomposition and Applications

Doob's decomposition

Lemma 1.1.3 *Any integrable* $\mathbb{H}$*-adapted process* X *is a special* $\mathbb{H}$*-semimartingale*[1] *with (unique) decomposition* $X = M^{X,\mathbb{H}} + V^{X,\mathbb{H}}$ *where* $M^{X,\mathbb{H}}$ *is an* $\mathbb{H}$*-martingale and* $V^{X,\mathbb{H}}$ *is an* $\mathbb{H}$*-predictable process with* $V_0^{X,\mathbb{H}} = 0$*. Furthermore,*

$$\Delta V_n^{X,\mathbb{H}} = \mathbb{E}(\Delta X_n|\mathcal{H}_{n-1}), \forall n \ge 1\,.$$

The process $M^{X,\mathbb{H}}$ *is called the martingale part of* X *and* $V^{X,\mathbb{H}}$ *the predictable part of* X*.*

Proof. In the proof, $V := V^{X,\mathbb{H}}$ and $M := M^{X,\mathbb{H}}$. Setting $V_0 = 0$ and, for $n \ge 1$

$$V_n - V_{n-1} = \mathbb{E}(X_n - X_{n-1}|\mathcal{H}_{n-1}),$$

we construct an $\mathbb{H}$-predictable process. This leads to

$$\Delta M_n = \Delta X_n - \Delta V_n = X_n - \mathbb{E}(X_n|\mathcal{H}_{n-1}), \forall n \ge 1\,.$$

Setting $M_0 = X_0$, the process M is an $\mathbb{H}$-martingale from Proposition 1.1.1. □

In what follows, we shall also denote V^X (resp. $V^{\mathbb{H}}$) the $\mathbb{H}$-predictable part of X if there are no ambiguity on the choice of the filtration (resp. on the choice of the process).
As an immediate corollary, we obtain the Doob decomposition of supermartingales:

[1]A special semimartingale is an adapted process X such that $X = M + V$ where M is a martingale and V a predictable process.

Corollary 1.1.4 *If X is an $\mathbb{H}$-adapted supermartingale, it admits a unique decomposition*

$$X = M^X - A^X$$

where M^X is an $\mathbb{H}$-martingale and A^X is an increasing $\mathbb{H}$-predictable process with $A_0^X = 0$.

Proof. The supermartingale property of X implies that $\Delta V^X \leq 0$. It remains to set $A^X = -V^X$. □

Of course, a process of the form $M - A$ where M is a martingale and A an increasing process is a supermartingale.

Predictable brackets

Proposition 1.1.5 *If X and Y are square integrable $\mathbb{H}$-martingales, there exists a unique $\mathbb{H}$-predictable process $V^{X,Y}$ such that $V_0^{X,Y} = 0$ and $XY - V^{X,Y}$ is an $\mathbb{H}$-martingale. Furthermore*

$$\Delta V_n^{X,Y} = \mathbb{E}(Y_n \Delta X_n | \mathcal{H}_{n-1}) = \mathbb{E}(\Delta Y_n \Delta X_n | \mathcal{H}_{n-1}) \, , \; n \geq 1.$$

The process $\langle X, Y \rangle := V^{X,Y}$ is called the predictable bracket of the two martingales X and Y.

Proof. Indeed, from Lemma 1.1.3, and using the martingale property of X and Y, we have, for $n \geq 1$:

$$\begin{aligned}
\Delta V_n^{X,Y} = V_n^{X,Y} - V_{n-1}^{X,Y} &= \mathbb{E}(X_n Y_n - X_{n-1} Y_{n-1} | \mathcal{H}_{n-1}) \\
&= \mathbb{E}(Y_n \Delta X_n | \mathcal{H}_{n-1}) + \mathbb{E}(X_{n-1} \Delta Y_n | \mathcal{H}_{n-1}) = \mathbb{E}(Y_n \Delta X_n | \mathcal{H}_{n-1}) \\
&= \mathbb{E}(\Delta Y_n \Delta X_n | \mathcal{H}_{n-1}) \, .
\end{aligned}$$

We have used that, from the martingale property of Y,

$$\mathbb{E}(X_{n-1} \Delta Y_n | \mathcal{H}_{n-1}) = X_{n-1} \mathbb{E}(\Delta Y_n | \mathcal{H}_{n-1}) = 0 \, .$$

□

The predictable bracket of two semimartingales X, Y is defined in continuous time as the dual predictable projection of the covariation process, that is $\langle X, Y \rangle := [X, Y]^p$. For discrete time semimartingales, we adopt the same definition. The covariation process is

$$[X, Y]_0 = 0 \, , \; [X, Y]_n := \sum_{k=1}^{n} \Delta X_k \, \Delta Y_k, n \geq 1,$$

and $[X, Y]^p$ is the unique predictable (bounded variation) process null at time 0, such that $[X, Y] - [X, Y]^p$ is a martingale, i.e., $[X, Y]^p$ is the predictable part of the semimartingale $[X, Y]$.

Lemma 1.1.6 *Let X, Y be two $\mathbb{H}$-adapted processes (hence, semimartingales) square integrable. Then, the predictable bracket of the semimartingales X, Y, i.e., the predictable part of the semimartingale $[X, Y]$ is the process defined as*

$$\langle X, Y \rangle_0 = 0 \, , \; \Delta \langle X, Y \rangle_n = \mathbb{E}(\Delta X_n \, \Delta Y_n | \mathcal{H}_{n-1}) \, , \; n \geq 1 \, .$$

Proof. From Doob's decomposition (Lemma 1.1.3), for $n \geq 1$,

$$(\Delta [X, Y]^p)_n = \mathbb{E}([X, Y]_n - [X, Y]_{n-1} | \mathcal{H}_{n-1}) = \mathbb{E}(\Delta X_n \, \Delta Y_n | \mathcal{H}_{n-1}) \, .$$

□

Note that the predictable bracket depends on the filtration, which is not the case for continuous semimartingales in a continuous time setting.

Stochastic integral of adapted processes and martingale property

Definition 1.1.7 *The* ***stochastic integral*** *of a process Y w.r.t. a process X is the process $Y \centerdot X$ defined as $(Y \centerdot X)_n := \sum_{k=1}^n Y_k \Delta X_k,\ n \geq 0$. In the case where $X = (X^1, \cdots, X^d)$ is a d-dimensional process, for a d-dimensional process Y, one defines $(Y \centerdot X)_n := \sum_{k=1}^n \sum_{j=1}^d Y_k^j \Delta X_k^j,\ n \geq 1$.*

Proposition 1.1.8 ***Integration by parts formula.*** *For two processes X and Y*

$$XY = X_0 Y_0 + X_- \centerdot Y + Y_- \centerdot X + [X, Y] = X_0 Y_0 + X_- \centerdot Y + Y \centerdot X\,.$$

Proof. This equality is based on

$$\Delta(XY)_n = X_{n-1}\Delta Y_n + Y_{n-1}\Delta X_n + \Delta X_n\, \Delta Y_n = X_{n-1}\Delta Y_n + Y_n \Delta X_n.$$

□

Lemma 1.1.9 *If X is a square integrable $\mathbb{H}$-martingale and H an $\mathbb{H}$-predictable square integrable process, then the process $H \centerdot X$ is an $\mathbb{H}$-martingale.*

Proof. For H predictable,

$$\mathbb{E}(H_n \Delta X_n | \mathcal{H}_{n-1}) = H_n \mathbb{E}(\Delta X_n | \mathcal{H}_{n-1}) = \mathbb{E}(\Delta M_n^H \Delta X_n | \mathcal{H}_{n-1}) = 0$$

and the result is obvious. □

Lemma 1.1.10 *If X and Y are two square integrable $\mathbb{H}$-martingales then $XY - [X, Y]$ is an $\mathbb{H}$-martingale. In particular, $X^2 - [X]$ is an $\mathbb{H}$-martingale for a square-integrable $\mathbb{H}$-martingale X.*

Proof. This is a direct consequence of integration by parts formula and the fact that X_- and Y_- are predictable. □

Definition 1.1.11 *Two square integrable martingales X and Y are said to be orthogonal if their product is a martingale, i.e. if $\mathbb{E}(\Delta(XY)_n | \mathcal{H}_{n-1}) = 0$. This condition is equivalent to any of the following assertions*
(a) $\mathbb{E}(\Delta Y_n \Delta X_n | \mathcal{H}_{n-1}) = 0$
(b) $\mathbb{E}(Y_n \Delta X_n | \mathcal{H}_{n-1}) = 0$
(c) $[X, Y]$ *is a martingale*
(d) $\langle X, Y \rangle = 0$.

Proof. [of the various equivalence conditions] From integration by parts formula, the orthogonality is equivalent to $[X, Y]$ is a martingale, which is equivalent to the two other conditions, due to $\Delta(XY)_n = X_{n-1}\Delta Y_n + Y_n \Delta X_n$, and the fact that $X_- \centerdot Y$ and $Y_- \centerdot X$ are martingales. □

Lemma 1.1.12 *If X is a square integrable $\mathbb{H}$-martingale and H an $\mathbb{H}$-adapted square integrable process with $\mathbb{H}$-martingale part orthogonal to X, then the process $H \centerdot X$ is an $\mathbb{H}$-martingale.*

Proof. Let $H = M^H + V^H$. Since

$$\mathbb{E}(H_n \Delta X_n | \mathcal{H}_{n-1}) = \mathbb{E}(M_n^H \Delta X_n | \mathcal{H}_{n-1}) + V_n^H \mathbb{E}(\Delta X_n | \mathcal{H}_{n-1}) = \mathbb{E}(\Delta M_n^H \Delta X_n | \mathcal{H}_{n-1}) = 0$$

the result is obvious. If H is predictable, $M^H = 0$, therefore $H \centerdot X$ is an $\mathbb{H}$-martingale. □

We now give the isometry formula, similar to the one in continuous time.

Proposition 1.1.13 *If X is a martingale and XY square integrable, and if $Y \centerdot X$ is a martingale, then $\mathbb{E}(((Y \centerdot X)_n)^2) = \sum_{k=1}^n \mathbb{E}(Y_k^2 (\Delta X_k)^2)$.*

Proof. Since $Y \centerdot X$ is a martingale, $\mathbb{E}\big(\Delta((Y \centerdot X)^2)_k \big| \mathcal{H}_{k-1}\big) = \mathbb{E}\big((\Delta(Y \centerdot X)_k)^2 \big| \mathcal{H}_{k-1}\big)$. Then, using $\Delta(Y \centerdot X)_k = Y_k \Delta X_k$ and taking expectations, we get

$$\mathbb{E}\big(\Delta((Y \centerdot X)^2)_k\big) = \mathbb{E}\big(Y_k^2 (\Delta X_k)^2\big) \; .$$

Finally, taking the sum for k from 0 to n, we obtain

$$\mathbb{E}\big((Y \centerdot X)_n^2\big) = \mathbb{E}\Big(\sum_{k=1}^{n} Y_k^2 (\Delta X_k)^2\Big) \; , \quad \forall n \geq 0 \; .$$

□

1.1.3 Projections

In this section, $\mathbb{H}$ is a filtration, and X is a process, not assumed to be $\mathbb{H}$ adapted. For future use, we mimic two definitions important in continuous time. In particular, we introduce optional projections, even if in discrete time, optional means adapted. We keep the continuous time denomination to make easier the comparison between both results for readers aware about continuous time.

Definition 1.1.14 *The $\mathbb{H}$-optional projection of an integrable process X is the $\mathbb{H}$-adapted process ${}^oX$ defined as ${}^oX_n = \mathbb{E}(X_n|\mathcal{H}_n)$. The $\mathbb{H}$-predictable projection of a process X is the $\mathbb{H}$-predictable process ${}^pX$ defined as ${}^pX_n = \mathbb{E}(X_n|\mathcal{H}_{n-1})$.*

Definition 1.1.15 *The $\mathbb{H}$-dual optional projection of an increasing process X is the increasing adapted process X^o defined as $\Delta X_n^o = \mathbb{E}(\Delta X_n|\mathcal{H}_n)$ for $n \geq 1$ and $X_0^o = \mathbb{E}(X_0|\mathcal{H}_0)$.*
The $\mathbb{H}$-dual predictable projection of an increasing process X is the increasing predictable process X^p defined as $\Delta X_n^p = \mathbb{E}(\Delta X_n|\mathcal{H}_{n-1})$ for $n \geq 1$ and $X_0^p = 0$. If X is an $\mathbb{H}$-adapted process, $X^p = V$ where V is the predictable part of the semimartingale X.

The dual $\mathbb{H}$-optional projection of X, satisfies

$$\mathbb{E}\Big((Y \centerdot X)_{\infty^-}\Big) = \mathbb{E}\Big((Y \centerdot X^o)_{\infty^-}\Big) \tag{1.1}$$

for any non negative bounded $\mathbb{H}$-adapted process Y. The dual $\mathbb{H}$-predictable projection of X, satisfies

$$\mathbb{E}\Big((Y \centerdot X)_{\infty^-}\Big) = \mathbb{E}\Big((Y \centerdot X^p)_{\infty^-}\Big) \tag{1.2}$$

for any non negative bounded $\mathbb{H}$-predictable process Y.
Using that any process X can be written as $X = X^\uparrow + X^\downarrow$ where $X^\uparrow$ (resp. $X^\downarrow$) is increasing (resp. decreasing), one can also define dual projections for any process X as $(X^\uparrow)^p - (X^\downarrow)^p$. Indeed, $X^\uparrow$ and $X^\downarrow$ can be constructed as follows.

$$X_n^\uparrow = \sum_{k=1}^{n} (X_k - X_{k-1})^+, \quad X_n^\downarrow = \sum_{k=1}^{n} (X_k - X_{k-1})^F -$$

are increasing and since

$$(X_k - X_{k-1})^+ - (X_k - X_{k-1})^- = \Delta X_k \, ,$$

the result is proved.

Exercise 1.1.16 Prove that for a pair of processes (X, Y), the following duality formulae hold

$$\begin{aligned} \mathbb{E}((X \centerdot Y^p)_{\infty^-}) &= \mathbb{E}(({}^pX \centerdot Y)_{\infty^-}) \\ \mathbb{E}((X \centerdot Y^p)_{\infty^-}) &= \mathbb{E}(({}^pX \centerdot Y)_{\infty^-}) \, . \end{aligned}$$

Proposition 1.1.17 *The processes ${}^oX - X^o$ and ${}^oX - X^p$ are martingales*

Proof. Using the tower property

$$\begin{aligned}\mathbb{E}(\Delta({}^oX - X^o)_n|\mathcal{H}_{n-1}) &= \mathbb{E}\Big(\mathbb{E}(X_n|\mathcal{H}_n) - \mathbb{E}(X_n|\mathcal{H}_{n-1}) - \mathbb{E}(X_n - X_{n-1}|\mathcal{H}_n)|\mathcal{H}_{n-1}\Big) \\ &= \mathbb{E}\Big(- \mathbb{E}(X_n|\mathcal{H}_{n-1}) + \mathbb{E}(X_{n-1}|\mathcal{H}_n)|\mathcal{H}_{n-1}\Big) = 0\,.\end{aligned}$$

The proof that ${}^oX - X^p$ is a martingale is left to the reader. □

1.1.4 Multiplicative decomposition

Theorem 1.1.18 *Let X be an $\mathbb{H}$-adapted integrable positive process, then X can be represented in a unique form as*

$$X = K^X N^X,$$

where K^X is an $\mathbb{H}$-predictable process with $K^X_0 = 1$ and N^X is an $\mathbb{H}$-martingale. More precisely,

$$\begin{aligned} N^X_0 &= X_0, \qquad & N^X_n &= X_0 \prod_{k=1}^{n} \frac{X_k}{\mathbb{E}(X_k|\mathcal{H}_{k-1})}, \quad \forall n \geq 1\,, \\ K^X_0 &= 1, \qquad & K^X_n &= \prod_{k=1}^{n} \frac{\mathbb{E}(X_k|\mathcal{H}_{k-1})}{X_{k-1}}, \quad \forall n \geq 1. \end{aligned}$$

Proof. For each $n \geq 1$ fixed, the positive random variable N^X_n, is integrable since by recurrence $\mathbb{E}[N^X_n] = \mathbb{E}\big[N^X_{n-1}\mathbb{E}\big(\frac{X_n}{\mathbb{E}(X_n|\mathcal{H}_{n-1})}|\mathcal{H}_{n-1}\big)\big] = \mathbb{E}\big(N^X_{n-1}\big) = X_0$, and from Proposition 1.1.1, N^X is a martingale.

In the other hand, the process K^X, defined by

$$K^X_n = \frac{X_n}{X_0 \prod_{k=1}^n \frac{X_k}{\mathbb{E}(X_k|\mathcal{H}_{k-1})}} = \prod_{k=1}^{n} \frac{\mathbb{E}(X_k|\mathcal{H}_{k-1})}{X_{k-1}}$$

is an $\mathbb{H}$-predictable process. □

Remark 1.1.19 In terms of Doob's decomposition $X = M^X + V^X$, one has

$$K^X_n = \frac{M^X_{n-1} + V^X_n}{X_{n-1}} K^X_{n-1} = X_0 \prod_{k=1}^{n} \frac{M^X_{k-1} + V^X_k}{X_{k-1}}, \;\; N^X_n = X_n / K^X_n,\; n \geq 1\,.$$

Corollary 1.1.20 *Any positive $\mathbb{H}$-supermartingale Y admits a unique multiplicative predictable decomposition $Y = N^Y D^Y$ where N^Y is an $\mathbb{H}$-martingale and D^Y an $\mathbb{H}$-predictable decreasing process with $D^Y_0 = 1$. Conversely, any process of the form $Y = XD$ where X is an $\mathbb{H}$-martingale and D an $\mathbb{H}$-predictable decreasing process is a supermartingale.*

Proof. The process $D = K^Y$ is indeed decreasing if Y is a supermartingale. □

1.1.5 Stochastic Exponential Process

Given a process X, we define the stochastic exponential of X denoted by $\mathcal{E}(X)$ as the solution of the following equation in differences:

$$\begin{cases} \Delta\mathcal{E}(X)_n &= \mathcal{E}(X)_{n-1}\Delta X_n\,, \quad \forall n \geq 1\,, \\ \mathcal{E}(X)_0 &= 1\,. \end{cases} \tag{1.3}$$

Proposition 1.1.21 *The solution of (1.3), is given by*

$$\mathcal{E}(X)_n := \Pi_{k=1}^n (\Delta X_k + 1)\,, \quad \forall n \geq 1\,. \tag{1.4}$$

If X is a martingale, $\mathcal{E}(X)$ is a martingale. If $\Delta X_n > -1$, for all $n \geq 0$, then $\mathcal{E}(X)$ is positive.

Proof. The equality is obtained by recurrence. The martingale property is easily checked from (1.3). □

Proposition 1.1.22 *If Y is a positive process with $Y_0 = 1$, there exists a unique process X such that $Y = \mathcal{E}(X)$.*

Proof. Set $\Delta X_n = \frac{\Delta Y_n}{Y_{n-1}}$. □

We now give the obvious relation between stochastic exponential and exponential

Proposition 1.1.23 *If X is a process such that $\Delta X > -1$ then $\mathcal{E}(X)_n = e^{U_n}$ where $U_0 = 0$ and $\Delta U_n = \log(1 + \Delta X_n)$.*

Proof. Note that $e^{U_n} = e^{U_{n-1}}(1 + \Delta X_n)$. □

Lemma 1.1.24 *Let ψ and γ be predictable and M and N be two processes. Then*

$$\mathcal{E}(\psi \centerdot M)\mathcal{E}(\gamma \centerdot N) = \mathcal{E}(\psi \centerdot M + \gamma \centerdot N + \psi\gamma \centerdot [M, N])\,.$$

Proof. By definition, the two sides are equal to 1 at time 0. For $n \geq 1$, the left-hand side $K_n := \mathcal{E}(\psi \centerdot M)_n \mathcal{E}(\gamma \centerdot N)_n$ satisfies $K_n = K_{n-1}(1 + \psi_n \Delta M_n)(1 + \gamma_n \Delta N_n)$. The right-hand side $J_n := \mathcal{E}(\psi \centerdot M + \gamma \centerdot N + \psi\gamma \centerdot [M, N])_n$ satisfies $J_n = J_{n-1}(1 + \psi_n \Delta M_n + \gamma_n \Delta N_n + \psi_n \gamma_n \Delta M_n \Delta N_n)$. Assuming by recurrence that $K_{n-1} = J_{n-1}$, the result follows. □

This result is known in continuous time as Yor's equality.

1.1.6 Stopping Times and Local Martingales

Random and Stopping times

A random time is a random variable valued in $\mathbb{N} \cup \{+\infty\}$, it is an $\mathbb{H}$-stopping time if $\{\tau \leq n\} \in \mathcal{H}_n$, for any $n \geq 0$. A random time τ is an $\mathbb{H}$-predictable time if $\{\tau = 0\} \in \mathcal{H}_0$ and $\{\tau \leq n\} \in \mathcal{H}_{n-1}$, for any $n \geq 0$. A predictable time is, obviously, a stopping time. If τ is an $\mathbb{H}$-stopping time, we define the two σ-algebra of events before τ and strictly before τ

$$\begin{aligned} \mathcal{H}_\tau &= \{F \in \mathcal{H}_\infty \,:\, F \cap \{\tau \leq n\} \in \mathcal{H}_n, \forall n\} \\ \mathcal{H}_{\tau-} &= \sigma\{F \cap \{n < \tau\} \text{ for } F \in \mathcal{H}_n\}. \end{aligned}$$

Obviously $\mathcal{H}_{\tau-} \subset \mathcal{H}_\tau$. In discrete time, this inclusion can be strict. If τ is a random time, the **stopped** process X^τ is defined as $X_n^\tau = X_{\tau \wedge n}$.

Lemma 1.1.25 *If $A \in \mathcal{H}_\infty$, then $A \cap \{\tau = +\infty\}$ is $\mathcal{H}_{\tau-}$-measurable.*

Proof. For n fixed and $B \in \mathcal{H}_n$, one has $B \cap \{\tau = +\infty\} \in \mathcal{H}_{\tau-}$ as $\{\tau = +\infty\} = \cap_{m \geq n}\{\tau > m\}$. It follows that, if $A \in \mathcal{H}_\infty$, $\forall n$, $(A \cap \{\tau \leq n\}) \cap \{\tau = +\infty\} \in \mathcal{H}_{\tau-}$. Then

$$A \cap \{\tau = +\infty\} = \cup (A \cap \{\tau \leq n\}) \cap \{\tau = +\infty\} \in H_{\tau-}$$

If τ is a random time,

$$\begin{aligned}\mathcal{H}_\tau &= \{X_\tau \,:\, X \text{ is an adapted process}\}\\ \mathcal{H}_{\tau-} &= \{X_\tau \,:\, X \text{ is a predictable process}\}.\end{aligned}$$

Note that, for $\tau \equiv n$, one has $\mathcal{H}_n = \mathcal{H}_\tau$ and $\mathcal{H}_{n-1} = \mathcal{H}_{\tau-}$. The **restriction** of a random time to a given set $F \in \mathcal{H}_\infty$ is defined as

$$\tau_F(\omega) := \begin{cases}\tau(\omega) & \omega \in F\\ \infty & \omega \notin F\,.\end{cases} \tag{1.5}$$

In particular $0_A = 0$ on A and ∞ elsewhere.

Proposition 1.1.26 *(a)If X is predictable, $X_\tau \mathbb{1}_{\tau<\infty}$ is $\mathcal{H}_{\tau-}$ measurable for any stopping time τ.*
(b) The process X is predictable iff $X_\tau \mathbb{1}_{\tau<\infty}$ is $\mathcal{H}_{\tau-}$ measurable for any predictable stopping time τ.
(c) If X is adapted $X_\tau \mathbb{1}_{\tau<\infty}$ is $\mathcal{H}_\tau$-measurable.
(d) A random variable ζ in $\mathcal{H}_\infty$ is $\mathcal{H}_{\tau-}$-measurable if and only if there exists a predictable process X such that $X_\tau = \zeta$ on $\{\tau < +\infty\}$.

Proof. a) As the predictable process are generated by the process $X = \mathbb{1}_{[\![0_A]\!]}$, $A \in \mathcal{H}_0$ and for $X = \mathbb{1}_{]S,T]}$, where S, T are two stopping times valued in $\mathbb{N}$, it is sufficient to prove it for these processes.
If $X = \mathbb{1}_{[\![0_A]\!]}$, then $X_\tau \mathbb{1}_{\{\tau<+\infty\}} = \mathbb{1}_{A\cap\{\tau=0\}}$ and the result is obvious. In the second case,

$$X_\tau \mathbb{1}_{\{\tau<+\infty\}} = (\mathbb{1}_{\{S<\tau\}} - \mathbb{1}_{\{T<\tau\}})\mathbb{1}_{\{\tau<+\infty\}}\,.$$

As $\mathbb{1}_{\{S<\tau\}}\mathbb{1}_{\{\tau<+\infty\}} = \sum_n \mathbb{1}_{\{S=n\}}\mathbb{1}_{\{n<\tau\}}\mathbb{1}_{\{\tau<+\infty\}}$, then $\mathbb{1}_{\{S<\tau\}} - \mathbb{1}_{\{T<\tau\}})\mathbb{1}_{\{\tau<+\infty\}}$ is $\mathcal{H}_{\tau-}$-measurable.
b) For the sufficient condition, we can choose $\tau \equiv n$ for a fixed n, τ is predictable and $X_\tau \mathbb{1}_{\{\tau<+\infty\}} = X_n \in \mathcal{H}_{n-1}$. Then X is predictable.
d) Let ζ in $\mathcal{H}_\infty$. We suppose that there exists a predictable process X such that $X_\tau = \zeta$ on $\{\tau < +\infty\}$. Then $\zeta = \zeta\mathbb{1}_{\{\tau=+\infty\}} + X_\tau \mathbb{1}_{\{\tau<+\infty\}}$. The random variable $\zeta\mathbb{1}_{\{\tau=+\infty\}}$ is $\mathcal{H}_{\tau-}$-measurable. The result follows from Lemma 1.1.25 and the first assertion. Conversely, it is sufficient to prove the assertion, if $\zeta = \mathbb{1}_B$ with $B \in \mathcal{H}_0$ or if $\zeta = \mathbb{1}_{B\cap\{n<\tau\}}$ with $B \in \mathcal{H}_n$. In this case, we can choose $X = \mathbb{1}_{0_A}$ in the first case and $X = \mathbb{1}_{0_A}$ in the second.

Lemma 1.1.27 *One has $\mathcal{H}_{\tau-} \subset \mathcal{H}_\tau$. In general the inclusion is strict.*

Proof. Take $X_n = \sum_{i=0}^n Y_i$ where Y_i are i.i.d. random variables, non constant and $\mathbb{F}$ the natural filtration of X. Let $\tau = \inf\{n \,:\, X_n \geq a\}$ and $\zeta = X_\tau \mathbb{1}_{\zeta<\infty}$ is $\mathcal{F}_\tau$ measurable, but is not $\mathcal{F}_{\tau-}$ measurable. □

Exercise 1.1.28 Prove that, for any $\mathbb{H}$-stopping time T valued in $\mathbb{N}$, and any process X,

$$\mathbb{E}(X_T \mathbb{1}_{\{T<\infty\}}) \;=\; \mathbb{E}({}^oX_T \mathbb{1}_{\{T<\infty\}})$$

and that for any predictable stopping time S,

$$\mathbb{E}(X_S \mathbb{1}_{\{S<\infty\}}) = \mathbb{E}({}^pX_S \mathbb{1}_{\{S<\infty\}})\,.$$

Local martingales

We collect classical results on local martingales (see [21, 19]).

Definition 1.1.29 *The process X is a local martingale if it is adapted and if there exists an increasing sequence $(\tau_k, k \geq 1)$ of $\mathbb{H}$-stopping times such that $\mathbb{P}(\lim \tau_k = \infty) = 1$ and for any k, the stopped process X^{τ_k} is a martingale.*

We denote by $\mathcal{M}_{loc}$ the set of local martingales.

We borrow from [17] an example of a process which is a local martingale but not a martingale.

Let Y be a random variable which is integrable but not square integrable and U a random variable taking the value 1 and the value -1 with probability $1/2$, independent of Y. We define $\mathcal{F}_0 = \sigma(Y)$ and $\mathcal{F}_n = \sigma(Y, U)$ for all $n \geq 1$. Let us consider $M_0 = Y$ and $M_n = Y + UY^2$ if $n \geq 1$. Then M is a local martingale which is not a martingale. Indeed M_1 is not integrable.
Let τ_k be the $\mathbb{F}$-stopping time $\tau_k = \min\{n, |M_n| \geq k\}$, τ_k goes to infinity a.s when k goes to infinity (as usual the minimum of an empty set if $+\infty$). We show that M^{τ_k} is a martingale.
Let k fixed, by definition, $M_0 = Y$ is integrable, hence $M_0^{\tau_k} = M_0$ is integrable and

$$\begin{aligned} E(|M_1^{\tau_k}|) &= E(|M_1^{\tau_k}|\mathbb{1}_{\{\tau_k=0\}}) + E(|M_1^{\tau_k}|\mathbb{1}_{\{\tau_k>0\}}) \\ &= E(|M_0|\mathbb{1}_{\{\tau_k=0\}}) + E(|M_1|\mathbb{1}_{\{\tau_k>0\}}) \leq E(|M_0|) + k + k^2 \end{aligned}$$

since $M_1 = M_0 + U(M_0)^2$ and $M_0 < k$ on $\tau_k > 0$. Then $M_n^{\tau_k}$ is integrable for all n. Moreover, to prove the martingale property, we just have to check it for $n = 1$

$$\begin{aligned} \mathbb{E}(M_1^{\tau_k}|\mathcal{F}_0) &= E(M_0^{\tau_k}\mathbb{1}_{\{\tau_k=0\}}|\mathcal{F}_0) + \mathbb{E}(M_1^{\tau_k}\mathbb{1}_{\{\tau_k>0\}}|\mathcal{F}_0) \\ &= M_0\mathbb{1}_{\{\tau_k=0\}} + \mathbb{1}_{\{\tau_k>0\}}(M_0 + M_0^2\mathbb{E}(U)) = M_0 = M_0^{\tau_k} \end{aligned}$$

where we used the fact that $\{\tau_k = 0\} \in \mathcal{F}_0$ and $\{\tau_k > 0\} \in \mathcal{F}_0$ and that $\mathbb{E}(U|\mathcal{F}_0) = \mathbb{E}(U) = 0$.

Generalized martingales

Let $\mathcal{F}$ and $\mathcal{G}$ be two σ-algebra with $\mathcal{F} \subset \mathcal{G}$. If ζ is a positive $\mathcal{G}$-measurable random variable, one can define $\mathbb{E}(\zeta|\mathcal{F})$ even if ζ is not integrable. Indeed, there exists an $\mathcal{F}$-measurable random variable $\widehat{\zeta}$ (which can take $+\infty$ value) such that $\mathbb{E}(\zeta\mathbb{1}_F) = \mathbb{E}(\widehat{\zeta}\mathbb{1}_F)$ for all $F \in \mathcal{F}$. We write $\widehat{\zeta} = \mathbb{E}(\zeta|\mathcal{F})$.

Definition 1.1.30 *The process X is a generalized martingale (G.M.) if it is adapted and if $\mathbb{E}(|X_{n+1}|\,|\mathcal{H}_n) < \infty$ for any n and $\mathbb{E}(X_{n+1}|\mathcal{H}_n) = X_n$ where $\mathbb{E}(X_{n+1}|\mathcal{H}_n) = \mathbb{E}(X_{n+1}^+|\mathcal{H}_n) - \mathbb{E}(X_{n+1}^-|\mathcal{H}_n)$.*

One can define optional projection in case of generalized expectation under the weaker hypothesis $\mathbb{E}(|X_n||\mathcal{H}_n) < \infty$.

We denote by $\mathcal{GM}$ the set of generalized martingales.

Proposition 1.1.31 *A G.M. X is a local martingale*
Let X be a local martingale such that $\mathbb{E}(X_n^-) < \infty$ for all n, or that $\mathbb{E}(X_n^+) < \infty$ for all n, then X is a martingale. In particular, if X is an integrable local martingale, it is a martingale.

Proposition 1.1.32 *A real valued process X is a local martingale iff the following two assertions are hold true.*
(i) $X_0 \in \mathbb{L}^1(\Omega, \mathcal{F}_0; \mathbb{P})$
(ii) $\mathbb{E}(|X_{n+1}||\mathcal{F}_n] < \infty$ a.s. and $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ for all n in the sense of generalized conditional expectations.
If X is a nonnegative local martingale with $\mathbb{E}(X_0) < \infty$, then it is a true martingale.

In particular, if X is a local martingale and $\mathcal{E}(X)$ is non negative, $\mathcal{E}(X)$ is a martingale.

Proposition 1.1.33 *If H is a predictable process and M a local martingale, then $H \centerdot M$ is a local martingale.*

Definition 1.1.34 *Two local martingales X and Y are orthogonal if XY is a local martingale.*

1.1.7 Change of probability

Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures defined on $\mathcal{H}_\infty$. The probability measure $\mathbb{Q}$ is locally absolutely continuous w.r.t. the probability measure $\mathbb{P}$ if $\mathbb{Q}_n \mathbb{P}_n$ for all $n \geq 0$. In this case we can define the $\mathcal{H}_n$-measurable integrable non negative random variable $L_n = d\mathbb{Q}_n/d\mathbb{P}_n$.The process L is called the density process and is a $\mathbb{P}$ martingale. This process can vanish, however, $L_n = 0$ on the set $\{L_{n-1} = 0\}$, and we take the convention that, on the set $\{L_{n-1} = 0\}$, $L_n/L_{n-1} = 0$.

Lemma 1.1.35 *Let $\mathbb{Q} << \mathbb{P}$ on a σ-algebra $\mathcal{G}$ with Radon-Nikodym derivative $L := d\mathbb{Q}/d\mathbb{P}$ and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{G}$. Suppose that X is $\mathbb{Q}$ integrable. Then, $\mathbb{Q}(\mathbb{E}[L|\mathcal{F}] > 0) = 1$ and one has*

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}] = \frac{\mathbb{E}[XL|\mathcal{F}]}{\mathbb{E}[L|\mathcal{F}]} \ \mathbb{Q} - a.s$$

Proof. In a first step, we prove that $\mathbb{Q}(\mathbb{E}[L|\mathcal{F}] > 0) = 1$ and the martingale L is positive $\mathbb{Q}$-as.. Indeed,

$$\mathbb{Q}(L = 0) = \mathbb{E}_P(L\mathbb{1}_{\{L=0\}}) = 0, .$$

In our setting, the Bayes formula states that, for any integrable $\mathcal{H}_N$ measurable random variable Y and $n \leq N$,

$$\mathbb{E}_{\mathbb{Q}}(Y|\mathcal{H}_n) = \frac{1}{L_n}\mathbb{E}_{\mathbb{P}}(YL_N|\mathcal{H}_n) \ \mathbb{Q} - a.s.$$

□

If $\mathbb{P}_n \sim \mathbb{Q}_n$ for all n, $\mathbb{P}$ and $\mathbb{Q}$ are called locally equivalent. In that case, L^{-1} is a $\mathbb{Q}$-martingale.

Proposition 1.1.36 *(i) Let X be an adapted process and assume $\mathbb{Q}$ locally absolutely continuous w.r.t. $\mathbb{P}$ with density process L. The process X is a martingale under $\mathbb{Q}$ if the process XL is a martingale under $\mathbb{P}$.*
(ii) The process X is a local martingale under $\mathbb{Q}$ if the process XL is a local martingale under $\mathbb{P}$.
(iii) If moreover $\mathbb{P}$ is also locally absolutely continuous w.r.t. $\mathbb{Q}$ (i.e., $\mathbb{P}$ and $\mathbb{Q}$ are locally equivalent), then the process XL is a local martingale under $\mathbb{P}$ iff the process X is a local martingale under $\mathbb{Q}$.

Proof. (i) From $\mathbb{E}_{\mathbb{Q}}(|X_n|) = \mathbb{E}_(|X_nL_n|$ we obtain that XL is $\mathbb{P}$-integrable. Bayes's formula $\mathbb{E}_{\mathbb{Q}}(X_n|\mathcal{H}_{n-1}) = \frac{\mathbb{E}(X_nL_n|\mathcal{H}_{n-1})}{L_{n-1}}$ leads to the result.
ii) This assertion is obtained by localization.
iii) If $\mathbb{P}$ and $\mathbb{Q}$ are locally equivalent, L^{-1} is a $\mathbb{Q}$-martingale. So the result follows form ii) applying with XL.

Proposition 1.1.37 *Let X be an $\mathbb{H}$-local martingale under $\mathbb{P}$ and let $\mathbb{Q}$ be locally absolutely continuous w.r.t. $\mathbb{P}$. The process $X^{\mathbb{Q}}$ defined as*

$$X_n^{\mathbb{Q}} = X_n - \sum_{k=1}^{n} \frac{1}{L_k}\Delta[X, L]_k$$

and the process $\widetilde{X}^{\mathbb{Q}}$ defined as

$$\widetilde{X}_n^{\mathbb{Q}} = X_n - \sum_{k=1}^{n} \frac{1}{L_{k-1}}\Delta\langle X, L\rangle_k \, ,$$

are well defined under $\mathbb{Q}$ and
(i) $X^{\mathbb{Q}}$ is a local martingale under $\mathbb{Q}$
(ii) if moreover $\mathbb{E}[|\Delta X_n|L_n|\mathcal{H}_{n-1}] < \infty$ a.s. for all n, then $\widetilde{X}^{\mathbb{Q}}$ is a local martingale under $\mathbb{Q}$.

Proof. Under $\mathbb{Q}$, the process L is a.s. positive since $\mathbb{Q}(L_n = 0) = \mathbb{E}_{\mathbb{P}}(L_n \mathbb{1}_{\{L_n=0\}}) = 0$.
i) We have

$$\Delta X_n^{\mathbb{Q}} = \Delta X_n - \frac{\Delta X_n \Delta L_n}{L_n} = \frac{\Delta X_n}{L_n}(L_n - \Delta L_n) = \frac{\Delta X_n}{L_n} L_{n-1}$$

From Bayes' formula and the martingale property of L

$$\mathbb{E}_{\mathbb{Q}}(\Delta X_n^{\mathbb{Q}}|\mathcal{H}_{n-1}) = \frac{\mathbb{E}_{\mathbb{P}}(L_n \Delta X_n^{\mathbb{Q}}|\mathcal{H}_{n-1})}{\mathbb{E}_{\mathbb{P}}(L_n|\mathcal{H}_{n-1})} = \frac{\mathbb{E}_{\mathbb{P}}(\Delta X_n L_{n-1}|\mathcal{H}_{n-1})}{L_{n-1}} = 0$$

ii) The result follows from the fact that the process $\widetilde{X}^{\mathbb{Q}}$ satisfies

$$\widetilde{X}_n^{\mathbb{Q}} = X_n - \sum_{k=1}^{n} \frac{1}{L_{k-1}} \mathbb{E}_{\mathbb{P}}[L_k \Delta X_k|\mathcal{H}_{k-1}] = X_n - \sum_{k=1}^{n} \mathbb{E}_{\mathbb{Q}}[\Delta X_k|\mathcal{H}_{k-1}]$$

hence is a $\mathbb{Q}$-martingale if X_n is $\mathbb{Q}$ integrable. If $\mathbb{E}[|\Delta X_n|L_n|\mathcal{F}_{n-1}] < \infty$, then $E_{\mathbb{Q}}[|\Delta X_k||\mathcal{H}_{k-1}] < +\infty$ and $\widetilde{X}^{\mathbb{Q}}$ is $\mathbb{Q}$-local-martingale.

Remark 1.1.38 If X is a martingale under $\mathbb{P}$ and $\mathbb{Q}$-integrable, then $\widetilde{X}^{\mathbb{Q}}$ is a martingale under $\mathbb{Q}$. This result can be proved directly using Doob's decomposition. Moreover, if $\mathbb{E}[|\Delta X_n|L_{n-1}] < \infty$, $X^{\mathbb{Q}}$ is a martingale under $\mathbb{Q}$. Indeed we have,

$$\Delta X_n^{\mathbb{Q}} = \Delta X_n - \frac{\Delta X_n \Delta L_n}{L_n} = \frac{\Delta X_n}{L_n}(L_n - \Delta L_n) = \frac{\Delta X_n}{L_n} L_{n-1}$$

From Bayes' formula and the martingale property of L

$$\mathbb{E}_{\mathbb{Q}}(\Delta X_n^{\mathbb{Q}}|\mathcal{H}_{n-1}) = \frac{\mathbb{E}_{\mathbb{P}}(L_n \Delta X_n^{\mathbb{Q}}|\mathcal{H}_{n-1})}{\mathbb{E}_{\mathbb{P}}(L_n|\mathcal{H}_{n-1})} = \frac{\mathbb{E}_{\mathbb{P}}(\Delta X_n L_{n-1}|\mathcal{H}_{n-1})}{L_{n-1}} = 0$$

Proposition 1.1.39 *In discrete time, for any local martingale there exists an equivalent probability measure so that it is a martingale.*

See Kabanov

Predictable representation property

An important property for finance purpose is the existence of a multidimensional martingale which enjoys the predictable representation property

Definition 1.1.40 *A martingale X (which can be d-dimensional) enjoys the predictable representation property in the filtration $\mathbb{H}$ is any one-dimensional $\mathbb{H}$-martingale has the form $Y = Y_0 + \varphi \centerdot X$ for a predictable process φ.*

Note that, the predictability of φ imply that $\varphi \centerdot X$ is a local martingale.

Proposition 1.1.41 *If there exists a martingale which enjoys PRP, the space Ω is finite.*

Proposition 1.1.42 *If X is an $\mathbb{H}$-martingale, then for any $\mathbb{H}$-martingale Y there exists a predictable process φ and an $\mathbb{H}$-martingale $X^\perp$ orthogonal to X such that $Y = Y_0 + \varphi \centerdot X + X^\perp$.*

Proof. Choosing a localization sequence $(T_m)_{m\in\mathbb{N}}$ such that $(X_n^\perp)^{T_m}$ and X^{T_m} are square integrable allows us to prove the result for square integrable martingales. Notice that if $\Delta\langle X, X\rangle_n = 0$, then $\Delta\langle X, Y\rangle_n = 0$. Let φ be the $\mathbb{H}$-predictable process $\varphi_n = \frac{\Delta\langle X,Y\rangle_n}{\Delta\langle X,X\rangle_n}\mathbb{1}_{\{\Delta\langle X,X\rangle_n>0\}}$[2]. Then $\varphi \centerdot X$ is a local martingale, and $X^\perp$ is a local martingale, as the difference of two local martingales. Since

$$\mathbb{E}(\Delta X_n^\perp \Delta X_n|\mathcal{H}_{n-1}) = \Delta\langle X, Y\rangle_n - \frac{\Delta\langle X, Y\rangle_n}{\Delta\langle X, X\rangle_n}\mathbb{1}_{\Delta\langle X,X\rangle_n>0}\Delta\langle X, X\rangle_n = 0$$

the orthogonality is proved. $\square$

We now check that the PRP is stable by change of probability measure. This property is well know in continuous time. The goal here is to give a direct proof.

Theorem 1.1.43 *Let X be a d-dimensional $(\mathbb{P}, \mathbb{H})$-martingale enjoying the $\mathbb{H}$-predictable representation property under $\mathbb{P}$. Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $\mathcal{H}_n$ for any $n \geq 0$, and $L_n := \frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{H}_n}$. The $\mathbb{Q}$-martingale $X^\mathbb{Q}$ defined as*

$$X^\mathbb{Q} = X - \frac{1}{L} \centerdot [X, L]$$

enjoys $(\mathbb{Q}, \mathbb{H})$-predictable representation property.

Proof. For a $\mathbb{Q}$-martingale Y, we have to show that there exists an $\mathbb{H}$-predictable process ϑ such that

$$\Delta Y_n = \vartheta_n \Delta X_n^\mathbb{Q}.$$

In a first step, we suppose that Y is of the form $Y_n = \mathbb{E}_\mathbb{Q}(\zeta|\mathcal{H}_n)$, for ζ an $\mathcal{H}_N$-measurable $\mathbb{Q}$-integrable random variable. We set

$$Y_n = \mathbb{E}_\mathbb{Q}(\zeta|\mathcal{H}_n) = \frac{1}{L_n}\mathbb{E}_\mathbb{P}(\zeta L_N|\mathcal{H}_n) =: \frac{\zeta_n}{L_n}$$

From integration by parts formula

$$\Delta Y_n = \Delta\Big(\frac{1}{L_n}\zeta_n\Big) = \zeta_{n-1}\Delta\Big(\frac{1}{L_n}\Big) + \frac{1}{L_n}\Delta\zeta_n\,. \tag{1.6}$$

By definition, the process ζ is a $\mathbb{P}$-martingale and using the PRP for X, we have the existence of an $\mathbb{H}$-predictable process ϕ such that

$$\Delta\zeta_n = \phi_n \Delta X_n. \tag{1.7}$$

From Girsanov's theorem, the process $X^\mathbb{Q}$ defined as

$$\Delta X_n^\mathbb{Q} = \Delta X_n - \frac{1}{L_n}\Delta X_n \Delta L_n = \frac{L_{n-1}}{L_n}\Delta X_n$$

is an $(\mathbb{H}, \mathbb{Q})$-local martingale. Inserting ΔX_n in (1.7), it follows that

$$\Delta\zeta_n = \phi_n \frac{L_n}{L_{n-1}}\Delta X_n^\mathbb{Q}. \tag{1.8}$$

Since X enjoys PRP, there exists an $\mathbb{H}$-predictable process ψ such that $\Delta L_n = \psi_n \Delta X_n$. Therefore,

$$\Delta\Big(\frac{1}{L_n}\Big) = -\frac{1}{L_n L_{n-1}}\Delta L_n = -\frac{1}{L_n L_{n-1}}\psi_n \Delta X_n = -\frac{\psi_n}{L_{n-1}^2}\Delta X_n^\mathbb{Q}. \tag{1.9}$$

[2] with the convention $\frac{a}{b}\mathbb{1}_{b>0} = 0$ if $b = 0$

Plugging (1.8) and (1.9) in (1.6) yields to

$$\begin{aligned}\Delta Y_n &= -\zeta_{n-1}\frac{\psi_n}{L_{n-1}^2}\Delta X_n^{\mathbb{Q}} + \phi_n \frac{1}{L_{n-1}}\Delta X_n^{\mathbb{Q}} \\ &= (\frac{\phi_n}{L_{n-1}} - \zeta_{n-1}\frac{\psi_n}{L_{n-1}^2})\Delta X_n^{\mathbb{Q}} = \theta_n \Delta X_n^{\mathbb{Q}}\end{aligned}$$

where

$$\theta_n = \frac{\phi_n}{L_{n-1}} - \zeta_{n-1}\frac{\psi_n}{L_{n-1}^2}$$

belongs to $\mathcal{H}_{n-1}$. If Y is not u.i., or is a local martingale, we localize it with a sequence of stopping times T_j so that the stopped processes are u.i. We apply PRP to the stopped processes and we pass to the limit.
Let j fixed, then there is a predictable process φ s.t.

$$Y_{n\wedge T_j} = \sum_{k=1}^{n} \varphi_{k\wedge T_j}\Delta X_k^{\mathbb{Q}}. \tag{1.10}$$

Indeed, since this equality is pathwise, and $Y_{n\wedge T_j} = Y_{n\wedge T_{j+1}\wedge T_j}$, the process φ for T_{j+1} stopped at T_j is the process φ for T_j. Then passing to the limit in Equation (1.10), we obtain that

$$Y_n = \sum_{k=1}^{n} \varphi_k \Delta X_k^{\mathbb{Q}}$$

Finally, $X^{\mathbb{Q}}$ enjoys PRP.

□

1.1.8 Arbitrages

We follow Jacod and Shiryaev [13].

We now consider a d-dimensional positive semimartingale S on a filtered probability space $(\Omega, \mathbb{H}, \mathbb{P})$, which represents the prices of d risky assets. We assume that there exists a risk free asset (the savings account) S^0 defined as $S_n^0 = (1+r)^n$, where r is the interest rate. (One can take as savings account any positive predictable process S^0.) A portfolio is a vector of processes (α, π) where $\pi = (\pi^i, i = 1, \cdots d)$, with α adapted and π predictable. The wealth associated to this portfolio is X such that $X_n = \alpha_n + \sum_{i=1}^d \pi_n^i S_n^i =: \alpha_n + \pi_n S_n$. The portfolio is said to be self-financing if $S_{n-1}^0 \Delta\alpha_n + S_{n-1}\Delta\pi_n = 0$ or equivalently $\Delta X_n = \alpha_n^0 \Delta S_n^0 + \pi_n \Delta S_n$.

It is standard to work with the discounted prices $\bar{S} = S/S^0$ and discounted wealth $\bar{X} = X/S^0$. Then, the self-financing condition can be written as $\Delta\bar{X} = \pi\Delta\bar{S}$. This allows us to consider only the part π of the portfolio and the initial wealth of the portfolio: the vector π being known, the process α which satisfies self financing condition is then $\alpha_n S_n^0 = X_n^{\pi,x} - \sum_{i=1}^d \pi_n^i S_n^i$ where $\bar{X}_n^{\pi,x} = x + \sum_{k=1}^n \pi_k \Delta\bar{S}_k$. We consider a model with horizon N, where N is given: processes are defined only up to time N, and there are no more transactions after time N.

We say that the model is arbitrage-free if for any π with $X_0^\pi = 0$ and $X_N^{\pi,0} \geq 0$, then $X_N^{\pi,0} = 0$. The model is said to be weakly arbitrage-free if for π with $X_0^\pi = 0$ and $X_n^{\pi,0} \geq 0, \forall n \geq 0$, then $X_N^{\pi,0} = 0$.
The model is said to be strongly arbitrage-free if for any π with $X_0^\pi = 0$ and $X_N^{\pi,0} \geq 0$, then $X_n^{\pi,0} = 0, \forall n \geq 0$.

We will denote by $\mathcal{Q}$ the class of all probability measures which are equivalent to $\mathbb{P}$ and under which the discounted process S is a martingale, and by $\mathcal{Q}_{loc}$ the set of all probability measures which are equivalent to $\mathbb{P}$ and under which the discounted process $\bar{S}$ is a local martingale. Finally $\mathcal{Q}_b$ is the set of all $\mathbb{Q}$ in $\mathcal{Q}$ such that the Radon-Nikodym density $L = d\mathbb{Q}/d\mathbb{P}$ is bounded.

Proposition 1.1.44 *There is equivalence between:*
a) The model is arbitrage-free.
b) The model is weakly arbitrage-free.
c) The model is strongly arbitrage-free.
d) The set $\mathcal{Q}$ is non-empty.
e) The set $\mathcal{Q}_b$ is non-empty.
f) The set $\mathcal{Q}_{loc}$ is non-empty.

If $\mathcal{Q}$ is not empty, any Radon-Nikodym density is called a deflator.

1.1.9 Enlargement of filtration

In continuous time, a difficult problem is to give conditions such that an $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale for two filtrations satisfying $\mathbb{F} \subset \mathbb{G}$, and, if it is the case, to give the $\mathbb{G}$-semimartingale decomposition of an $\mathbb{F}$-martingale. In discrete time, the following proposition is an easy consequence of Doob's decomposition and states that if $\mathbb{F} \subset \mathbb{G}$, then any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale and gives explicitly the decomposition of this semimartingale.

Proposition 1.1.45 *In a discrete time setting, any integrable process is a special semimartingale in any filtration with respect to which it is adapted: if $\mathbb{F} \subset \mathbb{G}$, and if X is an $\mathbb{F}$-martingale, it is a $\mathbb{G}$ special semimartingale with decomposition*

$$X = M^{\mathbb{G}} + V^{\mathbb{G}}$$

where $M^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $V^{\mathbb{G}}$ is $\mathbb{G}$-predictable, $V_0^{\mathbb{G}} = 0$, and

$$\Delta V_n^{\mathbb{G}} = \mathbb{E}(\Delta X_n | \mathcal{G}_{n-1}),\ n \geq 1.$$

The process $V^{\mathbb{G}}$ is often called the information drift of X relative to $\mathbb{G}$.

□

Definition 1.1.46 *Immersion is satisfied between the filtration $\mathbb{F}$ and a larger filtration $\mathbb{G}$ (or $\mathbb{F}$ is immersed in $\mathbb{G}$), if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. If this property is achieved, we will denote it by $\mathbb{F} \hookrightarrow \mathbb{G}$. In order to specify that the immersion is achieved with the probability measure $\mathbb{P}$, we will denote it by $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$.*

Immersion is also called $(\mathcal{H})$-hypothesis in the literature. The following result is well known and useful (see [7]).

Proposition 1.1.47 *Immersion hypothesis is equivalent to any of the following properties,where for a set A, we denote $\mathbb{P}(A|\mathcal{F}) = \mathbb{E}(\mathbb{1}_A|\mathcal{F})$:*

($\mathcal{H}$1) $\forall n \geq 0$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_n$ are conditionally independent given $\mathcal{F}_n$, i.e. if for all $n \geq 0$, for all sets $G_n \in \mathcal{G}_n$ and $F \in \mathcal{F}_\infty$ $\mathbb{P}(F \cap G_n|\mathcal{F}_n) = \mathbb{P}(F|\mathcal{F}_n)\mathbb{P}(G_n|\mathcal{F}_n)$.

($\mathcal{H}$2) $\forall n \geq 0$, $G_n \in \mathcal{G}_n$, $\mathbb{P}(G_n|\mathcal{F}_n) = \mathbb{P}(G_n|\mathcal{F}_\infty)$.

($\mathcal{H}$3) $\forall n \geq 0$, $F \in \mathcal{F}_\infty$, $\mathbb{P}(F|\mathcal{F}_n) = \mathbb{P}(F|\mathcal{G}_n)$.

Proof. We recall the proof for the ease of the reader.

- Immersion $\Rightarrow$ ($\mathcal{H}1$). Let $F \in \mathcal{F}_\infty$. Under immersion, the $\mathbb{F}$-martingale $(F_n)_{n\geq 0}$, defined by $F_n := \mathbb{E}(\mathbb{1}_F|\mathcal{F}_n)$ for all $n \geq 0$, is a $\mathbb{G}$-martingale. Hence, for any $n \geq 0$ and any $G_n \in \mathcal{G}_n$,

$$\mathbb{P}(F \cap G_n|\mathcal{F}_n) = \mathbb{E}\big[\mathbb{P}(F|\mathcal{G}_n)\mathbb{1}_{G_n}\big|\mathcal{F}_n\big] \overset{(\mathcal{H})}{=} \mathbb{E}\big[\mathbb{P}(F|\mathcal{F}_n)\mathbb{1}_{G_n}\big|\mathcal{F}_n\big] = \mathbb{P}(F|\mathcal{F}_n)\mathbb{P}(G_n|\mathcal{F}_n)\ ,$$

  which is ($\mathcal{H}1$).

- ($\mathcal{H}1$) $\Rightarrow$ ($\mathcal{H}2$). If ($\mathcal{H}1$) holds, then for any $F \in \mathcal{F}_\infty$ and any $G_n \in \mathcal{G}_n$

$$\mathbb{E}\big[\mathbb{1}_F\mathbb{P}(G_n|\mathcal{F}_n)\big] = \mathbb{E}\big[\mathbb{P}(F|\mathcal{F}_n)\mathbb{P}(G_n|\mathcal{F}_n)\big] \overset{(\mathcal{H}1)}{=} \mathbb{E}\big[\mathbb{P}(F \cap G_n|\mathcal{F}_n)\big] = \mathbb{P}(F \cap G_n)\ ,$$

  which is exactly ($\mathcal{H}2$).

- ($\mathcal{H}2$) $\Rightarrow$ ($\mathcal{H}3$). Suppose ($\mathcal{H}2$) and let $F \in \mathcal{F}_\infty$ and $G_n \in \mathcal{G}_n$ for any $n \geq 0$, then

$$\mathbb{E}\big[\mathbb{P}(F|\mathcal{F}_n)\mathbb{1}_{G_n}\big] = \mathbb{E}\big[\mathbb{1}_F\mathbb{P}(G_n|\mathcal{F}_n)\big] \overset{(\mathcal{H}2)}{=} \mathbb{E}\big[\mathbb{1}_F\mathbb{P}(G_n|\mathcal{F}_\infty)\big] = \mathbb{P}(F \cap G_n)\ ,$$

  which implies ($\mathcal{H}3$).

- ($\mathcal{H}3$) $\Rightarrow$ immersion. Consider an $\mathcal{F}$-martingale $(F_n)_{n\geq 0}$ of the form $F_n := \mathbb{P}(F|\mathcal{F}_n)$. Then, $(G_n)_{n\geq 0}$ defined by $G_n := \mathbb{P}(F_\infty|\mathcal{G}_n)$ for all $n \geq 0$ is an $\mathbb{G}$-martingale. Then under ($\mathcal{H}3$), we have that $F_n = G_n$ for all $n \geq 0$, therefore immersion is satisfied for u.i. martingales. The extension to all martingales is standard.

□

Our goal is to compute more explicitly the semimartingale decomposition in some specific cases, and to show, with elementary computations, that we recover the classical general formulae established in the literature in continuous time.

Comment 1.1.48 Note that results in continuous time can be directly applied to discrete time: if $\mathbb{F}$ is a discrete time filtration and X a discrete time process, one can study the continuous on right jumping filtration $\widetilde{\mathbb{F}}$ defined in continuous time for $n \leq t < n+1$ as $\widetilde{\mathcal{F}}_t = \mathcal{F}_n$, and the càdlàg process $\widetilde{X}_t = \sum_n X_n \mathbb{1}_{\{n\leq t<n+1\}}$. One interest of our computations relies on the fact that we do not need hypotheses done in continuous time and that our proofs are simple.

Another goal of this paper is to study how enlarging the filtration may introduce arbitrages. We start with a general result, valid for any filtration $\mathbb{H}$:

Lemma 1.1.49 *Let Y be an integrable $\mathbb{H}$-semimartingale. If there exists a positive $\mathbb{H}$-adapted process ψ such that*

$$\mathbb{E}(Y_n\psi_n|\mathcal{H}_{n-1}) = Y_{n-1}\mathbb{E}(\psi_n|\mathcal{H}_{n-1}),\ \forall n \geq 1\,,$$

there exists a positive $\mathbb{H}$-martingale L such that LY is an $\mathbb{H}$-martingale.

Proof. Let Y be a $(\mathbb{P},\mathbb{H})$-semimartingale with decomposition $Y = M^Y + V^Y$, with $\Delta V^Y_n = \mathbb{E}(\Delta Y_n|\mathcal{H}_{n-1})$ and where M^Y is a $(\mathbb{P},\mathbb{H})$-martingale. Define, for a given ψ, the $(\mathbb{P},\mathbb{H})$-martingale L

$$L_0 = 1,\ L_n = \prod_{k=1}^{n} \frac{\psi_k}{\mathbb{E}(\psi_k|\mathcal{H}_{k-1})} = L_{n-1}\frac{\psi_n}{\mathbb{E}(\psi_n|\mathcal{H}_{n-1})}\,, n \geq 1,$$

then, setting $d\mathbb{Q} = Ld\mathbb{P}$, the process M^Y decomposes as $M^Y = m^M + V^M$ where m^M is a $(\mathbb{Q},\mathbb{H})$-martingale and

$$\begin{aligned}
\Delta V^M_n &= \mathbb{E}_\mathbb{Q}(\Delta M^Y_n|\mathcal{H}_{n-1}) = \frac{1}{L_{n-1}}\mathbb{E}_\mathbb{P}(L_n\Delta M^Y_n|\mathcal{H}_{n-1}) \\
&= \frac{1}{L_{n-1}}(\mathbb{E}_\mathbb{P}(L_nM^Y_n|\mathcal{H}_{n-1}) - L_{n-1}M^Y_{n-1}) = \frac{1}{\mathbb{E}_\mathbb{P}(\psi_n|\mathcal{H}_{n-1})}\mathbb{E}_\mathbb{P}(\psi_n\Delta M^Y_n|\mathcal{H}_{n-1})\,.
\end{aligned}$$

The process Y is a $(\mathbb{Q},\mathbb{H})$-martingale if $V^M + V^Y = 0$ or equivalently $\Delta V^Y + \Delta V^M = 0$, that is

$$\mathbb{E}(\psi_n \Delta M_n^Y | \mathcal{H}_{n-1}) + \mathbb{E}(\psi_n|\mathcal{H}_{n-1})\mathbb{E}(\Delta Y_n|\mathcal{H}_{n-1}) = 0\,.$$

We develop and use that $\Delta M_n^Y = Y_n - \mathbb{E}(Y_n|\mathcal{H}_{n-1})$ and obtain, after simplification

$$\mathbb{E}(\psi_n Y_n|\mathcal{H}_{n-1}) = \mathbb{E}(\psi_n|\mathcal{H}_{n-1})Y_{n-1}\,.$$

□

In the setting of enlargement of filtration, we introduce the following definition of viable enlargement:

Definition 1.1.50 *Let $\mathbb{F} \subset \mathbb{G}$, we say that the enlargement $(\mathbb{F},\mathbb{G},\mathbb{P})$ is viable if there exists a positive $(\mathbb{P},\mathbb{G})$-martingale L with $L_0 = 1$ (called a deflator) such that, for any $(\mathbb{P},\mathbb{F})$-martingale X, the process XL is a $(\mathbb{P},\mathbb{G})$-martingale.*

Our definition implies that, if there is a discounted price process S, which is a $(\mathbb{P},\mathbb{F})$-martingale, then the market $(S,\mathbb{G})$ is arbitrage free. The study of necessary and sufficient conditions so that, for a given $(\mathbb{P},\mathbb{F})$-martingale S, there exists a deflator, can be found in Choulli and Deng [9].

1.2 Initial Enlargement

The filtration $\mathbb{F}^{\sigma(\xi)} = (\mathcal{F}_n^{\sigma(\xi)}, n \geq 0)$ is an initial enlargement of $\mathbb{F}$ with a random variable ξ taking values in $\mathbb{R}$ if $\mathcal{F}_n^{\sigma(\xi)} := \mathcal{F}_n \vee \sigma(\xi), n \geq 0$.

1.2.1 Bridge

We study the following particular example. Let $(Y_i, i \geq 1)$ a sequence of i.i.d. random variables with zero mean and the process X of the form $X_0 := 0$, $X_n := \sum_{i=1}^n Y_i$, $n \geq 1$. We note that X is an $\mathbb{F}$-martingale. We denote by $\mathbb{F}$ the natural filtration of X. For N fixed, we choose $\xi := X_N$.
We need to compute $\Delta V_n = \mathbb{E}(\Delta X_n|\mathcal{F}_{n-1}^{\sigma(\xi)}) = \mathbb{E}(\Delta X_n|\mathcal{F}_{n-1} \vee \sigma(X_N))$. Using the fact that $(Y_i, i \geq 1)$ are i.i.d, we have, for $n \leq j \leq N$

$$(Y_j, X_1, \cdots, X_{n-1}, X_N) \overset{\text{law}}{=} (Y_n, X_1, \cdots, X_{n-1}, X_N)\,,$$

hence

$$\begin{aligned}
\mathbb{E}(Y_n|\mathcal{F}_{n-1} \vee \sigma(X_N)) &= \mathbb{E}(Y_j|\mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{1}{N-(n-1)}\mathbb{E}(Y_n + \cdots + Y_j + \cdots + Y_N|\mathcal{F}_{n-1} \vee \sigma(X_N)) \\
&= \frac{1}{N-(n-1)}\mathbb{E}(X_N - X_{n-1}|\mathcal{F}_{n-1} \vee \sigma(X_N)) = \frac{X_N - X_{n-1}}{N-(n-1)}\,.
\end{aligned}$$

Therefore, the process $\widetilde{X}$ defined as

$$\widetilde{X}_n g = X_n - \sum_{k=1}^n \frac{X_N - X_{k-1}}{N-(k-1)}\,,\; n \geq 0$$

is a $\mathbb{F}^{\sigma(\xi)}$-martingale.

Comment 1.2.1 This formula is similar to the one obtained for Lévy bridges: if X is an integrable Lévy process in continuous time (e.g. a Brownian motion) with natural filtration $\mathbb{F}^X$, setting $\mathbb{G} = \mathbb{F}^X \vee \sigma(X_T)$ leads to

$$X_t^{\mathbb{G}} = X_t - \int_0^t \frac{X_T - X_s}{T-s}ds,\; 0 \leq t \leq T,$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale.

1.2.2 Initial enlargement with ξ, a $\mathbb{Z}$-valued random variable

Let X be an $\mathbb{F}$-martingale, ξ be a r.v. taking values in $\mathbb{Z}$ and, for any $j \in \mathbb{Z}$, let $p(j)$ be the $\mathbb{F}$-martingale defined as $p_n(j) = \mathbb{P}(\xi = j|\mathcal{F}_n)$.

Proposition 1.2.2 *The process $\widehat{X}$ defined as*

$$\widehat{X}_n = X_n - \sum_{k=1}^{n} \frac{\Delta\langle X, p(j)\rangle_k^{\mathbb{F}}|_{j=\xi}}{p_{k-1}(\xi)} \tag{1.11}$$

is an $\mathbb{F}^{\sigma(\xi)}$-martingale. On the set $\{\xi = j\}$, one has $p_n(j) \neq 0, \forall n \geq 0$.

Proof. The Doob decomposition of X in $\mathbb{F}^{\sigma(\xi)}$ is $X = M + V$ where M is a $\mathbb{F}^{\sigma(\xi)}$-martingale and, for $n \geq 1$, $\Delta V_n = \mathbb{E}(\Delta X_n|\mathcal{F}_{n-1} \vee \sigma(\xi))$ so that

$$\begin{aligned}(\Delta V_n)\mathbb{1}_{\{\xi=j\}} &= \mathbb{1}_{\{\xi=j\}}\frac{\mathbb{E}(\mathbb{1}_{\{\xi=j\}}\Delta X_n|\mathcal{F}_{n-1})}{\mathbb{P}(\xi = j|\mathcal{F}_{n-1})}\\ &= \mathbb{1}_{\{\xi=j\}}\frac{\mathbb{E}(p_n(j)\Delta X_n|\mathcal{F}_{n-1})}{p_{n-1}(j)} = \mathbb{1}_{\{\xi=j\}}\frac{\Delta\langle X, p(j)\rangle_n^{\mathbb{F}}}{p_{n-1}(j)}\,, \end{aligned} \tag{1.12}$$

where we have used the tower property in the second equality.
On the set $\{\xi = j\}$, one has $p_n(j) \neq 0, \forall n \geq 0$. Indeed,

$$\mathbb{E}(\mathbb{1}_{\{p_n(j)=0\}}\mathbb{1}_{\{\xi=j\}}) = \mathbb{E}(\mathbb{1}_{\{p_n(j)=0\}}\mathbb{E}(\mathbb{1}_{\{\xi=j\}}|\mathcal{F}_n)) = \mathbb{E}(\mathbb{1}_{\{p_n(j)=0\}}p_n(j)) = 0\,.$$

Proposition 1.2.3 *(a) The process $\frac{1}{p(\xi)}$ is an $\mathbb{F}^{\sigma(\xi)}$-supermartingale. If $p(k) > 0$ for any k, it is an $\mathbb{F}^{\sigma(\xi)}$-martingale.*
(b) If X is an $\mathbb{F}$-martingale and $p(k) > 0, \forall k$, $X/p(\xi)$ is an $\mathbb{F}^{\sigma(\xi)}$-martingale.

Proof. (a) In a first step, we note that $p(\xi) > 0$. Indeed, $p_n(\xi) = \sum_{k=-\infty}^{\infty} \mathbb{1}_{\{\xi=k\}}p_n(k)$ and $p_n(k) > 0$ on $\{\xi = k\}$.

$$\begin{aligned}\mathbb{E}(\frac{1}{p_n(\xi)}|\mathcal{F}_{n-1} \vee \sigma(\xi)) &= \sum_{k=-\infty}^{\infty} \mathbb{1}_{\{\xi=k\}}\frac{\mathbb{E}(\mathbb{1}_{\{\xi=k\}}\mathbb{1}_{\{p_n(k)>0\}}\frac{1}{p_n(k)}|\mathcal{F}_{n-1})}{\mathbb{E}(\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_{n-1})}\\ &= \sum_{k=-\infty}^{\infty} \mathbb{1}_{\{\xi=k\}}\frac{\mathbb{E}(\mathbb{1}_{\{p_n(k)>0\}}|\mathcal{F}_{n-1})}{p_{n-1}(k)} \leq \sum_{k=-\infty}^{\infty} \mathbb{1}_{\{\xi=k\}}\frac{1}{p_{n-1}(k)} = \frac{1}{p_{n-1}(\xi)}.\end{aligned}$$

If $p(k) > 0$ for any k, the last inequality is in fact an equality.
(b) For a $\mathbb{P}$ martingale X, if $p(k) > 0$, one has

$$\begin{aligned}\mathbb{E}(\frac{X_n}{p_n(\xi)}|\mathcal{F}_{n-1} \vee \sigma(\xi)) &= \sum_{k=-\infty}^{\infty} \mathbb{1}_{\xi=k}\frac{\mathbb{E}(\mathbb{1}_{\{\xi=k\}}\frac{X_n}{p_n(k)}|\mathcal{F}_{n-1})}{\mathbb{E}(\mathbb{1}_{\{\xi=k\}}|\mathcal{F}_{n-1})}\\ &= \sum_{k=-\infty}^{\infty} \mathbb{1}_{\{\xi=k\}}\frac{\mathbb{E}(X_n|\mathcal{F}_{n-1})}{p_{n-1}(k)} = \frac{X_{n-1}}{p_{n-1}(\xi)}\,.\end{aligned}$$

□

It follows that, if $p(k) > 0, \forall k$, the enlargement $(\mathbb{F}, \mathbb{F}^{\sigma(\xi)})$ is viable a deflator being $1/p(\zeta)$.

Lemma 1.2.4 *If $p(k) > 0$, the process $L_n := \frac{p_0(\zeta)}{p_n(\zeta)}$ is a positive martingale with expectation 1. Define $d\mathbb{Q} = L_n d\mathbb{P}$. Then ζ is independent from $\mathcal{F}_n$ under $\mathbb{Q}$, and $\mathbb{Q}|_{\mathcal{F}_n} = \mathbb{P}|_{\mathcal{F}_n}$, $\mathbb{Q}_{\sigma(\zeta)} = \mathbb{P}_{\sigma(\zeta)}$.*

Proof. For $X_n \in \mathcal{F}_n$,

$$\mathbb{E}_{\mathbb{Q}}(h(\zeta)X_n) = \mathbb{E}_{\mathbb{P}}(L_n h(\zeta)X_n) = \mathbb{E}(\sum_k \frac{p_0(k)}{p_n(k)} h(k) X_n \mathbb{1}_{\{\zeta=k\}}) = \mathbb{E}(\sum_k \frac{p_0(k)}{p_n(k)} h(k) X_n p_n(k))$$

If follows that

$$\mathbb{E}_{\mathbb{Q}}(h(\zeta)X_n) = \mathbb{E}(h(\zeta))\mathbb{E}(X_n)$$

Comment 1.2.5 In continuous time, under Jacod's hypothesis $\mathbb{P}(\tau \in du|\mathcal{F}_t) = p_t(u)\mathbb{P}(\xi \in du)$, the process $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale where

$$X_t^{\mathbb{G}} = X_t - \int_0^t \frac{d\langle X, p(u)\rangle_s^{\mathbb{F}}|_{u=\xi}}{p_{s-}(\xi)}, \quad \forall t \geq 0 .$$

If ξ takes discrete values, then Jacod's hypothesis (the conditional law of ξ w.r.t $\mathcal{F}_n$ is absolutely continuous w.r.t the law of ξ) is always true.

1.2.3 Supremum of random walk

We consider a particular case of the previous setting where we can compute the probabilities $p_n(j)$. Let $(Y_i, i \geq 1)$ a sequence of i.i.d. random variables taking values in $\mathbb{Z}$ and the process X of the form $X_0 := 0$, $X_n := \sum_{i=1}^n Y_i$, $n \geq 1$. For N fixed, we put $\xi := \sup_{0\leq n\leq N} X_n$ and we denote by $\mathbb{F}$ the natural filtration of X.
We denote by $g(n,k) = \mathbb{P}(\sup_{1\leq j\leq n} X_j = k)$ and by $h(n,k) = \mathbb{P}(\sup_{1\leq j\leq n} X_j \leq k)$. Then the probability $p_n(j)$ can be expressed as follow.
We note that

$$\{\sup_{k\leq N} X_k = j\} = \{\sup_{k\leq n} X_k = j, \sup_{1\leq k\leq N-n} X_{n+k} < j\} \cup \{\sup_{k\leq n} X_k \leq j, \sup_{1\leq k\leq N-n} X_{n+k} = j\}$$

and that , setting $\widetilde{X}_k = X_{n+k} - X_n = \sum_{i=1}^{N-n} \widetilde{Y}_i$ where $\widetilde{Y}$ are copies of Y, independent from $\mathcal{F}_n$

$$\begin{aligned}
\mathbb{P}(\sup_{1\leq k\leq N-n} X_{n+k} < j|\mathcal{F}_n) &= \mathbb{P}(\sup_{1\leq k\leq N-n} X_{n+k} - X_n < j - X_n|\mathcal{F}_n) \\
&= \mathbb{P}(\sup_{1\leq k\leq N-n} \widetilde{X}_k < j - X_n|\mathcal{F}_n) = h(N-n, j-X_n) \\
\mathbb{P}(\sup_{1\leq k\leq N-n} X_{n+k} = j|\mathcal{F}_n) &= \mathbb{P}(\sup_{1\leq k\leq N-n} \widetilde{X}_k = j - X_n|\mathcal{F}_n) = g(N-n, j-X_n))
\end{aligned}$$

Then

$$p_n(j) = \mathbb{1}_{\{\sup_{0\leq k\leq n} X_k = j\}} h(N-n, j-X_n) + \mathbb{1}_{\{\sup_{0\leq k\leq n} X_k \leq j\}} g(N-n, j-X_n).$$

1.2.4 Arbitrages

Proposition 1.2.6 *The process $Y(\zeta)$ is an $\mathbb{F}^{\sigma(\zeta)}$-martingale iff $Y_n(k)p_n(k), n \geq 0$ is an $\mathbb{F}$-martingale for any k.*

Proof. Let $Y_n(\zeta)$ be an $\mathbb{F}^{\sigma(\zeta)}$-martingale. Then,

$$\mathbb{E}(Y_n(\zeta)|\mathcal{F}_m^{\sigma(\zeta)}) = Y_m(\zeta)$$

or, equivalently, for any k

$$\mathbb{E}(Y_n(\zeta)|\mathcal{F}_m^{\sigma(\zeta)})\mathbb{1}_{\{\zeta=k\}} = Y_m(\zeta)\mathbb{1}_{\{\zeta=k\}}$$

and taking the $\mathcal{F}_m$ conditional expectation of both sides

$$\mathbb{E}(Y_n(k)\mathbb{1}_{\{\zeta=k\}}|\mathcal{F}_m) = Y_m(k)p_m(k)$$

hence, by tower property

$$\mathbb{E}(Y_n(k)p_n(k)|\mathcal{F}_m) = Y_m(k)p_m(k)\,.$$

Conversely, if $(Y_n(k)p_n(k), n \geq 0)$ is an $\mathbb{F}$-martingale for any k

$$\mathbb{E}(Y_n(\zeta)|\mathcal{F}_m^{(\sigma(\zeta)}) = \sum_k \mathbb{E}(Y_n(\zeta)\mathbb{1}_{\{\zeta=k\}}|\mathcal{F}_m^{\sigma(\zeta)}) = \sum_k \mathbb{1}_{\{\zeta=k\}}\mathbb{E}(Y_n(k)|\mathcal{F}_m^{\sigma(\zeta)})\,.$$

Then, using the fact that

$$\mathbb{1}_{\{\zeta=k\}}\mathbb{E}(Y_n(\zeta)|\mathcal{F}_m^{\sigma(\zeta)}) = \mathbb{1}_{\{\zeta=k\}}\frac{\mathbb{E}(Y_n(k)p_n(k)|\mathcal{F}_m)}{p_m(k)} = \mathbb{1}_{\{\zeta=k\}}\mathbb{E}(Y_n(\zeta)|\mathcal{F}_m^{\sigma(\zeta)})$$

we conclude □

Lemma 1.2.7 *If ξ is an $\mathcal{F}_N$-measurable r.v. for some N and ξ is not $\mathcal{F}_0$ measurable, the enlargement $(\mathbb{F}, \mathbb{G}, \mathbb{P})$ is not viable.*

Proof. Let $X_n = \mathbb{E}(\xi|\mathcal{F}_n)$. If a $\mathbb{G}$-deflator L exists, the process XL would be a $\mathbb{G}$-martingale, and $X_nL_n = \mathbb{E}(X_NL_N|\mathcal{G}_n)$. Using the fact that $X_N = \xi$ is $\mathcal{G}_n$-measurable for $0 \leq n \leq N$, we obtain $\mathbb{E}(X_NL_N|\mathcal{G}_n) = X_NL_n$, in particular $X_NL_0 = X_0L_0$ which is not possible since $X_N = \xi$ is not $\mathcal{F}_0$ measurable. □

Proposition 1.2.8 *For any $\mathbb{Z}$-valued random variable ζ, the following are equivalent.*
(a) The set $\{p_n(k) = 0 < p^k_{n-1}\}$ is negligeable, for all k and n.
(b) For any $\mathbb{F}$-adapted integrable process X satisfying NA($\mathbb{F}$), X satifies NA($\mathbb{F}^{\sigma(\zeta)}$).

1.3 Progressive Enlargement

We assume that τ is a random variable valued in $\mathbb{N} \cup \{+\infty\}$, and introduce the filtration $\mathbb{G}$ where, for $n \geq 0$, we set $\mathcal{G}_n = \mathcal{F}_n \vee \sigma(\tau \wedge n)$. In other words, $\mathbb{G}$ is the smallest filtration which contains $\mathbb{F}$ and makes τ a stopping time. In particular $\{\tau = 0\} \in \mathcal{G}_0$, so that, in general $\mathcal{G}_0$ is not trivial.
In continuous time, many results are obtained under the hypothesis that τ avoids $\mathbb{F}$-stopping times, or that all $\mathbb{F}$-martingales are continuous, which is not the case here. We present here some basic results. We assume that $\mathcal{F}_0$ is trivial. Recall that, in this setting, as written in Comment 1.2.5, τ satisfies the absolutely continuous Jacod's hypothesis.
In this section, the indicator of τ is

$$H_n = \mathbb{1}_{\{\tau \leq n\}}\,. \tag{1.13}$$

1.3.1 General results

Basic Properties

Lemma 1.3.1 *1) If Y is a $\mathbb{G}$-adapted process, there exists an $\mathbb{F}$-adapted process y such that*

$$Y_n\mathbb{1}_{\{n<\tau\}} = y_n\mathbb{1}_{\{n<\tau\}},\ \forall n \geq 0. \tag{1.14}$$

2) If Y is a $\mathbb{G}$-predictable process, there exists an $\mathbb{F}$-predictable process y such that

$$Y_n\mathbb{1}_{\{n\leq\tau\}} = y_n\mathbb{1}_{\{n\leq\tau\}},\ \forall n \geq 0. \tag{1.15}$$

Proof. In a first step, one proves (1.14) for $Y_n = X_n h(\tau \wedge n)$ where $X_n \in \mathcal{F}_n$ and h a bounded Borel function. In that case, the result if obvious, with $y_n = X_n h(n)$. The general case follows from monotone class theorem.
If Y is a $\mathbb{G}$-predictable process, then the process S defined by $S_0 = Y_0$ and $S_n = Y_{n+1}$ is adapted and so there exists an $\mathbb{F}$-adapted process s s.t.

$$Y_n \mathbb{1}_{\{n \leq \tau\}} = S_{n-1} \mathbb{1}_{\{n-1 < \tau\}} = s_{n-1} \mathbb{1}_{\{n-1<\tau\}} = y_n \mathbb{1}_{\{n \leq \tau\}}$$

where $y_n = s_{n-1}$ □

We introduce the supermartingale

$$Z_n = \mathbb{P}(\tau > n | \mathcal{F}_n), \forall n \geq 0$$

and its Doob's decomposition

$$Z = M - A \tag{1.16}$$

with

$$A_0 = 0,\ \Delta A_n = -\mathbb{E}(\Delta Z_n | \mathcal{F}_{n-1}) = \mathbb{P}(\tau = n | \mathcal{F}_{n-1})\,,\ \forall n \geq 1\,. \tag{1.17}$$

Note that A is, as it must be, the dual predictable projection of the process H defined as $H_n = \mathbb{1}_{\tau \leq n}$.

In particular, since $\mathcal{F}_0$ is trivial, $Z_0 = M_0 = \mathbb{P}(\tau > 0)$.

We also introduce the supermartingale

$$\widetilde{Z}_n = \mathbb{P}(\tau \geq n | \mathcal{F}_n),\ \forall n \geq 0 \tag{1.18}$$

and its Doob's decomposition

$$\widetilde{Z} = \widetilde{M} - \widetilde{A} \tag{1.19}$$

where $\widetilde{M}$ is an $\mathbb{F}$-martingale and $\widetilde{A}$ the $\mathbb{F}$-predictable increasing process satisfying $\widetilde{A}_0 = 0,\ \Delta\widetilde{A}_n = \mathbb{P}(\tau = n-1 | \mathcal{F}_{n-1}), \forall n \geq 1$.
We shall often use the trivial equalities

$$\widetilde{Z}_n = \mathbb{P}(\tau > n-1 | \mathcal{F}_n) = Z_n + \mathbb{P}(\tau = n | \mathcal{F}_n),\ Z_n = \mathbb{P}(\tau \geq n+1 | \mathcal{F}_n),\ \mathbb{E}(\widetilde{Z}_n | \mathcal{F}_{n-1}) = Z_{n-1}\,.$$

Proposition 1.3.2 *On the set $\{n \leq \tau\}$, the random variables $\widetilde{Z}_n$ and Z_{n-1} are positive. On the set $\{n > \tau\}$, the random variables $\widetilde{Z}_n$ and Z_{n-1} are strictly smaller than 1. One has $Z_\tau < 1$.*

Proof. The first assertion is obtained from the two following equalities:

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\{n \leq \tau\}} \mathbb{1}_{\{Z_{n-1}=0\}}) &= \mathbb{E}(\mathbb{P}(n \leq \tau | \mathcal{F}_{n-1}) \mathbb{1}_{\{Z_{n-1}=0\}}) = \mathbb{E}(Z_{n-1} \mathbb{1}_{\{Z_{n-1}=0\}}) = 0\,, \\
\mathbb{E}(\mathbb{1}_{\{n \leq \tau\}} \mathbb{1}_{\{\widetilde{Z}_n=0\}}) &= \mathbb{E}(\mathbb{P}(n \leq \tau | \mathcal{F}_n) \mathbb{1}_{\{\widetilde{Z}_n=0\}}) = \mathbb{E}(\widetilde{Z}_n \mathbb{1}_{\{\widetilde{Z}_n=0\}}) = 0\,.
\end{aligned}$$

The second assertion is left to the reader . By definition,

$$Z_\tau \mathbb{1}_{\{\tau<\infty\}} = \sum_n \mathbb{1}_{\{\tau=n\}} \mathbb{P}(\tau > n | \mathcal{F}_n) \leq 1 - \sum_n \mathbb{1}_{\{\tau=n\}} \mathbb{P}(\tau \leq n | \mathcal{F}_n)$$

and $1 - Z_n = \mathbb{P}(\tau \leq n | \mathcal{F}_n) \geq \mathbb{P}(\tau = n | \mathcal{F}_n) = p_n(n)$. The quantity $p_n(n)$ being positive on $\{\tau = n\}$, the result follows. □

We give a useful lemma known as key lemma. The proof of a) is standard, the proof of b) can be found in Aksamit et al. [5] for continuous time. For the ease of the reader, we recall these proofs.

Lemma 1.3.3 *One has, for any random time τ,*
a) if the random variable ζ is integrable

$$\mathbb{E}(\zeta | \mathcal{G}_n) \mathbb{1}_{\{\tau>n\}} = \mathbb{1}_{\{\tau>n\}} \frac{\mathbb{E}(\zeta \mathbb{1}_{\{\tau>n\}} | \mathcal{F}_n)}{Z_n}\ ,\forall n \geq 0\,.$$

b) for any integrable and $\mathcal{F}_n$-measurable r.v. Y_n,

$$\begin{aligned}
\mathbb{E}(Y_n|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau\geq n\}} &= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\widetilde{Z}_n|\mathcal{F}_{n-1}),\ \forall n\geq 1\\
\mathbb{E}(\frac{Y_n}{\widetilde{Z}_n}|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau\geq n\}} &= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\mathbb{1}_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1})\,,\ \forall n\geq 1. \qquad (1.20)
\end{aligned}$$

Proof.
a) Taking $Y_n=\mathbb{E}(\zeta|\mathcal{G}_n)$ in (1.14), and taking expectation w.r.t. $\mathcal{F}_n$ we obtain

$$\mathbb{E}(\zeta\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n)=\mathbb{E}(\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n)y_n=Z_ny_n\,,$$

hence, taking into account that Z_n is positive on the set $\{\tau>n\}$, $y_n\mathbb{1}_{\{\tau>n\}}=\mathbb{1}_{\{\tau>n\}}\frac{\mathbb{E}(\zeta\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n)}{Z_n}$.
b) The first part of b) follows from the equality (1.15). Only the second equality requires a proof. For $n\geq 1$, we have

$$\begin{aligned}
\mathbb{E}(\frac{Y_n}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\frac{1}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_{n-1})\\
&= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\frac{1}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}\mathbb{1}_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1})\\
&= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\mathbb{1}_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1})\,.
\end{aligned}$$

□

We give an immediate and important consequence in order to define the process y which satisfies (1.14) and the process y which satisfies (1.15) on the whole space.

Lemma 1.3.4 *The process y, which satisfies (1.14) can be chosen as $y_n=\frac{1}{Z_n}\mathbb{E}(Y_n\mathbb{1}_{\{n\leq\tau\}}|\mathcal{F}_n)\mathbb{1}_{\{Z_n>0\}}$. The process y, which satisfies (1.15) can be chosen as*

$$y_n=\mathbb{E}(Y_n\widetilde{Z}_n|\mathcal{F}_{n-1})\mathbb{1}_{\tilde{Z}_{n-1}>0}\,.$$

Proof. The proof is a consequence on Proposition 1.3.2 and Lemma 1.3.3 . □

Proposition 1.3.5 *The process $\Upsilon:=(1-H)\frac{1}{Z}$ is a $\mathbb{G}$-supermartingale (where, by convention, $(1-H_n)\frac{1}{Z_n}=0$ on the set $\{\tau\leq n\}=\{H_n=1\}$). If Z is positive, Υ is a $\mathbb{G}$-martingale.*
If X is an $\mathbb{F}$-martingale and Z positive, $(1-H)X\frac{1}{Z}$ is a $\mathbb{G}$-martingale.

Proof.

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{\{\tau>n\}}\frac{1}{Z_n}|\mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{\{\tau>n\}}\frac{1}{Z_n}|\mathcal{G}_{n-1})\\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{\{\tau>n\}}\mathbb{1}_{Z_n>0}\frac{1}{Z_n}|\mathcal{F}_{n-1})\\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{Z_n>0}\frac{1}{Z_n}\mathbb{E}(\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n)|\mathcal{F}_{n-1})\\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{Z_n>0}|\mathcal{F}_{n-1})\leq\mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}
\end{aligned}$$

In the first equality, we have used that, due to the fact that τ is a $\mathbb{G}$-stopping time, $\{\tau\leq n-1\}\in\mathcal{G}_{n-1}$, hence $\mathbb{E}(\mathbb{1}_{\{\tau>n\}}\frac{1}{Z_n}|\mathcal{G}_{n-1})\mathbb{1}_{\{\tau\leq n-1\}}=0$, in the second equality that $\{\tau>n\}\subset\{Z_n>0\}$. Hence the result, noting that one has equality in the last line if $Z>0$. If X is an $\mathbb{F}$-martingale, the same kind of proof establishes that, for $Z>0$,

$$\mathbb{E}(\mathbb{1}_{\{\tau>n\}}\frac{X_n}{Z_n}|\mathcal{G}_{n-1})=\mathbb{1}_{\{\tau>n-1\}}\frac{X_{n-1}}{Z_{n-1}}\,.$$

Lemma 1.3.6 *Let $H_n = \mathbb{1}_{\{\tau \le n\}}, n \ge 0$, and Λ be the $\mathbb{F}$-predictable process defined as*

$$\Lambda_0 = 0,\ \Delta\Lambda_n := \frac{\Delta A_n}{Z_{n-1}}\mathbb{1}_{\{Z_{n-1}>0\}},\ n \ge 1\,,$$

where A is defined in 1.17. The process N defined as

$$N_n := H_n - \Lambda_{n\wedge\tau} = H_n - \sum_{k=1}^{n\wedge\tau}\lambda_k,\ n \ge 0 \tag{1.21}$$

where $\lambda_n := \Delta\Lambda_n$ is a $\mathbb{G}$-martingale.

Proof. It suffices to find the Doob decomposition of the $\mathbb{G}$-semimartingale H. The predictable part of this decomposition is K with

$$\begin{aligned}\Delta K_n &= \mathbb{E}(\Delta H_n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau\le n-1\}}0 + \mathbb{1}_{\{\tau>n-1\}}\frac{\mathbb{E}(\Delta H_n|\mathcal{F}_{n-1})}{Z_{n-1}}\\ &= \mathbb{1}_{\{\tau\ge n\}}\frac{\mathbb{E}(Z_{n-1}-Z_n|\mathcal{F}_{n-1})}{Z_{n-1}} = \mathbb{1}_{\{\tau\ge n\}}\frac{A_n - A_{n-1}}{Z_{n-1}},\ n\ge 1.\end{aligned}$$

We conclude, as from Proposition 1.3.2 $Z_{n-1} > 0$ on $\{1 \le n \le \tau\}$, so that, on $\{n \le \tau\}$, one has $\Delta K_n = \lambda_n$ where $\lambda_n = \frac{\Delta A_n}{Z_{n-1}}\mathbb{1}_{\{Z_{n-1}>0\}}$ is $\mathbb{F}$-predictable.
Note for future use that $0 \le \lambda_n \le \mathbb{E}(1 - \frac{Z_n}{Z_{n-1}}\mathbb{1}_{\{Z_{n-1}>0\}}|\mathcal{F}_{n-1}) < 1$ (except if $\tau = equiv0!$). This will be useful to obtain that $\mathcal{E}(-\Lambda)$ is a positive process. □

Proposition 1.3.7 *Suppose Z positive. The multiplicative predictable decomposition of Z is given by $Z_n = N^Z_n\mathcal{E}\big(-\Lambda\big)_n$, $n \ge 0$ where N^Z is a positive $\mathbb{F}$-martingale and Λ is defined in Lemma 1.3.6.*

Proof. We have seen that there exist an $\mathbb{F}$-martingale N^Z and an $\mathbb{F}$-predictable process K^Z such that $Z = N^Z K^Z$ with

$$K^Z_n = \prod_{k=1}^n\frac{\mathbb{E}(Z_k|\mathcal{F}_{k-1})}{Z_{k-1}} = \prod_{k=1}^n\Big[-\frac{Z_{k-1}-\mathbb{E}(Z_k|\mathcal{F}_{k-1})}{Z_{k-1}}+1\Big]\,,\qquad \forall n\ge 1\,. \tag{1.22}$$

From Lemma 1.3.6 and the positivity of Z, we have

$$\Delta\Lambda_n = \frac{Z_{n-1}-\mathbb{E}\big(Z_n|\mathcal{F}_{n-1}\big)}{Z_{n-1}}\ ,\ \ \forall n\ge 1\,, \tag{1.23}$$

then by definition of the exponential process, we get that $K^Z = \mathcal{E}\big(-\Lambda\big)$. □

Lemma 1.3.8 *If $\widetilde{Z}$ is predictable and Z is positive, then $\mathbb{E}(\Delta N_n|\mathcal{F}_n) = -\Delta M_n$ for all $n \ge 0$, where M is the martingale part in the Doob decomposition of Z defined in 1.16 and N is defined in (1.21).*

Proof. By definition of N, we have that, for $n \ge 0$,

$$\begin{aligned}\mathbb{E}(\Delta N_n|\mathcal{F}_n) &= \mathbb{E}(\mathbb{1}_{\{\tau\le n\}} - \mathbb{1}_{\{\tau\le n-1\}} - \lambda_n\mathbb{1}_{\{\tau\ge n\}}|\mathcal{F}_n)\\ &= \mathbb{E}(-\mathbb{1}_{\{\tau>n\}} + \mathbb{1}_{\{\tau\ge n\}}|\mathcal{F}_n) - \lambda_n\mathbb{E}(\mathbb{1}_{\{\tau\ge n\}}|\mathcal{F}_n)\\ &= -Z_n + \widetilde{Z}_n - \lambda_n\widetilde{Z}_n = -\Delta Z_n - \lambda_n Z_{n-1}\,,\end{aligned}$$

where we have used that, since $\widetilde{Z}$ is predictable, $\widetilde{Z}_n = Z_{n-1}$. Finally, using that $\Delta Z_n + \Delta A_n = \Delta M_n$ and, Z being positive, $\lambda_n = \frac{\Delta A_n}{Z_{n-1}}$ which implies $\lambda_n Z_{n-1} = \Delta A_n$, we get $\mathbb{E}(\Delta N_n|\mathcal{F}_n) = -\Delta M_n$. □

Let H^o be the $\mathbb{F}$-dual optional projection of H. From definition 1.1.15, H^o is defined by

$$H^o_n := \sum_{k=0}^{n} \mathbb{E}(\Delta H_k|\mathcal{F}_k) = \sum_{k=0}^{n} \mathbb{P}(\tau = k|\mathcal{F}_k)\ , \quad \forall n \geq 0,$$

and satisfies

$$\mathbb{E}(Y_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}(\sum_{n=0}^{\infty} Y_n \Delta H_n) = \mathbb{E}(\sum_{n=0}^{\infty} Y_n \mathbb{E}(\Delta H_n|\mathcal{F}_n)) = \mathbb{E}(\sum_{n=0}^{\infty} Y_n \Delta H^o_n) \tag{1.24}$$

for any $\mathbb{F}$-adapted bounded process Y. We define $H^o_\infty := H^o_{\infty-} + \mathbb{P}(\tau = \infty|\mathcal{F}_\infty)$ where $H^o_{\infty-} = \lim_{n\to\infty} H^o_n := \sum_{k=0}^{\infty} \mathbb{P}(\tau = k|\mathcal{F}_k)$. Note that $\Delta\widetilde{A}_n = H^o_{n-1} - H^o_{n-2}$, where $\widetilde{A}$ is the predictable part of $\widetilde{M}$ and since $\widetilde{A}_1 = H^o_0$ we have $\widetilde{A}_n = H^o_{n-1}$, hence

$$Z_n + H^o_n = Z_n + \Delta H^o_n + H^o_{n-1} = \widetilde{Z}_n + \widetilde{A}_n = \widetilde{M}_n\,.$$

Furthermore, since $\lim_{n\to\infty} Z_n = \mathbb{1}_{\{\tau=\infty\}}$, and $\mathbb{E}(H^o_{\infty-}) = \lim \mathbb{E}(H^0_n) \leq 1$, one has

$$\widetilde{M}_n = Z_n + H^o_n = \mathbb{E}(\mathbb{1}_{\{\tau=\infty\}} + H^o_{\infty-}|\mathcal{F}_n)\,. \tag{1.25}$$

Proposition 1.3.9 *Let* $\Pi := H - (\widetilde{Z})^{-1}\mathbb{1}_{[0,\tau]} \centerdot H^o = H - \Gamma_{\cdot\wedge\tau}$*, with*

$$\Delta\Gamma_n = (\widetilde{Z}_n)^{-1}\mathbb{1}_{\{\widetilde{Z}_n>0\}}\Delta H^o_n,\ \Gamma_0 = 0\,. \tag{1.26}$$

Then, for any integrable $\mathbb{F}$*-adapted process* Y*, the process* $Y \centerdot \Pi$ *is a* $\mathbb{G}$*-martingale. In particular,* Π *is a* $\mathbb{G}$*-martingale.*

Proof. From $\Delta\Pi_n = \mathbb{1}_{\{\tau=n\}} - \frac{1}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}\mathbb{P}(\tau = n|\mathcal{F}_n)$, $n \geq 1$, one has $\Delta\Pi_n \mathbb{1}_{\{\tau<n\}} = 0$ and, from Lemma 1.3.3 (1.20),

$$\begin{aligned}
\mathbb{E}(Y_n\Delta\Pi_n|\mathcal{G}_{n-1}) &= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}(Y_n\Delta\Pi_n\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_{n-1}) + \mathbb{E}(Y_n\Delta\Pi_n\mathbb{1}_{\{\tau<n\}}|\mathcal{G}_{n-1}) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}\left(Y_n\left(\mathbb{P}(\tau = n|\mathcal{F}_n) - \frac{1}{\widetilde{Z}_n}\mathbb{P}(\tau = n|\mathcal{F}_n)\mathbb{P}(\tau \geq n|\mathcal{F}_n)\mathbb{1}_{\{\widetilde{Z}_n>0\}}\right)|\mathcal{F}_{n-1}\right) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}\left(Y_n\mathbb{P}(\tau = n|\mathcal{F}_n)\left(1 - \mathbb{1}_{\{\widetilde{Z}_n>0\}}\right)|\mathcal{F}_{n-1}\right) \\
&= \mathbb{1}_{\{\tau>n-1\}}\frac{1}{Z_{n-1}}\mathbb{E}\left(Y_n\mathbb{P}(\tau = n|\mathcal{F}_n)\mathbb{1}_{\{\widetilde{Z}_n=0\}}|\mathcal{F}_{n-1}\right),
\end{aligned}$$

where the fact that Y is $\mathbb{F}$-adapted has been used in the second equality. It remains to note that, on $\{\widetilde{Z}_n = 0\}$, one has $\mathbb{P}(\tau = n|\mathcal{F}_n) = 0$ to obtain $\mathbb{E}(Y_n\Delta\Pi_n|\mathcal{G}_{n-1}) = 0$. □

Note that, if Y is an $\mathbb{F}$-martingale, then, from lemma 1.1.9, the $\mathbb{G}$-martingale part of Y is orthogonal to Π. This result is similar to the one obtained in continuous time by Choulli et al. [9].

There are obviously infinitely many nondecreasing $\mathbb{G}$-adapted processes Θ such that $\mu := H - \Theta$ is a martingale stopped at time τ, e.g., $\Theta = H$ or any convex combination between $\Lambda_{\cdot\wedge\tau}$ and $\Gamma_{\cdot\wedge\tau}$, where Λ is the $\mathbb{F}$-predictable process defined in Lemma 1.3.6 and Γ the $\mathbb{F}$-optional process defined in (1.26). Assume that μ is a $\mathbb{G}$-martingale stopped at τ (so that $\Delta\mu_n\mathbb{1}_{\{\tau<n\}} = 0$) of the form $\mu = H - K\mathbb{1}_{[0,\tau]} \centerdot J$ where K, J are $\mathbb{F}$-adapted process and $Y \centerdot \mu$ is a $\mathbb{G}$-martingale for any integrable $\mathbb{F}$-adapted Y. As we show below, the property that $Y \centerdot \mu$ is a martingale for any $\mathbb{F}$-adapted Y characterizes the pair of processes (K, J) and implies that $\mu = \Pi$.

One can write

$$\begin{aligned}
0 &= \mathbb{E}(Y_n \Delta\mu_n | \mathcal{G}_{n-1}) = 1\!\!1_{\{\tau>n-1\}} \frac{1}{Z_{n-1}} \mathbb{E}(1\!\!1_{\{\tau>n-1\}} Y_n \Delta\mu_n | \mathcal{F}_{n-1}) \\
&= 1\!\!1_{\{\tau>n-1\}} \frac{1}{Z_{n-1}} \Big(\mathbb{E}(1\!\!1_{\{\tau\geq n\}} Y_n \Delta H_n | \mathcal{F}_{n-1}) - \ \mathbb{E}(1\!\!1_{\{\tau\geq n\}} Y_n K_n \Delta J_n | \mathcal{F}_{n-1}) \Big) \\
&= 1\!\!1_{\{\tau\geq n\}} \frac{1}{Z_{n-1}} \Big(\mathbb{E}(\mathbb{E}(1\!\!1_{\{\tau\geq n\}} \Delta H_n | \mathcal{F}_n) Y_n | \mathcal{F}_{n-1}) - \mathbb{E}(\mathbb{E}(1\!\!1_{\{\tau\geq n\}} | \mathcal{F}_n) Y_n K_n \Delta J_n | \mathcal{F}_{n-1}) \Big) \\
&= 1\!\!1_{\{\tau\geq n\}} \frac{1}{Z_{n-1}} \Big(\mathbb{E}(\mathbb{E}(\Delta H_n | \mathcal{F}_n) Y_n | \mathcal{F}_{n-1}) - \mathbb{E}(\widetilde{Z}_n Y_n K_n \Delta J_n | \mathcal{F}_{n-1}) \Big) \\
&= 1\!\!1_{\{\tau\geq n\}} \frac{1}{Z_{n-1}} \mathbb{E}(Y_n (\Delta H_n^o - \widetilde{Z}_n K_n \Delta J_n) | \mathcal{F}_{n-1})
\end{aligned}$$

where we used the fact that $1\!\!1_{\{\tau\geq n\}} \Delta H_n = \Delta H_n$. Then, taking conditional expectation w.r.t. $\mathcal{F}_{n-1}$, one obtains, for any Y_n

$$1\!\!1_{Z_{n-1}>0} (\mathbb{E}(Y_n (\Delta H_n^o - \widetilde{Z}_n K_n \Delta J_n) | \mathcal{F}_{n-1}) = 0\,,$$

hence

$$1\!\!1_{Z_{n-1}>0} (\Delta H_n^o - \widetilde{Z}_n K_n \Delta J_n) = 0\,.$$

It follows that $\mu = \Pi$.

Proposition 1.3.10 *Assume that Z is positive. Then, Z admits an "optional" multiplicative decomposition*

$$Z = \widetilde{N} \mathcal{E}(-\Gamma)$$

where $\widetilde{N}$ is an $\mathbb{F}$-martingale.

Proof. In the case $Z > 0$, we have $1 - \Delta\Gamma_n = 1 - (\widetilde{Z}_n)^{-1} \Delta H_n^o = 1 - (\widetilde{Z}_n)^{-1} \mathbb{P}(\tau = n | \mathcal{F}_n) > 0$, and the stochastic exponential $\mathcal{E}(-\Gamma)$ is positive.
We check that $Z/\mathcal{E}(-\Gamma)$ is a martingale.

$$\mathbb{E}\Big(\frac{Z_n}{\mathcal{E}(-\Gamma)_n} - \frac{Z_{n-1}}{\mathcal{E}(-\Gamma)_{n-1}} | \mathcal{F}_{n-1}\Big) \quad = \quad \frac{1}{\mathcal{E}(-\Gamma)_{n-1}} \mathbb{E}\Big(\frac{Z_n}{1 - \Delta\Gamma_n} - Z_{n-1} | \mathcal{F}_{n-1}\Big)$$

From $\widetilde{Z}_n - \Delta H_n^o = Z_n$, we obtain

$$\mathbb{E}\Big(\frac{Z_n}{1 - \Delta\Gamma_n} - Z_{n-1} | \mathcal{F}_{n-1}\Big) = \mathbb{E}\Big(\frac{Z_n \widetilde{Z}_n}{\widetilde{Z}_n - \Delta H_n^o} - Z_{n-1} | \mathcal{F}_{n-1}\Big) = \mathbb{E}(\widetilde{Z}_n - Z_{n-1} | \mathcal{F}_{n-1}) = 0$$

and the martingale property follows. □

$\mathbb{G}$-martingales versus $\mathbb{F}$-martingales

We first give a characterization for all $\mathbb{G}$-martingales in term of $\mathbb{F}$-martingales. We denote as before $\mathbb{P}(\tau = k | \mathcal{F}_n) = p_n(k)$.

Proposition 1.3.11 *A $\mathbb{G}$-adapted process of the form $Y := y 1\!\!1_{[\![0,\tau[\![} + y(\tau) 1\!\!1_{[\![\tau,\infty[\![}$ where y and $y(k)$ are $\mathbb{F}$-adapted processes, is a $\mathbb{G}$-martingale if and only if the following two conditions are satisfied*
(a) for any k, the process $\big(y_n(k) p_n(k), n \geq k\big)$ is an $\mathbb{F}$-martingale,
(b) the process $Y^{\mathbb{F}}$ is an $\mathbb{F}$-martingale, where

$$Y_n^{\mathbb{F}} := \mathbb{E}(Y_n | \mathcal{F}_n) = y_n Z_n + \sum_{k=0}^{n} y_n(k) p_n(k)\,. \tag{1.27}$$

We first give a result of $\mathbb{G}$-conditional expectation which be used in the proof of the Proposition.

Lemma 1.3.12 *Let $k \in \mathbb{N}$ and $n \geq k$. Consider $V_n(j), j \leq n$ a family of $\mathbb{F}$-measurable random variables, then*

$$\mathbb{1}_{\{\tau \leq k\}} \mathbb{E}\left(V_n(\tau)|\mathcal{G}_k\right) = \mathbb{1}_{\{\tau \leq k\}} \frac{1}{p_k(\tau)} \mathbb{E}\big(V_n(j)p_n(j)|\mathcal{F}_k\big)_{|j=\tau}.$$

Proof. From

$$\mathbb{1}_{\{\tau \leq k\}} \mathbb{E}\left(V_n(\tau)|\mathcal{G}_k\right) = \sum_{i=0}^{k} \mathbb{1}_{\{\tau = i\}} \mathbb{E}\left(V_n(i)|\mathcal{G}_k\right)$$

and using the fact that there exists an $\mathcal{F}_k$-measurable random variable $v_k(i)$ such that, for $i \leq k$

$$\mathbb{1}_{\{\tau = i\}} \mathbb{E}\left(V_n(i)|\mathcal{G}_k\right) = \mathbb{E}\left(\mathbb{1}_{\{\tau = i\}} V_n(i)|\mathcal{G}_k\right) = \mathbb{1}_{\{\tau = i\}} v_k(i)\,,$$

taking the conditional expectation w.r.t $\mathcal{F}_k$, we obtain that

$$\mathbb{E}\left(V_n(i)\mathbb{1}_{\{\tau = i\}}|\mathcal{F}_k\right) = \mathbb{E}\left(V_n(i)p_n(i)|\mathcal{F}_k\right) = v_k(i)\mathbb{P}\left(\tau = i|\mathcal{F}_k\right)$$

where we made use of the tower property to obtain the first equality. It follows that

$$\mathbb{1}_{\{\tau \leq k\}} \mathbb{E}\left(V_n(\tau)|\mathcal{G}_k\right) = \sum_{i=0}^{k} \mathbb{1}_{\{\tau = i\}} \frac{\mathbb{E}\left(V_n(i)p_n(i)\right)|\mathcal{F}_k)}{p_k(i)}$$

□

Proof. [of Proposition 1.3.11] For the necessity, in a first step, we show that we can reduce our attention to the case where Y is u.i. Indeed, let Y be a $\mathbb{G}$-martingale and $(T_j)_{j\geq 0}$ be a $\mathbb{G}$-localizing sequence such that, for each j, the associated stopped martingale $(Y_{n\wedge T_j}, n \geq 0)$ is u.i. Assuming that the result is established for u.i. martingales will prove that the processes in (a) and (b) are martingales up to T_j for each j. Since $T_j \to \infty$ as $j \to \infty$, the result follows.
Assume, then, that Y is a u.i. $\mathbb{G}$-martingale, hence the terminal value of this martingale is a $\mathcal{G}_\infty$ measurable random variable that one can write as $Y(\tau)$ where for any k, the random variable $Y(k)$ is $\mathcal{F}_\infty$ measurable, and $Y_n = \mathbb{E}(Y(\tau)|\mathcal{G}_n)$ has the form $Y = \; y\mathbb{1}_{[\![0,\tau[\![} + y(\tau)\mathbb{1}_{[\![\tau,\infty[\![}$.

• Assuming that Y is a $\mathbb{G}$-martingale, one has $\mathbb{E}(Y_n|\mathcal{G}_{n-1}) = Y_{n-1}$, hence, $\mathbb{E}(Y_n|\mathcal{G}_{n-1}))\mathbb{1}_{\{\tau=k\}} = Y_{n-1}\mathbb{1}_{\{\tau=k\}}$ and, for $k \leq n-1$, which leads to, writing $Y_n\mathbb{1}_{\{\tau=k\}} = y_n(k)\mathbb{1}_{\{\tau=k\}}$,

$$\mathbb{E}(y_n(k)\mathbb{1}_{\{\tau=k\}}|\mathcal{G}_{n-1}) = y_{n-1}(k)\mathbb{1}_{\{\tau=k\}}\,,$$

and taking conditional expectation w.r.t. $\mathcal{F}_{n-1}$ $\mathbb{E}(y_n(k)p_n(k)|\mathcal{F}_{n-1}) = y_{n-1}(k)p_{n-1}(k)$.
If Y is a $\mathbb{G}$ martingale, $Y^{\mathbb{F}}$ is an $\mathbb{F}$ martingale. The form of $Y^{\mathbb{F}}$ follows from

$$\begin{aligned}
\mathbb{E}(y_n\mathbb{1}_{\{n<\tau\}} + y_n(\tau)\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_n) &= y_nZ_n + \sum_{k=0}^{n} \mathbb{E}(y_n(\tau)\mathbb{1}_{\{\tau=k\}}|\mathcal{F}_n) \\
&= y_nZ_n + \sum_{k=0}^{n} y_n(k)p_n(k)\,.
\end{aligned}$$

• Conversely, assuming (a) and (b), we shall verify that $\mathbb{E}(Y_n|\mathcal{G}_k) = Y_k$ for $k \leq n$. Let us first note that

$$\mathbb{E}(Y_n|\mathcal{G}_k) = \mathbb{1}_{\{k<\tau\}} \frac{1}{Z_k} \mathbb{E}(Y_n\mathbb{1}_{\{k<\tau\}}|\mathcal{F}_k) + \mathbb{1}_{\{\tau\leq k\}} \mathbb{E}(Y_n\mathbb{1}_{\{\tau\leq k\}}|\mathcal{G}_k)\,. \tag{1.28}$$

We then compute the two conditional expectations in (1.28):

$$\begin{aligned}
\mathbb{E}(Y_n \mathbb{1}_{\{k<\tau\}}|\mathcal{F}_k) &= \mathbb{E}(Y_n|\mathcal{F}_k) - \mathbb{E}(Y_n \mathbb{1}_{\{\tau \le k\}}|\mathcal{F}_k) \\
&= \mathbb{E}(Y_n^{\mathbb{F}}|\mathcal{F}_k) - \mathbb{E}\left[\mathbb{E}(y_n(\tau)\mathbb{1}_{\{\tau\le k\}}|\mathcal{F}_n)|\mathcal{F}_k\right] = Y_k^{\mathbb{F}} - \mathbb{E}\left[\sum_{j=0}^{k} y_n(j)p_n(j)|\mathcal{F}_k\right] \\
&= y_k Z_k + \sum_{j=0}^{k} y_k(j)p_k(j) - \sum_{j=0}^{k} y_k(j)p_k(j) = y_k Z_k
\end{aligned}$$

where we have used the condition (a) and the equality derived in condition (b) to obtain the next-to-last identity. Furthermore, using Lemma 1.3.12

$$\begin{aligned}
\mathbb{E}(Y_n \mathbb{1}_{\{\tau\le k\}}|\mathcal{G}_k) &= \mathbb{E}(y_n(\tau)\mathbb{1}_{\{\tau\le k\}}|\mathcal{G}_k) = \mathbb{1}_{\{\tau\le k\}}\frac{1}{p_k(\tau)}\mathbb{E}\big(y_n(i)p_n(i)|\mathcal{F}_k\big)_{|i=\tau} \\
&= \mathbb{1}_{\{\tau\le k\}}\frac{1}{p_k(\tau)} y_k(\tau)p_k(\tau) = \mathbb{1}_{\{\tau\le k\}} y_k(\tau)
\end{aligned}$$

where the next-to-last identity holds in view of the condition (a).
Finally, $\mathbb{E}(Y_n|\mathcal{G}_k) = \mathbb{1}_{\{k<\tau\}}\frac{1}{Z_k} y_k Z_k + \mathbb{1}_{\{\tau\le k\}} y_k(\tau) = Y_k$. □

Immersion in progressive enlargement

We recall that $\mathbb{F}$ is immersed in $\mathbb{G}$ (we shall write $\mathbb{F} \hookrightarrow \mathbb{G}$) if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

Lemma 1.3.13 *$\mathbb{F}$ is immersed in $\mathbb{G}$ is equivalent to $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_\infty) = \mathbb{P}(\tau > n|\mathcal{F}_k)$ for any $k \ge n \ge 0$.*

Proof. First, suppose that immersion holds. The equality $Z_n = \mathbb{P}(\tau > n|\mathcal{F}_\infty) = \mathbb{P}(\tau > n|\mathcal{F}_k)$ is valid as soon as

$$Z_n = \mathbb{P}(\tau > n|\mathcal{F}_\infty) \tag{1.29}$$

and this is equivalent to $\mathbb{E}(X\mathbb{1}_{\tau>n}) = \mathbb{E}(XZ_n)$ for any bounded $\mathcal{F}_\infty$-measurable random variable X.

$$\mathbb{E}(X\mathbb{1}_{\tau>n}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G}_n)\mathbb{1}_{\tau>n}) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_n)\mathbb{1}_{\tau>n})$$

where the second equality comes from immersion. The equality

$$\mathbb{E}(\mathbb{E}(X|\mathcal{F}_n)\mathbb{1}_{\tau>n}) = \mathbb{E}(X\mathbb{E}(\mathbb{1}_{\tau>n})|\mathcal{F}_n)) = \mathbb{E}(XZ_n)$$

implies the result.
Conversely, assuming $\mathbb{P}(\tau > n|\mathcal{F}_k) = \mathbb{P}(\tau > n|\mathcal{F}_n)$ for $k \ge n$, we prove that any $\mathbb{F}$-martingale X is a $\mathbb{G}$-martingale, i.e., $\mathbb{E}(X_n|\mathcal{G}_k) = X_k$ for $k \le n$, or equivalently for any $\mathcal{F}_k$ measurable r.v. U_k, for any j,

$$\mathbb{E}(X_n U_k \mathbb{1}_{\tau\wedge k=j}) = \mathbb{E}(X_k U_k \mathbb{1}_{\tau\wedge k=j}).$$

This equality is obvious for $k < j$. For $j \le k$

$$\mathbb{E}(X_n U_k \mathbb{1}_{\tau\wedge k=j}) = \mathbb{E}(X_n U_k \mathbb{P}(\tau\wedge k = j|\mathcal{F}_n))) = \mathbb{E}(X_n U_k \mathbb{P}(\tau = j|\mathcal{F}_n))) = \mathbb{E}(X_n U_k \mathbb{P}(\tau = j|\mathcal{F}_k)))$$

where we have used the hypothesis in the last equality. It follows that, using the $\mathbb{F}$-martingale property of X

$$\mathbb{E}(X_n U_k \mathbb{P}(\tau = j|\mathcal{F}_k))) = \mathbb{E}(X_k U_k \mathbb{P}(\tau = j|\mathcal{F}_k))) = \mathbb{E}(X_k U_k \mathbb{1}_{\tau\wedge k=j})$$

□

Proposition 1.3.14 *Assume that Z is positive and that immersion holds. Then, $Z = \mathcal{E}(-\Gamma)$ where Γ is defined in* (1.26)*.*

Proof. If immersion holds, the same kind of computation as the one in the proof of Proposition 1.3.10 leads to

$$\frac{Z_n}{\mathcal{E}(-\Gamma)_n} = \frac{1}{\mathcal{E}(-\Gamma)_{n-1}} \frac{Z_n}{1-\Delta\Gamma_n} = \frac{Z_n}{\mathcal{E}(-\Gamma)_{n-1}} \frac{\widetilde{Z}_n}{\widetilde{Z}_n - \Delta H^o_n} = \frac{\widetilde{Z}_n}{\mathcal{E}(-\Gamma)_{n-1}}$$

and using the fact that immersion implies that $\widetilde{Z}_n = Z_{n-1}$, we obtain, by recursion, that $\frac{Z_n}{\mathcal{E}(-\Gamma)_n} = 1$. □

Lemma 1.3.15 *$\mathbb{F}$ is immersed in $\mathbb{G}$ if and only if $\widetilde{Z}$ is predictable and $\widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)$, $n \geq 0$.*

Proof. Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$. Then, for $n \geq 0$,

$$\widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_n) \quad = \quad \mathbb{P}(\tau > n-1|\mathcal{F}_n) = \mathbb{P}(\tau > n-1|\mathcal{F}_\infty) = \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)\,,$$

where the third equality follow from immersion assumption. The equality $\widetilde{Z}_n = \mathbb{P}(\tau > n-1|\mathcal{F}_{n-1}) = Z_{n-1}$ establishes the predictability of $\widetilde{Z}$.
Assume now that $\widetilde{Z}$ is predictable and $\widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_\infty)$. Then, $\widetilde{Z}_n = \mathbb{P}(\tau \geq n|\mathcal{F}_{n-1})$ and

$$\mathbb{P}(\tau > n|\mathcal{F}_n) \quad = \quad \mathbb{P}(\tau \geq n+1|\mathcal{F}_n) = \widetilde{Z}_{n+1} = \mathbb{P}(\tau > n|\mathcal{F}_\infty)\,.$$

The immersion property follows. □

Remark 1.3.16 We will see in the proof of Theorem 1.3.48 that $\tilde{Z}$ predictable implies that τ is a pseudo-stopping time, hence Z (and $\widetilde{Z}$) is decreasing.

Lemma 1.3.17 *Under immersion $p_n(k) = p_k(k)$ for $n \geq k$.*

Theorem 1.3.18 *Suppose $\mathbb{F} \hookrightarrow \mathbb{G}$ and $Z > 0$. Let N be defined in (1.21). Then the following assertions are equivalent*
(i) *Z is $\mathbb{F}$-predictable.*
(ii) *For any $\mathbb{G}$-predictable process U, one has $\mathbb{E}\Big((U_{\bullet}N)_n\Big) = 0$, $\forall n \geq 1$, in particular $\mathbb{E}\Big(\Delta N_n\big|\mathcal{F}_n\Big) = 0$, $\forall n \geq 1$.*
(iii) *Any $\mathbb{F}$-martingale X is orthogonal to N.*

Proof. (i)⇒ (ii). By uniqueness of Doob's decomposition and the predictability of Z, $Z_n = M_0 - A_n$, hence $\Delta M_n = 0$.
By Lemma 1.3.8 and 1.3.15, we have that

$$\mathbb{E}(\Delta N_n|\mathcal{F}_n) = -\Delta M_n$$

and $\mathbb{E}\Big(\Delta N_n\big|\mathcal{F}_n\Big) = 0$.
For $k \leq n$, let $\bar{U}_k$ be an $\mathcal{F}_{k-1}$-measurable r.v. be such that $\bar{U}_k 1\!\!1_{\{\tau > k-1\}} = U_k 1\!\!1_{\{\tau>k-1\}}$, then

$$\mathbb{E}(U_k\Delta N_k|\mathcal{F}_n) = \bar{U}_k\left[\mathbb{E}(-1\!\!1_{\{\tau>k\}} + 1\!\!1_{\{\tau\geq k\}}|\mathcal{F}_n) - \lambda_k\mathbb{E}(1\!\!1_{\{\tau\geq k\}}|\mathcal{F}_n)\right]\,,$$

which, using immersion propetry

$$\begin{aligned}\mathbb{E}(U_k\Delta N_k|\mathcal{F}_n) &= \bar{U}_k\left[\mathbb{E}(-1\!\!1_{\{\tau>k\}} + 1\!\!1_{\{\tau\geq k\}}|\mathcal{F}_k) - \lambda_k\mathbb{E}(1\!\!1_{\{\tau\geq k\}}|\mathcal{F}_k)\right] \\ &= \bar{U}_k\mathbb{E}(\Delta N_k|\mathcal{F}_k) = 0\,,\end{aligned}$$

taking the sum over all $k \le n$, and noting that, from immersion $\mathbb{E}(\bar{U}_k \Delta N_k|\mathcal{F}_k) = \mathbb{E}(\bar{U}_k \Delta N_k|\mathcal{F}_n)$, we obtain $\mathbb{E}\Big(\sum_{k=1}^n U_k \Delta N_k \big|\mathcal{F}_n\Big) = 0$, that is the desired result.

(ii) $\Rightarrow$ (iii). In the proof, we suppose that X is square integrable. The general case follows by localization. We prove that $\mathbb{E}(\Delta X_n\ \Delta N_n|\mathcal{G}_{n-1}) = 0$ for all $n \ge 1$.
From the Lemma 1.3.3, we have that

$$\mathbb{E}(\Delta X_n\ \Delta N_n|\mathcal{G}_{n-1})1\!\!1_{\{\tau\ge n\}} = \frac{1}{Z_{n-1}}\mathbb{E}[\Delta X_n 1\!\!1_{\{\tau>n-1\}}\Delta N_n|\mathcal{F}_{n-1}]1\!\!1_{\{\tau\ge n\}}\ ,$$

since ΔX_n is $\mathcal{F}_n$-measurable and $1\!\!1_{\{\tau>n-1\}}$ is $\mathcal{G}_{n-1}$-measurable we have, from (ii)

$$\mathbb{E}(\Delta X_n 1\!\!1_{\{\tau>n-1\}}\Delta N_n|\mathcal{F}_{n-1}) = \mathbb{E}\big[\Delta X_n \mathbb{E}(1\!\!1_{\{\tau>n-1\}}\Delta N_n|\mathcal{F}_n)|\mathcal{F}_{n-1}\big] = 0$$

hence

$$\mathbb{E}(\Delta X_n \Delta N_n|\mathcal{G}_{n-1})1\!\!1_{\{\tau\ge n\}} = 0\ .$$

On the set $\{\tau < n\}$, using that $\{\tau < n\} \in \mathcal{G}_{n-1}$, we obtain

$$\mathbb{E}(\Delta X_n\ \Delta N_n|\mathcal{G}_{n-1})1\!\!1_{\{\tau<n\}} = \mathbb{E}[\Delta X_n(1\!\!1_{\{\tau=n\}} - \lambda_n 1\!\!1_{\{\tau\ge n\}})1\!\!1_{\{\tau<n\}}|\mathcal{G}_{n-1}] = 0\ .$$

Finally, we get $\mathbb{E}[\Delta(X_n N_n)|\mathcal{G}_{n-1}] = 0$.

(iii) $\Rightarrow$ (i). By (iii), we have in the one hand, for $n \ge 1$, $\mathbb{E}(\Delta X_n \Delta N_n|\mathcal{G}_{n-1}) = 0$, then $\mathbb{E}(\Delta X_n \Delta N_n) = 0$. In the other hand, $\mathbb{E}(\Delta X_n \Delta N_n) = \mathbb{E}[\Delta X_n \mathbb{E}(\Delta N_n|\mathcal{F}_n)]$. In the case $X = M$, applying Lemma 1.3.8, we obtain $\mathbb{E}(\Delta N_n \Delta M_n) = -\mathbb{E}(|\Delta M_n|^2)$, which implies $\mathbb{E}(|\Delta M_n|^2) = 0$. Therefore $\Delta M_n = 0 =$, or equivalently $\mathbb{E}(Z_n|\mathcal{F}_{n-1}) = Z_n$, which is equivalent to the predictability of Z. □

Example 1.3.19 Assume that $\tau = \inf\{n : V_n \ge \Theta\}$ where V is an increasing $\mathbb{F}$-adapted process and Θ is independent from $\mathbb{F}$, with an exponential law. Then, $Z_n = \mathbb{P}(V_n > \Theta|\mathcal{F}_n) = e^{-V_n}$ and immersion property holds (and $\widetilde{Z}_n = \mathbb{P}(\tau > n-1|\mathcal{F}_n) = \mathbb{P}(V_{n-1} > \Theta|\mathcal{F}_n) = e^{-V_{n-1}} = Z_{n-1}$).
If V is predictable, the Doob decomposition of Z is $Z_n = 1 - A_n = 1 - (1 - e^{-V_n})$, and $Z_n = \mathcal{E}(-\Lambda)$ with $\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta V_n}$, where Λ was in Lemma 1.3.6. Note that, from Proposition 1.3.14, $\Lambda = \Gamma$. Moreover, Z is predictable and assertions of Theorem 1.3.18 hold.
If V is not predictable,

$$\Delta\Lambda_n = \frac{\Delta A_n}{Z_{n-1}} = \frac{\mathbb{E}(-\Delta Z_n|\mathcal{F}_{n-1})}{Z_{n-1}} = e^{-V_{n-1}}\frac{1 - \mathbb{E}(e^{-\Delta V_n}|\mathcal{F}_{n-1})}{Z_{n-1}} = 1 - \mathbb{E}(e^{-\Delta V_n}|\mathcal{F}_{n-1})$$

and $\Delta\Gamma_n = 1 - e^{-\Delta V_n}$.

Equivalent probability measures

Proposition 1.3.20 *Suppose $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$. Let $\mathbb{Q}$ be a probability measure which is equivalent to $\mathbb{P}$ and let L be its Radon-Nikodym density. If L is $\mathbb{F}$-adapted, then*

$$\mathbb{Q}(\tau > n|\mathcal{F}_n) = \mathbb{P}(\tau > n|\mathcal{F}_n) = Z_n\ ,\ \forall n \ge 0$$

and $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$. Consequently, the predictable compensator of H is unchanged under such equivalent changes of probability measures, i.e. N is a $\mathbb{G}$-martingale under $\mathbb{P}$ and $\mathbb{Q}$.

Proof. Let X be an $(\mathbb{F}, \mathbb{Q})$-martingale, then, L being $\mathbb{F}$ adapted, $(X_n L_n, n \ge 0)$ is an $(\mathbb{F}, \mathbb{P})$-martingale, and since $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$ we have that $(X_n L_n, n \ge 0)$ is a $(\mathbb{G}, \mathbb{P})$-martingale which implies that X is a $(\mathbb{G},\mathbb{Q})$-martingale, i.e. $\mathbb{F} \overset{\mathbb{Q}}{\hookrightarrow} \mathbb{G}$. We have for each $n \le k$, using Bayes' formula

$$\mathbb{Q}(\tau \le n|\mathcal{F}_k) = \frac{\mathbb{E}^{\mathbb{P}}(L_k \mathbb{1}_{\{\tau\le n\}}|\mathcal{F}_k)}{\mathbb{E}^{\mathbb{P}}(L_k|\mathcal{F}_k)} = \mathbb{P}(\tau \le n|\mathcal{F}_k) \ ,$$

in particular, $\mathbb{Q}(\tau \le n|\mathcal{F}_n) = \mathbb{P}(\tau \le n|\mathcal{F}_n)$, then by $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$, $\mathbb{Q}(\tau \le n|\mathcal{F}_n) = \mathbb{Q}(\tau \le n|\mathcal{F}_k)$ and the assertion follows.

□

Immersion property is not stable by change of probability. We give here the result obtained in a continous time setting in [**?**, Th. 6.32].

Theorem 1.3.21 *Assume that $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ and that Z is $\mathbb{F}$-predictable and positive. Let X be an $(\mathbb{F},\mathbb{P})$-martingale and let ψ be an integrable $\mathbb{G}$-predictable process such that$\mathcal{E}(\psi \centerdot X)$ is a positive $\mathbb{G}$-martingale. Let φ be an integrable $\mathbb{F}$-predictable process such that $\mathcal{E}(\varphi \centerdot N)$ is a positive $\mathbb{G}$-martingale. Let us introduce the $\mathbb{G}$-martingale*

$$L_n := \mathcal{E}(\psi \centerdot X)_n \mathcal{E}(\varphi \centerdot N)_n \ , \quad \forall n \ge 0 \ ,$$

where N is defined in (1.21) and assume that L is uniformly integrable.
Define

$$d\mathbb{Q} = L_n d\mathbb{P} \quad \text{on } \ \mathcal{G}_n \ , \quad \forall n \ge 0 \ .$$

Then, the $\mathbb{Q}$-Azéma supermartingale associated with τ has the following multiplicative decomposition:

$$Z^{\mathbb{Q}}_n = \mathbb{Q}(\tau > n|\mathcal{F}_n) = \mathcal{E}\Big((\bar{\psi} - {}^p\psi) \centerdot X^{\mathbb{Q}}\Big)_n \ \mathcal{E}\Big(-\varphi \centerdot \Lambda\Big)_n Z_n \ , \quad \forall n \ge 0 \ ,$$

where

- *Λ is defined in Lemma 1.3.6*
- *${}^p\psi$ is the $\mathbb{F}$-predictable projection of the process ψ under the probability $\mathbb{Q}$, i.e. ${}^p\psi := \psi_0$ and ${}^p\psi_n := \mathbb{E}^{\mathbb{Q}}(\psi_n|\mathcal{F}_{n-1})$, for all $n \ge 1$,*
- *$\bar{\psi}$ is an $\mathbb{F}$-predictable process such that $\psi_n \mathbb{1}_{\{\tau>n-1\}} = \bar{\psi}_n \mathbb{1}_{\{\tau>n-1\}}$, for all $n \ge 0$ and*
- *$X^{\mathbb{Q}}$ defined by $X^{\mathbb{Q}}_0 := X_0$ and $X^{\mathbb{Q}}_n := X_n - \sum_{k=1}^n \frac{\Delta[X,\ell]_k}{\ell_k}$ for all $n \ge 0$, is an $(\mathbb{F},\mathbb{Q})$-martingale with $\ell_k = \mathbb{E}(L_k|\mathcal{F}_k)$, for all $k \ge 0$.*

Furthermore, the process

$$H_n - \sum_{k=1}^{n\wedge\tau} \Big(1 - \frac{Z_k}{Z_{k-1}}\varphi_k\Big)\Delta\Lambda_k \ , \quad \forall n \ge 0,$$

is the compensated $\mathbb{Q}$-martingale associated with H. In particular, if the process ψ is $\mathbb{F}$-predictable, then:

$$Z^{\mathbb{Q}}_n = \mathbb{Q}(\tau > n|\mathcal{F}_n) = \mathcal{E}(-\varphi \centerdot \Lambda)_n Z_n \ , \quad \forall n \ge 0$$

and the immersion property holds under $\mathbb{Q}$.

Proof. Let $n \ge 0$ be fixed. Note that L is a martingale: this is a local martingale by orthogonality of X and N and a martingale by Proposition 1.1.31.
• In a first step, we compute $\ell_n := \mathbb{E}^{\mathbb{P}}(L_n|\mathcal{F}_n)$. By Lemma 1.1.24, we have that for all $k \ge 1$,

$$\Delta L_k = L_{k-1}(\psi_k \Delta X_k + \varphi_k \Delta N_k + \psi_k\varphi_k \Delta[X,N]_k) \ .$$

Then, taking the sum for $1 \le k \le n$, and since $L_0 = 1$,

$$L_n = 1 + \big((L_-\psi) \centerdot X)_n + \big((L_-\varphi) \centerdot N\big)_n + \big((L_-\psi\varphi) \centerdot [X,N]\big)_n \ . \tag{1.30}$$

Using that $\Delta X_k \in \mathcal{F}_n$, $L_{k-1}\psi_k\varphi_k \in \mathcal{G}_{n-1}$ for all $1 \le k \le n$ and Z predictable, we have that by Theorem 1.3.18

$$\mathbb{E}\Big(\big((L_-\psi\varphi) \centerdot [X,N]\big)_n \big| \mathcal{F}_n\Big) = \sum_{k=1}^{n} \Delta X_k \mathbb{E}(L_{k-1}\psi_k\varphi_k\Delta N_k|\mathcal{F}_n) = 0 \ , \tag{1.31}$$

and again by Theorem 1.3.18,

$$\mathbb{E}\big((L_-\varphi \centerdot N)_n \big| \mathcal{F}_n\big) = 0 \ . \tag{1.32}$$

Taking the conditional expectation of (1.30) and using (1.31) and (1.32), we obtain

$$\ell_n = 1 + \mathbb{E}\big((L_-\psi \centerdot X)_n \big| \mathcal{F}_n\big) = 1 + \sum_{k=1}^{n} \mathbb{E}\big(L_{k-1}\psi_k|\mathcal{F}_n\big)\Delta X_k \ .$$

Then, since $L_{k-1}\psi_k \in \mathcal{G}_{k-1}$ and immersion holds, we obtain from Proposition 1.1.47 ($\mathcal{H}2$) that $\mathbb{E}(L_{k-1}\psi_k|\mathcal{F}_n) = \mathbb{E}(L_{k-1}\psi_k|\mathcal{F}_{k-1})$ for all $k \le n$, and

$$\ell_n := 1 + \sum_{k=1}^{n} \mathbb{E}(L_{k-1}\psi_k|\mathcal{F}_{k-1})\Delta X_k = \ell_{n-1} + \mathbb{E}(L_{n-1}\psi_n|\mathcal{F}_{n-1})\Delta X_n \ .$$

From $\mathbb{E}(L_{n-1}\psi_n|\mathcal{F}_{n-1}) = \mathbb{E}^{\mathbb{Q}}(\psi_n|\mathcal{F}_{n-1})\ell_{n-1}$, we obtain

$$\ell_n = \ell_{n-1}(1 + \mathbb{E}^{\mathbb{Q}}(\psi_n|\mathcal{F}_{n-1})\Delta X_n) = \ell_{n-1}(1 +^p \psi_n \Delta X_n) \tag{1.33}$$

where ${}^p\psi$ is the $\mathbb{F}$-predictable projection of ψ under the probability $\mathbb{Q}$. Finally

$$\ell_n = \mathcal{E}({}^p\psi \centerdot X)_n \ .$$

• In a second step, we compute $\mathbb{E}(\mathbb{1}_{\{\tau>n\}}L_n|\mathcal{F}_n)$. In the one hand, there exists an $\mathbb{F}$-predictable process $\bar{\psi}$ such that $\psi_k\mathbb{1}_{\{\tau>n\}} = \bar{\psi}_k\mathbb{1}_{\{\tau>n\}}$ for all $k \le n$. It follows that

$$\mathcal{E}(\psi \centerdot X)_n\mathbb{1}_{\{\tau>n\}} = \Pi_{k=1}^{n}(1 + \bar{\psi}_k\Delta X_k)\mathbb{1}_{\{\tau>n\}} \ . \tag{1.34}$$

In the other hand, we have that

$$\mathcal{E}(\varphi \centerdot N)_n\mathbb{1}_{\{\tau>n\}} = \Pi_{k=1}^{n}(1 + \varphi_k(\Delta H_k - \Delta\Lambda_k\mathbb{1}_{\{\tau\ge k\}})\mathbb{1}_{\{\tau>n\}} = \Pi_{k=1}^{n}(1 - \varphi_k\Delta\Lambda_k)\mathbb{1}_{\{\tau>n\}} \ . \tag{1.35}$$

Then, using (1.34), we get that

$$\mathbb{E}(\mathbb{1}_{\{\tau>n\}}L_n|\mathcal{F}_n) = \mathcal{E}(\bar{\psi} \centerdot X)_n\mathbb{E}\big(\mathcal{E}(\varphi \centerdot N)_n\mathbb{1}_{\{\tau>n\}}\big|\mathcal{F}_n\big) \ .$$

From (1.35) and the fact that $\varphi_k\Delta\Lambda_k \in \mathcal{F}_n$ for all $1 \le k \le n$, we obtain

$$\mathcal{E}(\mathbb{1}_{\{\tau>n\}}L_n|\mathcal{F}_n) = \mathcal{E}(\bar{\psi} \centerdot X)_n\mathcal{E}(-\varphi \centerdot \Lambda)_n Z_n \ . \tag{1.36}$$

Replacing (1.33) and (1.36) in the formula $Z_n^{\mathbb{Q}} = \mathbb{E}^{\mathbb{P}}(\mathbb{1}_{\{\tau>n\}}L_n|\mathcal{F}_n)/\ell_n$ leads to

$$Z_n^{\mathbb{Q}} = \mathcal{E}(-\varphi \centerdot \Lambda)_n \frac{\mathcal{E}(\bar{\psi} \centerdot X)_n}{\mathcal{E}({}^p\psi \centerdot X)_n} Z_n \ .$$

By definition of the exponential, it follows that

$$\frac{\mathcal{E}(\bar{\psi} \centerdot X)_n}{\mathcal{E}({}^p\psi \centerdot X)_n} = \mathcal{E}\Big(\sum_k^{\cdot} \frac{(\bar{\psi}_k - {}^p\psi_k)\Delta X_k}{{}^p\psi_k\Delta X_k + 1}\Big)_n \ . \tag{1.37}$$

From $\frac{\Delta \ell_k}{\ell_{k-1}} = {}^p\psi_k \Delta X_k$, we have that

$$\frac{1}{{}^p\psi_k \Delta X_k + 1} = \frac{\ell_{k-1}}{\ell_k} = 1 - \frac{\Delta \ell_k}{\ell_k} \ , \ \forall 0 \leq k \leq n \ . \tag{1.38}$$

Then, replacing (1.38) in (1.37), we obtain

$$\frac{\mathcal{E}(\bar{\psi} \centerdot X)_n}{\mathcal{E}({}^p\psi \centerdot X)_n} = \mathcal{E}\big((\bar{\psi} - {}^p\psi) \centerdot X^{\mathbb{Q}}\big)_n \ ,$$

where $X^{\mathbb{Q}}$ defined by $\Delta X_k^{\mathbb{Q}} = \Delta X_k - \frac{\Delta \ell_k \Delta X_k}{\ell_k}$, for all $1 \leq k \leq n$ is an $(\mathbb{F}, \mathbb{Q})$-martingale by Proposition 1.1.37. Therefore

$$Z_n^{\mathbb{Q}} = \mathcal{E}(-\varphi \centerdot \Lambda)_n \mathcal{E}\big((\bar{\psi} - {}^p\psi) \centerdot X^{\mathbb{Q}}\big)_n Z_n \ .$$

It follows that for all $k \geq 1$, $\{Z_{k-1}^{\mathbb{Q}} > 0\} = \{Z_{k-1} > 0\} = \Omega$ and

$$\frac{Z_k^{\mathbb{Q}}}{Z_{k-1}^{\mathbb{Q}}} = (1 - \varphi_k \Delta \Lambda_k)\big(1 + (\bar{\psi}_k - {}^p\psi_k)\Delta X_k^{\mathbb{Q}}\big)\frac{Z_k}{Z_{k-1}} \ , \tag{1.39}$$

then taking conditional expectation under $\mathbb{Q}$ in (1.39), and using that ${}^p\psi$, $\bar{\psi}$, φ, Λ and Z are $\mathbb{F}$-predictable and that $X^{\mathbb{Q}}$ is an $(\mathbb{F}, \mathbb{Q})$-martingale, we get

$$\mathbb{E}^{\mathbb{Q}}\Big(\frac{Z_k^{\mathbb{Q}}}{Z_{k-1}^{\mathbb{Q}}}\Big|\mathcal{F}_{k-1}\Big) = (1 - \varphi_k \Delta \Lambda_k)\frac{Z_k}{Z_{k-1}} \ , \quad \forall k \geq 1 \ . \tag{1.40}$$

The compensated $\mathbb{Q}$-martingale of H is

$$H_n - \sum_{k=1}^{n \wedge \tau} \frac{Z_{k-1}^{\mathbb{Q}} - \mathbb{E}^{\mathbb{Q}}(Z_k^{\mathbb{Q}}|\mathcal{F}_{k-1})}{Z_{k-1}^{\mathbb{Q}}} = H_n - \sum_{k=1}^{n \wedge \tau}\Big(1 - \mathbb{E}^{\mathbb{Q}}\Big(\frac{Z_k^{\mathbb{Q}}}{Z_{k-1}^{\mathbb{Q}}}\Big|\mathcal{F}_{k-1}\Big)\Big) =: \nu_n \ . \tag{1.41}$$

Therefore, using (1.40),

$$\nu_n = H_n - \sum_{k=1}^{n \wedge \tau}\Big(1 - (1 - \varphi_k \Delta \Lambda_k)\frac{Z_k}{Z_{k-1}}\Big) = H_n - \sum_{k=1}^{n \wedge \tau}\Big(\frac{Z_{k-1} - Z_k}{Z_{k-1}} - \frac{\varphi_k Z_k \Delta \Lambda_k}{Z_{k-1}}\Big) \ , \tag{1.42}$$

since Z is predictable, we have that $\Delta \Lambda_k = \frac{Z_{k-1} - Z_k}{Z_{k-1}}$ for all $k \geq 1$, then (1.42) is equivalent to

$$\nu_n = H_n - \sum_{k=1}^{n \wedge \tau}\Big(1 - \varphi_k \frac{Z_k}{Z_{k-1}}\Big)\Delta \Lambda_k \ .$$

In particular, if ψ is $\mathbb{F}$-predictable, $\bar{\psi} = {}^p\psi = \psi$ then

$$Z_n^{\mathbb{Q}} \ = \ \mathcal{E}(-\varphi \centerdot \Lambda)_n Z_n \ , \ \ell_n = 1 \, .$$

Let X be an $(\mathbb{F}, \mathbb{Q})$-martingale. Since $e \equiv 1$, the process X is an $(\mathbb{F}, \mathbb{P})$-martingale, hence, by immersion a $(\mathbb{G}, \mathbb{P})$ martingale. From Girsanov Theorem $X^{\mathbb{G}} = X - \sum_k \frac{1}{L_{k-1}}\Delta\langle X, L\rangle_k$ is a $\mathbb{G}$-martingale. From the fact that any $\mathbb{F}$ martingale is orthogonal to N, the bracket is null which implies that X is a $(\mathbb{G}, \mathbb{Q})$-martingale. □

Corollary 1.3.22 *Suppose that* $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ *and* Z *is* $\mathbb{F}$*-predictable. Define* $\mathbb{Q}$ *on* $\mathcal{G}_n$ *by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_n := \mathcal{E}(\psi \centerdot X)_n \ , \quad \forall n \geq 0 \ ,$$

with X *an* $\mathbb{F}$*-martingale and* ψ *a* $\mathbb{G}$*-predictable process such that* L *is a uniformly integrable* $\mathbb{G}$*-martingale. Then, under* $\mathbb{Q}$ *the process* $N = H - \Lambda^\tau$ *remains a* $\mathbb{G}$*-martingale.*

Corollary 1.3.23 *Suppose that $\mathbb{F} \overset{\mathbb{P}}{\hookrightarrow} \mathbb{G}$ and Z is $\mathbb{F}$-predictable. Define $\mathbb{Q}$ on $\mathcal{G}_n$ by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_n := \mathcal{E}(\psi \centerdot X)_n \ , \quad \forall n \geq 0 \ ,$$

with X an $\mathbb{F}$-martingale and ψ a $\mathbb{G}$-predictable process such that L is a uniformly integrable $\mathbb{G}$-martingale. Then, under $\mathbb{Q}$ the process $N = H - \Lambda^\tau$ remains a $\mathbb{G}$-martingale.

Proof. It suffices to take $\varphi = 0$ in Theorem 1.3.21. □

Theorem 1.3.18, Theorem 1.3.21 and Corollary 1.3.23 are the discrete version of Lemma 5.1, Theorem 6.4 and Corollary 6.5 in [10]. In continuous time the results holds under Assumption (A): the random time τ avoids every $\mathbb{F}$ stopping time T, i.e. $\mathbb{P}(\tau = T) = 0$, but in discrete time Assumption (A) does not hold, in order to have the same results we need the hypothesis that Z is predictable, instead Assumption (A).

1.3.2 Study before τ

Semimartingale decomposition

Proposition 1.3.24 *Any square integrable $\mathbb{F}$-martingale X stopped at τ is a $\mathbb{G}$-semimartingale with decomposition*

$$X_n^\tau = X_n^{\mathbb{G}} + \sum_{k=1}^{n \wedge \tau} \frac{1}{Z_{k-1}} \Delta \langle \widetilde{M}, X \rangle_k^{\mathbb{F}} \, ,$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale (stopped at τ). Here, $\widetilde{M}$ is the martingale part of the Doob decomposition of the supermartingale $\widetilde{Z}$.

Proof. We compute the predictable part of the $\mathbb{G}$-semimartingale X on the set $\{0 \leq n < \tau\}$ using Lemma 1.3.3

$$\mathbb{1}_{\{\tau > n\}} \mathbb{E}(\Delta X_{n+1} | \mathcal{G}_n) \quad = \quad \mathbb{1}_{\{\tau > n\}} \frac{1}{Z_n} \mathbb{E}(\widetilde{Z}_{n+1} \Delta X_{n+1} | \mathcal{F}_n) \, .$$

Using now the Doob decomposition of $\widetilde{Z}$, and the martingale property of X, we obtain

$$\begin{aligned} \mathbb{E}(\widetilde{Z}_{n+1} \Delta X_{n+1} | \mathcal{F}_n) &= \mathbb{E}((\widetilde{M}_{n+1} - \widetilde{A}_{n+1}) \Delta X_{n+1} | \mathcal{F}_n) \\ &= \mathbb{E}(\widetilde{M}_{n+1} \Delta X_{n+1} | \mathcal{F}_n) = \Delta \langle \widetilde{M}, X \rangle_{n+1}^{\mathbb{F}} \end{aligned}$$

and finally

$$\mathbb{1}_{\{\tau > n\}} \mathbb{E}(\Delta X_{n+1} | \mathcal{G}_n) = \mathbb{1}_{\{\tau > n\}} \frac{1}{Z_n} \Delta \langle \widetilde{M}, X \rangle_{n+1}^{\mathbb{F}} \, .$$

□

Remark 1.3.25 The result extends to any filtration $\mathbb{K}$ such that $\mathbb{F} \subset \mathbb{K}$ and, for any $\mathbb{K}$ predictable process U, there exists an $\mathbb{F}$ predictable process $U^{\mathbb{F}}$ such that $U_n \mathbb{1}_{\{n \leq \tau\}} = \widetilde{U}_n \mathbb{1}_{\{n \leq \tau\}}$.

Comment 1.3.26 From this result, we can hope that, in continuous time the $\mathbb{G}$-semimartingale decomposition formula of an $\mathbb{F}$-martingale X stopped at time τ will be

$$X_t^\tau = \widehat{X}_t + \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle \widetilde{M}, X \rangle_s^{\mathbb{F}} \, ,$$

where $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$, $\quad \widetilde{Z}_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t)$ and $\widetilde{M}$ is an $\mathbb{F}$-martingale and $\widetilde{A}$ is $\mathbb{F}$-predictable with $\widetilde{Z} = \widetilde{M} - \widetilde{A}$. This is indeed the case and known as the Jeulin formula. Note that, as $\widetilde{Z}$ is not càdlàg, the decomposition $\widetilde{Z} = \widetilde{M} - \widetilde{A}$ is not the standard Doob-Meyer decomposition established only for càdlàg supermartingales.

In the two following propositions, we present the discrete time version of well known decomposition in continuous time. The first result (Proposition 1.3.24) is a predictable decomposition, the second one (Proposition 1.3.28 an "optional" decomposition.

Proposition 1.3.27 *)*

$$X_n^\tau = X_n^{\mathbb{G}} + \sum_{k=1}^{n\wedge\tau} \frac{1}{Z_{k-1}} \big(\Delta\langle M, X\rangle_k^{\mathbb{F}} + (\Delta X_\tau \mathbb{1}_{\tau,\infty})_k^{p,\mathbb{F}}\big),$$

Proof. Note that

$$\mathbb{E}(\widetilde{Z}_{n+1}\Delta X_{n+1}|\mathcal{F}_n) = \mathbb{E}(Z_{n+1}\Delta X_{n+1}|\mathcal{F}_n) + \mathbb{E}(\mathbb{1}_{\tau=n+1}\Delta X_{n+1}|\mathcal{F}_n)$$

and that the decomposition can be written as stated. □

In continuous time, one has that every càdlàg $\mathbb{F}$-local martingale X stopped at time τ is a $\mathbb{G}$-special semimartingale with canonical decomposition

$$X_t^\tau = \widehat{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X,\mu\rangle_s^{\mathbb{F}} + dJ_s}{Z_{s-}}$$

where $\widehat{X}$ is a $\mathbb{G}$-local martingale, μ is the martingale part of the $\mathbb{F}$-Doob Meyer decomposition of Z and J is the $\mathbb{F}$-dual predictable projection of the process $(\Delta X_\tau)\mathbb{1}_{]\tau,\infty]}$.

We introduce two $\mathbb{F}$-stopping times which play an important role in the optional decomposition. Let $R =: \inf\{n \geq 0, Z_n = 0\}$ and $\widetilde{R} = R\mathbb{1}_{\{\widetilde{Z}_R=0<Z_{R-}\}} + \infty\mathbb{1}_{\{\widetilde{Z}_R=0<Z_{R-}\}^c}$. If T is a stopping time, we denote by $[\![T]\!] := \{(\omega,n), T(\omega)=n\}$ and it is called the graph of T.

Proposition 1.3.28 *Any $\mathbb{F}$-local-martingale X stopped at τ admits the following optional decomposition*

$$X_n^\tau = \widehat{X}_n^{\mathbb{G}} + \sum_{k=1}^{n\wedge\tau} \frac{1}{\widetilde{Z}_k}\Delta\left[\widetilde{M}, X\right]_k + \sum_{k=1}^{n\wedge\tau}\left(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)_k^{p,\mathbb{F}}, \tag{1.43}$$

where $\widehat{X}^{\mathbb{G}}$ is a $\mathbb{G}$-martingale and $\widetilde{M}$ is the martingale part of the Doob decomposition of the supermartingale $\widetilde{Z}$ defined in (1.18).

Proof. The proof is inspired of the proof in continuous time in [4]. We give it for the ease of the readers. We can remark that, for any $\mathbb{F}$-martingale Y, one has $\mathbb{E}(Y_\tau) = \mathbb{E}([Y,\widetilde{M}]_\infty)$. Indeed

$$\begin{aligned}\mathbb{E}(Y_\tau) &= \sum_{k\geq 0}\mathbb{E}(Y_k\mathbb{1}_{\{\tau=k\}}) = \sum_{k\geq 0}\mathbb{E}(Y_k\Delta\widetilde{Z}_k) \\ &= \sum_{k\geq 0}\mathbb{E}(\Delta Y_k\Delta\widetilde{Z}_k) = \mathbb{E}([Y,\widetilde{M}]_\infty)\end{aligned}$$

The second equality is a consequence of the tower property and the two last equalities are obtained using that Y is a $\mathbb{F}$-martingale.

It is sufficient to prove that for all $\mathbb{G}$-predictable process K, $\mathbb{E}((K \centerdot \widehat{X}^{\mathbb{G}})_\infty) = 0$. For any $\mathbb{G}$-predictable process K, there exists an $\mathbb{F}$ predictable process $\widetilde{K}$ such that $K\mathbb{1}_{[\![0,\tau]\!]} = \widetilde{K}\mathbb{1}_{[\![0,\tau]\!]}$. As mentionned in Lemma 1.3.4, we can choose $\widetilde{K}$ such that $\widetilde{K}\mathbb{1}_{Z_-=0} = 0$. Then

$$\begin{aligned}\mathbb{E}((K\centerdot X^\tau)_\infty) &= \mathbb{E}((\widetilde{K}\centerdot X)_\tau) = \mathbb{E}([\widetilde{K}\centerdot X,\widetilde{M}]_\infty) \\ &= \mathbb{E}\left(\sum_{k\geq 1}\frac{\widetilde{K}_k\widetilde{Z}_k}{\widetilde{Z}_k}\mathbb{1}_{\{\widetilde{Z}_k>0\}}\Delta[X,\widetilde{M}]_k\right) + \mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_k\mathbb{1}_{\{\widetilde{Z}_k=0\}}\Delta[X,\widetilde{M}]_k\right)\end{aligned} \tag{1.44}$$

As $\widetilde{K}$ and $\widetilde{M}$ are $\mathbb{F}$-adapted, one has, by tower property

$$\mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_k\widetilde{Z}_k\,(\widetilde{Z}_k)^{-1}\mathbb{1}_{\{\widetilde{Z}_k>0\}}\Delta[X,\widetilde{M}]_k\right)=\mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_k\mathbb{1}_{\{k\leq\tau\}}(\widetilde{Z}_k)^{-1}\mathbb{1}_{\{\widetilde{Z}_k>0\}}\Delta[X,\widetilde{M}]_k\right).$$

From Proposition 1.3.2, we obtain that the first term of the right hand side of (1.44)is equal to $\mathbb{E}\left(\sum_{k\geq 1}\frac{K_k\mathbb{1}_{\{k\leq\tau\}}}{\widetilde{Z}_k}\Delta[X,\widetilde{M}]_k\right)$.

For the second term of the right hand side of (1.44), we remark that, due to the choice of $\widetilde{K}$, one has

$$\mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_k\mathbb{1}_{\{\widetilde{Z}_k=0\}}\Delta[X,\widetilde{M}]_k\right)=\mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_k\mathbb{1}_{\{\widetilde{Z}_k=0<Z_{k-1}\}}\Delta[X,\widetilde{M}]_k\right)$$

and that $\{\widetilde{Z}=0<Z_-\}=[\widetilde{R}]$ and that $\Delta\widetilde{M}_{\widetilde{R}}=-Z_{\widetilde{R}}$ on $\{\widetilde{R}<\infty\}$. Then

$$\mathbb{1}_{\{\widetilde{Z}=0<Z_-\}}\Delta[X,\widetilde{M}]=-Z_{\widetilde{R}_-}\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}.$$

Hence

$$\mathbb{E}\left(\sum_{k\geq 1}K_k\mathbb{1}_{\{\widetilde{Z}_k=0<Z_{k-1}\}}\Delta[X,\widetilde{M}]_k\right)=-\mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_kZ_{k-1}(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![})_k\right)$$

Using that $\widetilde{K}Z_-$ is predictable, we obtain

$$\begin{aligned}\mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_k\mathbb{1}_{\{\widetilde{Z}_k=0\}}\Delta[X,\widetilde{M}]_k\right) &= -\mathbb{E}\left(\sum_{k\geq 1}\widetilde{K}_kZ_{k-1}\left(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^p_k\right)\\ &= -\mathbb{E}\left(\sum_{k\geq 1}K_k\mathbb{1}_{\{k-1<\tau\}}\left(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^p_k\right)\\ &= -\mathbb{E}\left(\sum_{k=1}^{\tau}K_k\left(\Delta X_{\widetilde{R}}\mathbb{1}_{[\![\widetilde{R},\infty[\![}\right)^p_k\right).\end{aligned}$$

It now follows that $\mathbb{E}((K\centerdot\widehat{X}^{\mathbb{G}})_\infty)=0$. $\square$

Arbitrages

In the case of progressive enlargement, we distinguish arbitrages which can occur before τ and those which can occur after τ.

Definition 1.3.29 *The enlargement $(\mathbb{F},\mathbb{G}^\tau)$ is viable if there exists a positive $\mathbb{G}$-martingale L such that, for any $\mathbb{F}$-martingale X, the process LX^τ is a martingale.*

Lemma 1.3.30 *Let $\mathbb{G}^{\tau-}$ be the filtration $\mathbb{G}$ "strictly before τ", i.e., $\mathcal{G}^{\tau-}_n=\mathcal{G}_{n\wedge(\tau-1)}$. There exists a positive $\mathbb{G}$-martingale L such that, for any $\mathbb{F}$-martingale X, the process $LX^{\tau-}$ is a martingale, where $X^{\tau-}_n=X_{n\wedge(\tau-1)}$.*

Proof. For any $\mathbb{F}$-martingale X we are looking for ψ such that on the set $\{1\leq n<\tau\}$ (strictly before τ)

$$\mathbb{1}_{\{n-1\leq\tau\}}\mathbb{E}(\psi_nX_n|\mathcal{G}_{n-1})=\mathbb{1}_{\{n-1\leq\tau\}}X_{n-1}\mathbb{E}(\psi_n|\mathcal{G}_{n-1})$$

that is

$$\mathbb{1}_{\{n-1\leq\tau\}}\frac{1}{\widetilde{Z}_{n-1}}\mathbb{E}(\psi_nX_nZ_n|\mathcal{F}_{n-1})=\mathbb{1}_{\{n-1\leq\tau\}}\frac{X_{n-1}}{\widetilde{Z}_{n-1}}\mathbb{E}(\psi_nZ_n|\mathcal{F}_{n-1}).$$

We are looking for a positive $\mathbb{F}$-adapted process ψ, satisfying

$$\mathbb{E}(\psi_n X_n Z_n|\mathcal{F}_{n-1}) = X_{n-1}\mathbb{E}(\psi_n Z_n|\mathcal{F}_{n-1})\,.$$

The choice $\psi = (1/Z)\mathbb{1}_{\{Z>0\}} + \mathbb{1}_{\{Z=0\}}$ provides a solution, valid for any martingale X. □

Theorem 1.3.31 *Assume that τ is not an $\mathbb{F}$-stopping time and denote by $\mathbb{G}^\tau$ the filtration $\mathcal{G}^\tau_n = \mathcal{G}_{\tau\wedge n}, n \geq 0$. Then, the enlargement $(\mathbb{F}, \mathbb{G}^\tau)$ is viable if and only, for any n, the set $\{0 = \widetilde{Z}_n < Z_{n-1}\}$ is empty.*

We mean here that, for any $\mathbb{F}$-martingale X, the stopped process X^τ admits a deflator. This result was established in Choulli and Deng [9] and is a particular case of the general results obtained in Aksamit et al.[5]. We give here a slightly different proof, by means of the two following propositions.

Proposition 1.3.32 *Assume that for any n, the set $\{\widetilde{Z}_n = 0 < Z_{n-1}\}$ is empty. The process $L = \mathcal{E}(Y)$, where Y is the $\mathbb{G}$-martingale defined by $\Delta Y_k = \mathbb{1}_{\{\tau\geq k\}}(\frac{Z_{k-1}}{\widetilde{Z}_k} - 1)$ for $k \geq 1$ and $Y_0 = 0$, is a positive $\mathbb{G}$-martingale. If X is an $\mathbb{F}$-martingale, the process $X^\tau L$ is a $(\mathbb{G}, \mathbb{P})$ martingale.*

Proof. The process Y is a martingale: for $n \geq 1$,

$$\begin{aligned}
\mathbb{E}(\Delta Y_n|\mathcal{G}_{n-1}) &= \mathbb{E}(\mathbb{1}_{\{\tau\geq n\}}\frac{Z_{n-1}-\widetilde{Z}_n}{\widetilde{Z}_n}|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{\{\widetilde{Z}_n>0\}}(Z_{n-1}-\widetilde{Z}_n)|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Z_{n-1}-\widetilde{Z}_n - \mathbb{1}_{\{\widetilde{Z}_n=0\}}(Z_{n-1}-\widetilde{Z}_n)|\mathcal{F}_{n-1}) \\
&= \mathbb{1}_{\{\tau\geq n\}}\frac{1}{Z_{n-1}}\mathbb{E}(Z_{n-1}-\widetilde{Z}_n|\mathcal{F}_{n-1}) = 0\,,
\end{aligned}$$

where we have used (1.20), the fact that $\mathbb{E}(\widetilde{Z}_n|\mathcal{F}_{n-1}) = Z_{n-1}$ and that, by assumption $\{\widetilde{Z}_n = 0\} \subset \{Z_{n-1} = 0\}$, hence $\mathbb{1}_{\{\widetilde{Z}_n=0\}}(Z_{n-1} - \widetilde{Z}_n) = 0$.

Hence L is a martingale. Note that the fact that $\{Z_{n-1} = 0\} \subset \{\widetilde{Z}_n = 0\}$ implies that the inclusion $\{\widetilde{Z}_n = 0\} \subset \{Z_{n-1} = 0\}$ is equivalent to $\{\widetilde{Z}_n = 0\} = \{Z_{n-1} = 0\}$, or to $\{\widetilde{Z}_n = 0 < Z_{n-1}\}$ is empty. On the set $\{\tau \geq k\}$, one has $Z_{k-1} > 0$ which implies that $\Delta Y_k = (\frac{Z_{k-1}}{\widetilde{Z}_k} - 1) \geq -1$, hence L is positive. Furthermore, for X an $\mathbb{F}$-martingale

$$\begin{aligned}
\mathbb{E}(X^\tau_{n+1}\frac{L_{n+1}}{L_n}|\mathcal{G}_n) &= \mathbb{E}(X_{(n+1)\wedge\tau}(1 + \mathbb{1}_{\{\tau\geq n+1\}}\frac{Z_n - \widetilde{Z}_{n+1}}{\widetilde{Z}_{n+1}})|\mathcal{G}_n) \\
&= \mathbb{E}(X_{n+1}\mathbb{1}_{\{\tau\geq n+1\}}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{G}_n) + \mathbb{E}(X_\tau\mathbb{1}_{\{\tau<n+1\}}|\mathcal{G}_n) \\
&= \mathbb{E}(X_{n+1}\mathbb{1}_{\{\tau\geq n+1\}}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{G}_n) + X_\tau\mathbb{1}_{\{\tau<n+1\}} \\
&= \mathbb{1}_{\{\tau>n\}}\frac{1}{Z_n}\mathbb{E}(X_{n+1}Z_n\mathbb{1}_{\{\widetilde{Z}_{n+1}>0\}}|\mathcal{F}_n) + X_\tau\mathbb{1}_{\{\tau\leq n\}} \\
&= \mathbb{1}_{\{\tau>n\}}\frac{1}{Z_n}\mathbb{E}(X_{n+1}Z_n(1 - \mathbb{1}_{\{\widetilde{Z}_{n+1}=0\}})|\mathcal{F}_n) + X_\tau\mathbb{1}_{\{\tau\leq n\}} \\
&= \mathbb{1}_{\{\tau>n\}}\frac{1}{Z_n}\mathbb{E}(X_{n+1}Z_n|\mathcal{F}_n) + X_\tau\mathbb{1}_{\{\tau\leq n\}} = X_{n\wedge\tau}\,,
\end{aligned}$$

where we have used that, by assumption, $Z_n\mathbb{1}_{\{\widetilde{Z}_{n+1}=0\}} = 0$. Hence the deflator property. □

Remark 1.3.33 In case of immersion, there are no arbitrages (indeed any e.m.m. in $\mathbb{F}$ will be an e.m.m. in $\mathbb{G}$). This can be also obtained using the previous result, since, under immersion hypothesis, one has $Z_{n-1} = \widetilde{Z}_n$.

Proposition 1.3.34 *If there exists $n \geq 1$ such that the set $\{0 = \widetilde{Z}_n < Z_{n-1}\}$ is not empty, and if τ is not an $\mathbb{F}$-stopping time, there exists an $\mathbb{F}$-martingale X such that X^τ is a $\mathbb{G}$-adapted increasing process with $X_0^\tau = 1$, $\mathbb{P}(X_\tau^\tau > 1) > 0$. Hence, the enlargement $(\mathbb{F}, \mathbb{G}^\tau)$ is not viable.*

Proof. The proof is the discrete time version of Acciaio et al. [1]. Let $\vartheta = \inf\{n : 0 = \widetilde{Z}_n < Z_{n-1}\}$. The random time ϑ is an $\mathbb{F}$-stopping time satisfying $\tau \leq \vartheta$ and $\mathbb{P}(\tau < \vartheta) > 0$. Let $I_n = \mathbb{1}_{\{\vartheta \leq n\}}$ and denote by D the $\mathbb{F}$-predictable process part of the Doob decomposition of I. One has $D_0 = 0$ and $\Delta D_n = \mathbb{P}(\vartheta = n|\mathcal{F}_{n-1})$. We introduce the $\mathbb{F}$-predictable increasing process U setting $U_n = \frac{1}{\mathcal{E}(-D)_n}$. Then,

$$\Delta U_n = \frac{1}{\mathcal{E}(-D)_{n-1}}\left(\frac{1}{1-\Delta D_n} - 1\right) = \frac{1}{\mathcal{E}(-D)_{n-1}}\frac{\Delta D_n}{1-\Delta D_n} = U_n \Delta D_n$$

We consider the process $X = UK$, where $K = 1 - I$,

$$\Delta X_n = -U_n \Delta I_n + K_{n-1}\Delta U_n = -U_n\left(\Delta I_n - K_{n-1}\Delta D_n\right)$$

and

$$\begin{aligned}\mathbb{E}(\Delta X_n|\mathcal{F}_{n-1}) &= -U_n \mathbb{E}(\Delta I_n - K_{n-1}\Delta D_n|F_{n-1}) = U_n\left(\mathbb{P}(\vartheta = n|\mathcal{F}_{n-1}) - K_{n-1}\mathbb{P}(\Delta D_n|F_{n-1})\right)\\ &= U_n K_{n-1}\left(\mathbb{P}(\vartheta = n|\mathcal{F}_{n-1}) - \mathbb{P}(\Delta D_n|F_{n-1})\right) = 0\,,\end{aligned}$$

where we have used that $K_{n-1}\mathbb{P}(\vartheta = n|\mathcal{F}_{n-1}) = \mathbb{E}(K_{n-1}\mathbb{1}_{\vartheta=n}|\mathcal{F}_{n-1}) = \mathbb{P}(\vartheta = n|\mathcal{F}_{n-1})$. Hence X is an $\mathbb{F}$-martingale.
We now prove that $X_\tau \geq 1$ and $\mathbb{P}(X_\tau > 1) > 0$, equivalently that $D_\tau \geq 0$ and $\mathbb{P}(D_\tau > 0) > 0$. For that, we compute

$$\begin{aligned}\mathbb{E}(D_\tau \mathbb{1}_{\tau<\infty}) &= \sum_{n=0}^{\infty}\mathbb{E}(D_n \mathbb{1}_{\{\tau=n\}}) = \sum_{n=0}^{\infty}\mathbb{E}(D_n \mathbb{P}(\tau = n|\mathcal{F}_n))\\ &= \sum_{n=1}^{\infty}\mathbb{E}(D_n\mathbb{P}(\tau > n|\mathcal{F}_n)) - \sum_{n=1}^{\infty}\mathbb{E}(D_n \mathbb{P}(\tau > n-1|\mathcal{F}_n)) + D_0\mathbb{P}(\tau = 0)\end{aligned}$$

Since D is predictable

$$\begin{aligned}\mathbb{E}(D_\tau \mathbb{1}_{\tau<\infty}) &= \sum_{n=1}^{\infty}\mathbb{E}(D_n\mathbb{P}(\tau > n|\mathcal{F}_n)) - \sum_{n=1}^{\infty}\mathbb{E}(D_n\mathbb{P}(\tau > n-1|\mathcal{F}_{n-1})) = -\sum_{n=1}^{\infty}\mathbb{E}(D_n\Delta Z_n)\\ &= \mathbb{E}\left(\sum_{n=1}^{\infty} Z_{n-1}\Delta D_n\right) = \mathbb{E}(Z_{\vartheta-1}\mathbb{1}_{\vartheta<\infty}) > 0\,,\end{aligned}$$

where, in the last inequality, we used that $\tau \leq \vartheta$ and $\mathbb{P}(\tau = \vartheta) < 1$. The process X^τ is then an increasing process and can not be turned in a martingale by change of probability. □

If $\widetilde{Z} > 0$, there are no arbitrages before τ.
We prove that if S is an $\mathbb{F}$ martingale, then, there exists a positive $\mathbb{G}$-martingale such that $S^\tau L$ is a local martingale.

Proposition 1.3.35 *If $\widetilde{Z} > 0$, the process*

$$L_n = \prod_{k=1}^{n}(1 + \Delta U_k) = L_{n-1}(1 + \Delta U_n)$$

where $\Delta U_k = \mathbb{1}_{\tau \geq k}(\frac{Z_{k-1}}{\widetilde{Z}_k} - 1)$ is a positive $\mathbb{G}$-martingale and the process $S^\tau L$ is a $(\mathbb{G}, \mathbb{P})$ martingale.

Proof. Indeed,

$$\begin{aligned}\mathbb{E}(1+\Delta U_n|\mathcal{G}_{n-1}) &= 1+\mathbb{E}(\mathbb{1}_{\tau\geq n}(\frac{Z_{n-1}}{\widetilde{Z}_n}-1)|\mathcal{G}_{n-1})\\ &= 1+\mathbb{1}_{\tau>n-1}\frac{1}{Z_{n-1}}\mathbb{E}(\mathbb{1}_{\tau\geq n}(\frac{Z_{n-1}}{\widetilde{Z}_n}-1)|\mathcal{F}_{n-1})\\ &= 1+\mathbb{1}_{\tau>n-1}\frac{1}{Z_{n-1}}(\mathbb{E}(\widetilde{Z}_n\frac{Z_{n-1}}{\widetilde{Z}_n}|\mathcal{F}_{n-1})-Z_{n-1})=1\end{aligned}$$

where we have used the positivity of $\tilde{Z}$ in the third equality. Then,

$$\begin{aligned}&\mathbb{E}(S_{(n+1)\wedge\tau}(1+\mathbb{1}_{\tau\geq n+1}(\frac{Z_n}{\widetilde{Z}_{n+1}}-1))|\mathcal{G}_n)\\ &=\mathbb{E}(S_{n+1}\mathbb{1}_{\tau\geq n+1}(1+\frac{Z_n}{\widetilde{Z}_{n+1}}-1)|\mathcal{G}_n)+\mathbb{E}(S_\tau\mathbb{1}_{\tau<n+1}|\mathcal{G}_n)\\ &=\mathbb{E}(S_{n+1}\mathbb{1}_{\tau\geq n+1}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{G}_n)+S_\tau\mathbb{1}_{\tau<n+1}\\ &=\mathbb{1}_{\tau>n}\frac{1}{Z_n}\mathbb{E}(S_{n+1}\widetilde{Z}_{n+1}\frac{Z_n}{\widetilde{Z}_{n+1}}|\mathcal{F}_n)+S_\tau\mathbb{1}_{\tau\leq n}=S_{n\wedge\tau}\end{aligned}$$

□

Comment 1.3.36 A necessary and sufficient condition can be found in Choulli-Deng for any S satisfying NA($\mathbb{F}$), S satisfies NA($\mathbb{G}$) is equivalent to $\{\tilde{Z}_n=0\}=\{Z_{n-1}=0\}$.

More generally

Lemma 1.3.37 *The following assertions hold.*
(a) The process $L^{\mathbb{G}}$ defined by

$$L^{\mathbb{G}}_n:=\prod_{k=1}^n(1+\Delta X_k)$$

where

$$\Delta X_n=-\mathbb{P}(\widetilde{Z}_n>0|\mathcal{F}_{n-1})\mathbb{1}_{\{\tau\geq n\}}+\frac{Z_{n-1}}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\geq n\}}$$

is a positive $\mathbb{G}$-martingale.
(b) The process $L^{\mathbb{F}}$ given by

$$L^{\mathbb{F}}_n:=\prod_{k=1}^n(1+\Delta Y_k)$$

and

$$\Delta Y_n=\widetilde{Z}_n\mathbb{P}(\widetilde{Z}_n=0|\mathcal{F}_{n-1})-Z_{n-1}\mathbb{1}_{\{\widetilde{Z}_n=0\}}$$

is a positive $\mathbb{F}$-martingale.

Proposition 1.3.38 *Suppose that S is an $\mathbb{F}$-martingale, and consider the equivalent probability measures $\mathbb{Q}^{\mathbb{F}}$ and $\mathbb{Q}^{\mathbb{G}}$ given by $\mathbb{Q}^{\mathbb{F}}:=L^{\mathbb{F}}\mathbb{P}$ and $\mathbb{Q}^{\mathbb{G}}:=L^{\mathbb{G}}\mathbb{P}$. Then, S is an $(\mathbb{F},\mathbb{Q}^{\mathbb{F}})$-martingale if and only if S^τ is a $\mathbb{G},\mathbb{Q}^{\mathbb{G}})$-martingale.*

See Choulli and Deng [9]

1.3.3 After τ

As we mentioned at the beginning, any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale (which is not the case in continuous time). In a progressive enlargement of filtration with a random time valued in $\mathbb{N}$, one can give the decomposition formula. We start with the general case, then we study the particular case where τ is honest, to provide comparison with the classical results. We also study the case of pseudo stopping times.

General case

Mixing the results obtained in initial enlargement and progressive enlargement before τ, for any $\mathbb{F}$-martingale X

$$X_n = X_n^{\mathbb{G}} + \sum_{k=1}^{n\wedge\tau} \frac{1}{Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k^{\mathbb{F}} + \sum_{k=\tau+1}^{n} \frac{\Delta\langle X, p(j)\rangle_k^{\mathbb{F}}|_{j=\tau}}{p_{k-1}(\tau)} . \tag{1.45}$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale.

Honest times

In continuous time, strong conditions are needed to keep the semimartingale property after τ, here it is no more the case. However, we now consider the case where τ is honest (and valued in $\mathbb{N}$). We recall the definition (see Barlow [6]) and some of the main properties.

Definition 1.3.39 *A random time is honest , if, for any $n \geq 0$, there exists an $\mathcal{F}_n$-measurable random variable $\tau(n)$ such that*

$$\mathbb{1}_{\{\tau\leq n\}}\tau = \mathbb{1}_{\{\tau\leq n\}}\tau(n) . \tag{1.46}$$

Remark 1.3.40 Following Jeulin, τ is honest if there exists an $\mathcal{F}_n$ measurable random variable $\widehat{\tau}(n)$, such that

$$\mathbb{1}_{\{\tau< n\}}\tau = \mathbb{1}_{\{\tau< n\}}\widehat{\tau}(n) . \tag{1.47}$$

The two definitions are equivalent. Indeed, starting with the equality (1.47), one can define $\tau(n) = \widehat{\tau}(n) \wedge n$; then on $\{\tau = n\}$, $\tau(n) = n$ and $\mathbb{1}_{\{\tau\leq n\}}\tau = \mathbb{1}_{\{\tau\leq n\}}\tau(n)$.

It follows that any $\mathbb{G}$-predictable process V can be written as $V_n = V_n^b \mathbb{1}_{\{n\leq\tau\}} + V_n^a \mathbb{1}_{\{\tau<n\}}$ where V^a, V^b are $\mathbb{F}$-predictable processes (the superscript a is for after τ and b for before).

Lemma 1.3.41 *If τ is honest, $Z_n = \widetilde{Z}_n$ on the set $\{n > \tau\}$ and $\widetilde{Z}_\tau = 1$.*
If $\widetilde{Z}_\tau = 1$, then τ is honest.

Proof. For any $n \geq 0$,

$$\begin{aligned}
\mathbb{P}(\tau = n|\mathcal{F}_n)\mathbb{1}_{\{n>\tau\}} &= \mathbb{P}(\tau = n|\mathcal{F}_n)\mathbb{1}_{\{n>\tau\}}\mathbb{1}_{\{n>\tau(n)\}} = \mathbb{E}(\mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{n>\tau(n)\}}|\mathcal{F}_n)\mathbb{1}_{\{n>\tau\}} \\
&= \mathbb{E}(\mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{n>\tau(n)\}}\mathbb{1}_{\{n>\tau\}}|\mathcal{F}_n)\mathbb{1}_{\{n>\tau\}} = 0 .
\end{aligned}$$

It follows that $Z_n\mathbb{1}_{\{\tau<n\}} = \widetilde{Z}_n\mathbb{1}_{\{\tau<n\}}$. Furthermore,

$$\begin{aligned}
\widetilde{Z}_n\mathbb{1}_{\{\tau=n\}} &= \mathbb{1}_{\{\tau=n\}}\mathbb{P}(\tau \geq n|\mathcal{F}_n) = \mathbb{1}_{\{\tau=n\}}\mathbb{1}_{\{\tau(n)=n\}}\mathbb{P}(\tau \geq n|\mathcal{F}_n) \\
&= \mathbb{1}_{\{\tau=n\}}\mathbb{E}(\mathbb{1}_{\{\tau(n)=n\}}\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_n) = \mathbb{1}_{\{\tau=n\}}
\end{aligned}$$

which implies $\widetilde{Z}_\tau = 1$.

If $\widetilde{Z}_\tau = 1$, let $\ell(n) = \sup\{k \leq n : \widetilde{Z}_k = 1\}$. Then, for any $n \geq 0$, Proposition 1.3.2 implies $\tau = \ell(n)$ on the set $\{\tau \leq n\}$, and τ is honest . ☐

Proposition 1.3.42 *If τ is honest then $\tau = \sup\{n : \widetilde{Z}_n = 1\}$*

Proof. It follows from the previous lemma and Proposition 1.3.2. $\square$

Proposition 1.3.43 *Let τ be an honest time and X an $\mathbb{F}$-martingale. Then,*

$$X_n = X_n^{\mathbb{G}} + \sum_{k=1}^{n\wedge\tau} \frac{1}{Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k^{\mathbb{F}} - \sum_{k=\tau+1}^{n} \frac{1}{1-Z_{k-1}} \Delta\langle \widetilde{M}, X\rangle_k^{\mathbb{F}}$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale.

Proof. Let $X = M^{\mathbb{G}} + V^{\mathbb{G}}$ be the $\mathbb{G}$-semimartingale decomposition of X. Let $n \geq 0$ be fixed. From the property of honest times, there exists $\widetilde{V}$, an $\mathbb{F}$-predictable process, such that

$$V_n^{\mathbb{G}} \mathbb{1}_{\{\tau\leq n\}} = \widetilde{V}_n \mathbb{1}_{\{\tau\leq n\}}\,.$$

Then,

$$\begin{aligned} \mathbb{1}_{\{\tau\leq n\}}(V_{n+1}^{\mathbb{G}} - V_n^{\mathbb{G}}) &= \mathbb{1}_{\{\tau\leq n\}}(\widetilde{V}_{n+1} - \widetilde{V}_n) = \mathbb{1}_{\{\tau\leq n\}}\mathbb{E}(X_{n+1} - X_n|\mathcal{G}_n) \\ &= \mathbb{E}(\mathbb{1}_{\{\tau\leq n\}}(X_{n+1} - X_n)|\mathcal{G}_n)\,. \end{aligned} \tag{1.48}$$

We now take the conditional expectation w.r.t. $\mathcal{F}_n$ in (1.48). Taking into account that $\widetilde{V}$ is $\mathbb{F}$-predictable, and the fact that $\mathbb{F} \subset \mathbb{G}$, we get

$$\begin{aligned} \mathbb{E}(\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_n)(\widetilde{V}_{n+1} - \widetilde{V}_n) &= \mathbb{E}(\mathbb{1}_{\{\tau\leq n\}}(X_{n+1} - X_n)|\mathcal{F}_n) \\ &= \mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_{n+1})(X_{n+1} - X_n)|\mathcal{F}_n)\,. \end{aligned}$$

Now, using the fact that

$$\begin{aligned} \mathbb{E}(\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_n) &= 1 - \mathbb{E}(\mathbb{1}_{\{\tau> n\}}|\mathcal{F}_n) = 1 - Z_n \\ \mathbb{E}(\mathbb{1}_{\{\tau\leq n\}}|\mathcal{F}_{n+1}) &= 1 - \mathbb{E}(\mathbb{1}_{\{\tau> n\}}|\mathcal{F}_{n+1}) = 1 - \mathbb{E}(\mathbb{1}_{\{\tau\geq n+1\}}|\mathcal{F}_{n+1}) = 1 - \widetilde{Z}_{n+1} \end{aligned}$$

and that X is an $\mathbb{F}$-martingale, we obtain, on the set $\{\tau \leq n\}$

$$(1 - Z_n)(\widetilde{V}_{n+1} - \widetilde{V}_n) = -\mathbb{E}(\widetilde{Z}_{n+1}(X_{n+1} - X_n)|\mathcal{F}_n) = -\Delta\langle \widetilde{M}, X\rangle \mathbb{F}_n\,.$$

$\square$

Remark 1.3.44 It seems important to note that the Doob decomposition of Z is not needed. Indeed, Equation (1.25) implies that Z admits the optional decomposition $Z = \widetilde{M} - H^o$ and hence, $\widetilde{M}$ can be viewed as the martingale part this optional decomposition. This "explains" why, in continuous time, such an optional decomposition of Z is required. However, since optional decompositions are not unique, we prefer to refer to $\widetilde{M}$ as the martingale part of the (unique) Doob-Meyer decomposition of $\widetilde{Z}$.

Comment 1.3.45 We recall, for the ease of the reader, the Jeulin formula in continuous time:

$$X_t = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} \Delta\langle \widetilde{M}, X\rangle_s^{\mathbb{F}} - \int_\tau^t \frac{1}{1-Z_{s-}} d\langle \widetilde{M}, X\rangle_s^{\mathbb{F}}\,.$$

where $\widetilde{M}$ is the martingale in the optional decomposition of Z as $Z = \widetilde{M} - H^o$.

Comment 1.3.46 Let τ an honest time. We have obtained a formula using Jacod's hypothesis in (1.45). In continuous time, one can show that an honest time satisfy equivalence Jacod's hypothesis if and only if it takes countably many values (see [2, lemma 4.11]) and one can not compare the two decompositions. In discrete time, honest times satisfy equivalence Jacod's hypothesis and one can check that the decompositions obtained in (1.45) and the one for honest times are the same. We proceed as in Aksamit [2]. Let $n \geq 1$ be fixed. On $\tau < n$, we have $\tau = \tau(n-1)$ where $\tau(n-1)$ is a $\mathcal{F}_{n-1}$-measurable r.v. and $\mathcal{F}_{n-1} \subset \mathcal{F}_n$. We now restrict our attention to $k < n$. On the one hand,

$$\begin{aligned}\mathbb{1}_{\{\tau=k\}}(1-Z_{n-1}) &= \mathbb{1}_{\{\tau=k=\tau(n-1)\}}\mathbb{P}(\tau \leq n-1|\mathcal{F}_{n-1}) = \mathbb{1}_{\{\tau=k\}}\mathbb{E}(\mathbb{1}_{\{\tau(n-1)=k\}}\mathbb{1}_{\{\tau\leq n-1\}}|\mathcal{F}_{n-1}) \\ &= \mathbb{1}_{\{\tau=k\}}\mathbb{E}(\mathbb{1}_{\{\tau(n-1)=k\}}\mathbb{1}_{\{\tau=k\}}|\mathcal{F}_{n-1}) = \mathbb{1}_{\{\tau=k\}}\mathbb{E}(\mathbb{1}_{\{\tau=k\}}|\mathcal{F}_{n-1}) = \mathbb{1}_{\{\tau=k\}}p_{n-1}(k)\end{aligned}$$

On the other hand

$$\begin{aligned}\mathbb{1}_{\{\tau=k\}}\mathbb{E}(\widetilde{M}_n\Delta X_n|\mathcal{F}_{n-1}) &= -\mathbb{1}_{\{\tau=k\}}\mathbb{E}((1-\widetilde{M}_n)\Delta X_n|\mathcal{F}_{n-1}) = -\mathbb{1}_{\{\tau=k=\tau(n-1)\}}\mathbb{E}((1-\widetilde{Z}_n)\Delta X_n|\mathcal{F}_{n-1}) \\ &= -\mathbb{1}_{\{\tau=k\}}\mathbb{E}(\mathbb{1}_{\{k=\tau(n-1)\}}\mathbb{1}_{\{\tau<n\}}\Delta X_n|\mathcal{F}_{n-1}) \\ &= -\mathbb{1}_{\{\tau=k\}}\mathbb{E}((\mathbb{E}(\mathbb{1}_{\{k=\tau\}}|\mathcal{F}_n)\Delta X_n|\mathcal{F}_{n-1}) = -\mathbb{1}_{\{\tau=k\}}\mathbb{E}(p_n(k)\Delta X_n|\mathcal{F}_{n-1})\,.\end{aligned}$$

Arbitrages before τ

Let τ be a bounded honest time which is not an $\mathbb{F}$-stopping time. The enlargement $(\mathbb{F},\mathbb{G})$ is not viable. Indeed, assuming the existence of a deflator L implies that $\widetilde{ML}$ is a $\mathbb{G}$-martingale. Since $\tilde{Z}_\tau = 1$, one has $\widetilde{M}_\tau \geq 1$, and $\mathbb{P}(\widetilde{M}_\tau > 1) > 0$. Therefore, using optional sampling theorem, $1 = \mathbb{E}(\widetilde{M}_\tau L_\tau) > \mathbb{E}(L_\tau) = 1$ yields to a contradiction and to existence of arbitrages.

We refer to Choulli and Deng [9] for a necessary and sufficient condition to avoid arbitrages after τ.

Links between progressive and initial enlargement

Proposition 1.3.47 *Let X be a process such that $X = \mathbb{1}_{]\!]\tau,\infty[\![} \centerdot X$. The process X is an $\mathbb{F}^{\sigma(\tau)}$ local martingale if and if it is a $\mathbb{G}$-local martingale.*

1.3.4 Pseudo-stopping times

We end the study of progressive enlargement with a specific class of random times. We assume that $\mathcal{F}_0$ is trivial. We recall that a random time τ is an $\mathbb{F}$-pseudo stopping time if $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$ for any bounded $\mathbb{F}$-martingale X (see [11]).

Theorem 1.3.48 *The following statements are equivalent:*

(i) τ *is an* $\mathbb{F}$*-pseudo stopping time.*

(ii) $H^o_{\infty-} = \mathbb{P}(\tau < \infty|\mathcal{F}_\infty)$, $H^o_\infty = 1$.

(iii) $\widetilde{M} \equiv 1$.

(iv) $\widetilde{Z}$ *is predictable.*

(v) Every $\mathbb{F}$*-martingale stopped at* τ *is a* $\mathbb{G}$*-martingale.*

Proof. (i) $\Rightarrow$ (ii) Using that the bounded martingale X is closed, $\lim_{n\to\infty} X_n =: X_\infty$ exists and one can write the integration by parts formula

$$H^o_{\infty-}X_\infty = \sum_{n=1}^{\infty} H^o_{n-1}\Delta X_n + \sum_{n=0}^{\infty} X_n\Delta H^o_n\,.$$

Taking expectation, and using the fact that X is an $\mathbb{F}$-martingale, we obtain

$$\mathbb{E}(H^o_{\infty-}X_\infty) = \mathbb{E}(\sum_{n=0}^{\infty} X_n \Delta H^o_n)$$

and, from property (1.24) of H^o , one has

$$\mathbb{E}(H^o_{\infty-}X_\infty) = \mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}})\,.$$

It follows that

$$\begin{aligned} X_0 &= \mathbb{E}(X_\tau) = \mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) + \mathbb{E}(X_\infty \mathbb{1}_{\{\tau=\infty\}}) = \mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) + \mathbb{E}(X_\infty) - \mathbb{E}(X_\infty \mathbb{1}_{\{\tau<\infty\}}) \\ &= \mathbb{E}(H^o_{\infty-}X_\infty) + X_0 - \mathbb{E}(X_\infty \mathbb{1}_{\{\tau<\infty\}})\,, \end{aligned}$$

hence $\mathbb{E}(H^o_{\infty-}X_\infty) = \mathbb{E}(X_\infty \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}(X_\infty \mathbb{P}(\tau<\infty|\mathcal{F}_\infty))$ which implies $H^o_{\infty-} = \mathbb{P}(\tau<\infty|\mathcal{F}_\infty)$.

(ii) $\Rightarrow$ (iii) Obvious

(iii) $\Rightarrow$ (iv) By definition of H^o, and (1.25) , we have that

$$\widetilde{M}_n = H^0_n + Z_n = H^0_{n-1} + \widetilde{Z}_n\,, \quad \forall n \geq 1\,, \tag{1.49}$$

therefore, by (iii), we deduce that $\widetilde{Z}_n = 1 - H^0_{n-1}$ which, since H^0 is $\mathbb{F}$-adapted, is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$, i.e. $\widetilde{Z}$ is $\mathbb{F}$-predictable.

(iv) $\Rightarrow$ (v) If $\widetilde{Z}$ is predictable, $\widetilde{M}$ is a predictable martingale, hence a constant (indeed, $\mathbb{E}(\widetilde{M}_n|\mathcal{F}_{n-1}) = \widetilde{M}_n = \widetilde{M}_{n-1}$) and for any $\mathbb{F}$ martingale X, $\Delta\langle X, \widetilde{M}\rangle_n = 0$ for all $n \geq 1$. The result follows from Proposition 1.3.24.

(v) $\Rightarrow$ (i) For any bounded $\mathbb{F}$-martingale X, the stopped process X^τ is a bounded (hence a uniformly integrable) $\mathbb{G}$-martingale. Then, as a consequence of the optional stopping theorem applied in $\mathbb{G}$ at time τ, we get $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$, hence, τ is an $\mathbb{F}$ pseudo-stopping time. □

Proposition 1.3.49 *If a pseudo time is honest, it is a stopping time.*

Proof. From Proposition 1.3.42 $\tau = \sup\{n : \widetilde{Z}_n = 1\}$ on $\{\tau < \infty\}$. The pseudo-stopping time property of τ implies $Z = 1 - H^o$. Moreover, $\widetilde{Z} - Z = \Delta H^o$. Then, on $\{\tau < \infty\}$ we obtain

$$\begin{aligned} \tau &= \sup\{n : \widetilde{Z}_n = 1\} = \sup\{n : Z_n + \Delta H^o_n = 1\} \\ &= \sup\{n : 1 - H^o_n + \Delta H^o_n = 1\} = \sup\{n : H^o_{n-1} = 0\} = \inf\{n : H^o_n > 0\}. \end{aligned}$$

So, τ equals an $\mathbb{F}$-stopping time on $\{\tau < \infty\}$. □

Obviously, pseudo-stopping times do not create arbitrages before τ. In continuous time, the links between pseudo-stopping times and immersion property are presented in [3], and it is proved that τ is a pseudo-stopping time if and only if $\widetilde{Z}$ is a càglàd decreasing process. In discrete time, we obtain a similar result, τ is a pseudo-stopping time if and only if $\widetilde{Z}$ is a predictable process (note that we do not require the deceasing assumption).

1.3.5 Martingale Representation in $\mathbb{G}$

Predictable representation

1.4 Other Enlargements

1.4.1 Enlargement with ζ, τ

A random variable ζ, valued in $\mathbb{Z}$ and a random time τ, valued in $\mathbb{N}$ are given. The filtration $\mathbb{G}$ is the smallest filtration with contains $\mathbb{F}$, makes τ a stopping time and ζ belongs to $\mathcal{G}_\tau$. We denote by $p_n(j,k) = \mathbb{P}(\tau = j, \zeta = k|\mathcal{F}_n)$. Let X be an $\mathbb{F}$-martingale, hence a $\mathbb{G}$-semimartingale with decomposition $X = V^{\mathbb{G}} + M^{\mathbb{G}}$ and

$$\Delta V_n^{\mathbb{G}} = \mathbb{E}(\Delta X_n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau\leq n-1\}}\mathbb{E}(\Delta X_n|\mathcal{G}_{n-1}) + \mathbb{1}_{\{n-1<\tau\}}\frac{1}{Z_{n-1}}\mathbb{E}(\Delta X_n\tilde{Z}_n|\mathcal{F}_{n-1})\,.$$

Before τ we can apply the results for progressive enlargement. We now compute the after τ part.

$$\mathbb{1}_{\{\tau\leq n-1\}}\mathbb{E}(\Delta X_n|\mathcal{G}_{n-1}) = \sum_{j=1,k=-\infty}^{j=n-1,k=\infty}\mathbb{1}_{\{\tau=j\}}\mathbb{1}_{\{\zeta=k\}}\mathbb{E}\left(\mathbb{1}_{\{\tau=j\}}\frac{1}{p_{n-1}(j,k)}\mathbb{1}_{\{\zeta=k\}}\Delta X_n|\mathcal{F}_{n-1}\right).$$

It follows that

$$X = X^{\mathbb{G}} + \mathbb{1}_{[0,\tau[}(Z_-)^{-1}\centerdot\langle X,\widetilde{M}\rangle^{\mathbb{F}} + \sum_{k=-\infty}^{k=\infty}\mathbb{1}_{[\tau,\infty]}\frac{1}{p_-(j,k)}\centerdot\langle X,p(j,k)\rangle^{\mathbb{F}}|_{k=\zeta,j=\tau}\,.$$

This decomposition is a mixed of Equation (1.45) and (1.11). The case of (1.11) corresponds to the particular case $\tau\equiv 0$ whereas (1.45) corresponds to $\zeta\equiv 0$

1.4.2 Enlargement with a process

Let Y be a process and consider the enlargement of $\mathbb{F}$ of the form $\mathcal{G}_n = \mathcal{F}_n \vee \sigma(Y_k, k = 0,\dots,n)$.[3]

We assume that X is a given $\mathbb{F}$-martingale. For $n\geq 0$ let $U_n(dy)$ be the regular conditional distribution of the random vector $\mathbf{Y_{n-1}} = (\mathbf{Y_0},\dots,\mathbf{Y_{n-1}})$ with respect to $\mathcal{F}_n$ and let $V_n(dy)$ be the regular conditional distribution of $\mathbf{Y_{n-1}}$ with respect to $\mathcal{F}_{n-1}$. In the following we will make the following assumption.

(A) $U_n(dy)$ is absolutely continuous w.r.t. $V_n(dy)$ for all $n\geq 1$.

If Condition (A) is satisfied, then we can define the density $d_n(y) = \frac{U_n(dy)}{V_n(dy)}$. We next show that we can express the information drift in terms of d.

Proposition 1.4.1 *Suppose that (A) holds true. Then the information drift of X w.r.t. to $\mathbb{G}$ is given by*

$$A_n = \sum_{k=1}^{n}\langle X, d(y)\rangle_k\big|_{y=(Y_0,\dots,Y_{k-1})}. \tag{1.50}$$

For the proof of Proposition (1.4.1) we need the following auxiliary result.

[3] The results of this section are based on some notes written by Stefan Ankirchner during his stay in Evry in september 2014.

Lemma 1.4.2 *Let $n \geq 1$ and $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}_{n-1}$-measurable non-negative map. Then*

$$f(\mathbf{Y_{n-1}}, \cdot) \quad = \quad \int f(y, \cdot) V_n(dy).$$

Proof. The lemma follows from a monotone class theorem. □

Moreover we have the following.

Lemma 1.4.3 *Let $n \geq 1$ and $\psi : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}_n$-measurable non-negative map. Then*

$$\mathbb{E}\left[\int \psi(y, \cdot)\, V_n(dy) \middle| \mathcal{F}_{n-1}\right] = \int \mathbb{E}\left[\psi(y, \cdot)|\mathcal{F}_{n-1}\right]\, V_n(dy). \tag{1.51}$$

Proof. We use a monotone class argument. Suppose first that $\psi(y, \omega) = \mathbb{1}_B(y)\mathbb{1}_C(\omega)$ with $B \in \mathcal{B}(\mathbb{R}^n)$ and $C \in \mathcal{F}_n$. Then we have

$$\begin{aligned}
& \mathbb{E}\left[\int \psi(y, \cdot)\, V_n(dy) \middle| \mathcal{F}_{n-1}\right] = \mathbb{E}\left[\int \mathbb{1}_B(y)\mathbb{1}_C(\omega)\, V_n(dy) \middle| \mathcal{F}_{n-1}\right] \\
= \; & \mathbb{E}\left[\mathbb{1}_C(\omega) V_n(B) | \mathcal{F}_{n-1}\right] = V_n(B)\mathbb{P}(C|\mathcal{F}_{n-1}) \\
= \; & \int \mathbb{1}_B(y)\mathbb{P}(C|\mathcal{F}_{n-1})\, V_n(dy) = \int \mathbb{E}\left[\psi(y, \cdot)|\mathcal{F}_{n-1}\right]\, V_n(dy).
\end{aligned}$$

The claim follows for arbitrary non-negative $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{F}_n$-measurable functions ψ via a monotone class theorem. □

Proof. [of Proposition 1.4.1] Let $A \in \mathcal{F}_{n-1}$ and $C \in \mathcal{B}(\mathbb{R}^n)$. Moreover, let $\psi(y, \omega) = \mathbb{1}_A(\omega)\mathbb{1}_C(y)\Delta X_n(\omega) d_n(y, \omega)$. Then Lemma 1.4.2 implies, with $f(y, \cdot) = \mathbb{E}[\psi(y, \cdot)|\mathcal{F}_{n-1}]$,

$$\begin{aligned}
& \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\mathbf{Y_{n-1}} \in \mathbf{C}\}} \mathbb{E}[\Delta X_n d_n(y)|\mathcal{F}_{n-1}]|_{y=\mathbf{Y_{n-1}}}] = \mathbb{E}[\mathbb{E}[\mathbb{1}_A \mathbb{1}_C(y) \Delta X_n d_n(y)|\mathcal{F}_{n-1}]|_{y=\mathbf{Y_{n-1}}}] \\
= \; & \mathbb{E}[\mathbb{E}[\psi(y, \cdot)|\mathcal{F}_{n-1}]|_{y=\bar{Y}_{n-1}}] = \mathbb{E}[\int \mathbb{E}[\psi(y, \cdot)|\mathcal{F}_{n-1}] V_n(dy)].
\end{aligned}$$

Now Lemma 1.4.3 further yields

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\mathbf{Y_{n-1}} \in \mathbf{C}\}} \mathbb{E}[\Delta X_n d_n(y)|\mathcal{F}_{n-1}]|_{y=\mathbf{Y_{n-1}}}] \quad &= \quad \mathbb{E}[\int \mathbb{1}_A \mathbb{1}_C(y) \Delta X_n U_n(dy)] \\
&= \quad \mathbb{E}[\mathbb{1}_A \Delta X_n \mathbb{P}(\{\mathbf{Y_{n-1}} \in \mathbf{C}\}|\mathcal{F}_\mathbf{n})] \\
&= \quad \mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\mathbf{Y_{n-1}} \in \mathbf{C}\}} \Delta X_n].
\end{aligned}$$

This shows $\mathbb{E}[\Delta X_n|\mathcal{G}_{n-1}] = \mathbb{E}[\Delta X_n d_n(y)|\mathcal{F}_{n-1}]|_{y=\mathbf{Y_{n-1}}}]$, and hence the result. □

1.5 Credit Risk

In the credit risk framework, one defines a random time τ to represent the default time. The information is such as τ is turned into a stopping time, therefore, the enlargement setting can be a useful tool. Then, one of the goal is to give the price of a claim of the form $\zeta \mathbb{1}_{\{\tau > T\}}$, given the information at hand.

1.5.1 Cox Model

The basic methodology (called the Cox model) consists of the following definitions. Let $\mathbb{F}$ be a given filtration and V an $\mathbb{F}$-adapted process. Assume that Θ is a positive random variable, independent of $\mathbb{F}$ and define

$$\tau := \inf\{n : V_n \geq \Theta\}$$

and $\mathbb{G}$ the progressive enlargement of $\mathbb{F}$ with τ. Then $\mathbb{P}(\tau > n|\mathcal{F}_n) = \mathbb{P}(V_n^* < \Theta|\mathcal{F}_n) = 1 - e^{-V_n^*}$, where $V_n^* = \sup_{k\leq n} V_k$. One of the tools is the intensity of τ, i.e., the predictable process $\Lambda^{\mathbb{G}}$ such that $H - \Lambda^{\mathbb{G}}$ is a $\mathbb{G}$ martingale (See Lemma 1.3.6). As we have seen, $\Lambda^{\mathbb{G}} = \Lambda^{\mathbb{F}}_{\cdot\wedge\tau}$ where $\Delta\Lambda^{\mathbb{F}}_n = \frac{\Delta A_n}{Z_{n-1}} 1\!\!1_{\{Z_{n-1}>0\}}$.

From now on, we assume that V is increasing (hence $Z_n = e^{-V_n}$ and Z is positive) and recall results of example 1.3.19.
If V is predictable, $Z_n = \mathcal{E}(-\Lambda^{\mathbb{F}})$ with $\Delta\Lambda^{\mathbb{F}}_n = 1 - e^{-\Delta V_n}$. Moreover, Z is predictable and assertions of Theorem 1.3.18 hold.
If V is not predictable,

$$\Delta\Lambda_n = 1 - \mathbb{E}(e^{-\Delta V_n}|\mathcal{F}_{n-1})$$

and $\Delta\Gamma_n = \frac{\Delta A_n}{Z_{n-1}} = 1 - e^{-\Delta V_n}$.

Let $\zeta \in \mathcal{F}_N$. Then, using that $Z = \mathcal{E}(-\Gamma)$ where Γ is defined in (1.26)

$$\mathbb{E}(\zeta 1\!\!1_{N<\tau}|\mathcal{G}_n) = 1\!\!1_{n<\tau}\frac{1}{\mathcal{E}(-\Gamma)_n}\mathbb{E}(\zeta\mathcal{E}(-\Gamma)_N|\mathcal{F}_n)$$

If V is predictable

$$\mathbb{E}(\zeta 1\!\!1_{N<\tau}|\mathcal{G}_n) = 1\!\!1_{n<\tau}\frac{1}{\mathcal{E}(-\Lambda)_n}\mathbb{E}(\zeta\mathcal{E}(-\Lambda)_N|\mathcal{F}_n)$$

If V is not predictable, $Z/\mathcal{E}(-\Lambda)$ is an $\mathbb{F}$-martingale. Let us denote $Z_n/\mathcal{E}(-\Lambda) = N^Z_n$, setting $\widehat{P} = N^Z\mathbb{P}$

$$\mathbb{E}(\zeta 1\!\!1_{N<\tau}|\mathcal{G}_n) = 1\!\!1_{n<\tau}\frac{1}{N^Z_n\mathcal{E}(-\Lambda)_n}\mathbb{E}(\zeta\mathcal{E}(-\Lambda)_N N^Z_N|\mathcal{F}_n) = 1\!\!1_{n<\tau}\frac{1}{\mathcal{E}(-\Lambda)_n}\widehat{\mathbb{E}}(\zeta\mathcal{E}(-\Lambda)_N|\mathcal{F}_n)$$

In the Cox model, one can assume that Θ is known. A first step is then to introduce $\mathbb{G} = \mathbb{F}^{(\sigma(\Theta)}$. Then, τ is a stopping time in $\mathbb{F}^{(\sigma(\Theta)}$. If V is predictable, τ is predictable as well. If V is not predictable, one can compute the process Λ such that $H-\Lambda$ is a martingale, by Doob's decomposition of H. We obtain

$$\Delta\Lambda_n = \mathbb{E}(\Delta H_n|\mathcal{G}_{n-1}) = \mathbb{P}(\tau = n|\mathcal{G}_{n-1}) = \mathbb{P}(V_n \geq \Theta > V_{n-1}|\mathcal{G}_{n-1}) = 1\!\!1_{\{\Theta>V_{n-1}\}}\mathbb{P}(V_n > x|\mathcal{F}_{n-1})|_{x=\Theta}$$

1.5.2 General model

As we have seen, $\Lambda^{\mathbb{G}} = \Lambda_{\cdot\wedge\tau}$ where $\Delta\Lambda^{\mathbb{F}}_n = \frac{\Delta A_n}{Z_{n-1}} 1\!\!1_{\{Z_{n-1}>0\}} = \mathbb{P}(\tau = n|\mathcal{F}_{n-1}, \tau > n-1) =: \lambda_n$ is the probability that the default occurs at time n, given the current information $\mathcal{F}_{n-1}$ and knowing that the default has not occurred at time $n-1$.

Proposition 1.5.1 $\mathbb{P}(\tau = n+1|\mathcal{F}_n, \tau > n) = 1 - \lambda_{n+1}$

Proof.

$$\mathbb{P}(\tau = n+1|\mathcal{F}_n, \tau > n)\frac{\mathbb{P}(\tau = n+1|\mathcal{F}_n)}{\mathbb{P}(\tau > n|\mathcal{F}_n)}1\!\!1_{Z_n>0} = \frac{M_n - A_{n+1}}{Z_n} = 1 - \frac{\Delta A_{n+1}}{Z_n}$$

□

For pricing the defaultable claim $\zeta 1\!\!1_{N<\tau}$

$$\mathbb{E}(\zeta 1\!\!1_{N<\tau}|\mathcal{G}_n) = 1\!\!1_{n<\tau}\frac{1}{Z_n}\mathbb{E}(XZ_N|\mathcal{F}_n)$$

1.5.3 Multidefaults

Let τ_1, τ_2 be two default and consider the progressive enlargement of $\mathbb{F}$ with the pair τ_1, τ_2, i.e. $\mathcal{G}_n = \mathcal{F}_n \vee \sigma(\tau_1 \wedge n) \vee \sigma(\tau_2 \wedge n)$. In a first step, we compute the $\mathbb{G}$ predictable process Λ^1 such that $H^1 - \Lambda^1$ is a $\mathbb{G}$ martingale. From Doob's decomposition $\Delta\Lambda^1_n = \mathbb{P}(\tau_1 = n|\mathcal{G}_{n-1})$. We denote by $\mathbb{G}^2$ the progressive enlargement of $\mathbb{F}$ with τ_2

$$\mathbb{P}(\tau_1 = n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau_1 > n-1\}} \frac{\mathbb{P}(\tau_1 = n|\mathcal{G}^2_{n-1})}{\mathbb{P}(\tau_1 > n-1|\mathcal{G}^2_{n-1})}$$

$$\mathbb{P}(\tau_1 = n|\mathcal{G}^2_{n-1}) = \mathbb{1}_{\{\tau_2 > n-1\}} \frac{\mathbb{P}(\tau_1 = n,\, \tau_2 > n-1|\mathcal{F}_{n-1})}{\mathbb{P}(\,\tau_2 > n-1|\mathcal{F}_{n-1})} + \sum_{k=0}^{n-1} \mathbb{1}_{\{\tau_2 = k\}} \frac{\mathbb{P}(\tau_1 = n,\, \tau_2 = k|\mathcal{F}_{n-1})}{\mathbb{P}(\tau_2 = k|\mathcal{F}_{n-1})}$$

We set $p_n(i,j) = \mathbb{P}(\tau_1 = i, \tau_2 = j|\mathcal{F}_n)$ and $p^{(2)}_n(k) = \mathbb{P}(\tau_2 = k|\mathcal{F}_n)$

$$\mathbb{P}(\tau_1 = n|\mathcal{G}^2_{n-1}) = \mathbb{1}_{\{\tau_2 > n-1\}} \frac{\sum_{k=n}^\infty p_{n-1}(n,k)}{\sum_{k=n}^\infty p^{(2)}_{n-1}(k)} + \sum_{k=0}^{n-1} \mathbb{1}_{\{\tau_2 = k\}} \frac{p_{n-1}(n,k)}{p^{(2)}_{n-1}(k)}$$

The same computations yield to

$$\mathbb{P}(\tau_1 > n-1|\mathcal{G}^2_{n-1}) = \mathbb{1}_{\{\tau_2 > n-1\}} \frac{\sum_{k=n}^\infty \sum_{i=n}^\infty p_{n-1}(i,k)}{\sum_{k=n}^\infty p^{(2)}_{n-1}(k)} + \sum_{k=0}^{n-1} \mathbb{1}_{\{\tau_2 = k\}} \frac{\sum_{i=n-1}^\infty p_{n-1}(i,k)}{p^{(2)}_{n-1}(k)}$$

Finally

$$\mathbb{P}(\tau_1 = n|\mathcal{G}_{n-1}) = \mathbb{1}_{\{\tau_2 > n-1\}} \frac{\sum_{k=n}^\infty p_{n-1}(n,k)}{\sum_{k=n}^\infty \sum_{i=n}^\infty p_{n-1}(i,k)} + \sum_{k=0}^{n-1} \mathbb{1}_{\{\tau_2 = k\}} \frac{p_{n-1}(n,k)}{\sum_{i=n-1}^\infty p_{n-1}(i,k)}$$

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Index