Lectures on
Interplay of Schramm-Loewner evolution curves with conformal field theory

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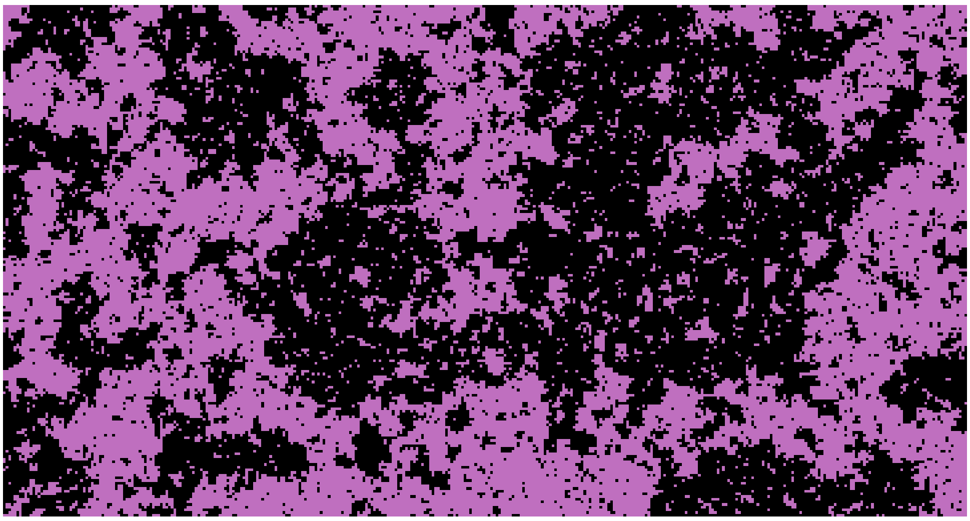
Abstract

In planar random geometry, a plethora of conformally invariant objects has emerged in the recent decades. Among these, particularly fruitful have been random fractal curves derived from one-dimensional Brownian motion: Schramm-Loewner Evolutions (SLE), Conformal Loop Ensembles (CLE), and their variants. Originally, they were introduced in the context of critical models in statistical physics to understand conformal invariance and critical phenomena upon taking the scaling limit. Indeed, not only do these objects describe critical interfaces in such models, but they also carry a deep connection to conformal field theory (CFT) — quantum field theory with conformal symmetry, conjecturally describing the full scaling limit of critical models. In these lectures, I will introduce models for conformally invariant random SLE paths, discuss their relation to critical models, CFT, and its algebraic content. (Note that these lectures reflect my personal perspective on this topic.)



A configuration of the critical Ising model.

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Contents

1 First Lecture: SLE and critical 2D interfaces

In this lecture, we introduce Schramm-Loewner evolutions (SLE) and describe how they are connected to planar lattice models in statistical physics and to conformal invariance. We focus on the case of planar domains with boundary and consider chordal interfaces, neglecting many (also interesting) phenomena in the bulk (e.g. radial and whole-plane variants of SLE). One of the first celebrated applications of SLE was the rigorous calculation of critical exponents [LSW01a, LSW01b, SW01, LSW02], in agreement with the earlier predictions in the physics literature [dN83, BPZ84a, BPZ84b, Car84, DF84, DS87, Nie87].

Let us here focus rather on the geometry of planar lattice models, of which critical exponents can also be viewed as a special case — SLEs describe interfaces, or domain walls, of critical planar lattice models in the scaling limit (i.e, as the lattice mesh tends to zero). In general, these models are believed to be described by conformally invariant quantum field theories — conformal field theory (CFT) — in the continuum. However, mathematical understanding of such a statement still remains to a large extent unclear, and is one of the major challenges in modern mathematical physics. On the other hand, martingale observables for SLE curves are closely related to certain (quite special) correlation functions in CFT.

1.1 Interfaces in critical lattice models

For concreteness let us consider the spin Ising model, which describes a magnet with a paramagnetic (disordered) and a ferromagnetic (ordered) phase. See, e.g., the lecture notes [DCS12] by Duminil-Copin & Smirnov for a detailed account.

On a finite graph $G = (V, E)$ with vertices V and edges E, a configuration in the Ising model consists of an assignment $\sigma : V \to \{\pm 1\}$ of spins $\sigma_x \in \{\pm 1\}$ to each vertex $x \in V$. The probability of a configuration σ is given by the *Boltzmann distribution* (the canonical ensemble)

$$\mathbb{P}[\sigma] = \frac{e^{-\beta H(\sigma)}}{Z}, \qquad H(\sigma) = -\sum_{(x,y)\in E} \sigma_x \sigma_y, \qquad Z = \sum_{\sigma} e^{-\beta H(\sigma)},$$

where $\beta = \frac{1}{T} > 0$ is the inverse-temperature and $H(\sigma)$ is the Hamiltonian. The Boltzmann distribution favors configurations where the neighboring spins are aligned. The behavior of the system is also highly dependent on the temperature: there is an order-disorder phase transition at a unique critical temperature $\beta_c = \frac{1}{T_c} \in (0, \infty)$. At the critical temperature, the scaling limit of the Ising model is believed (and in may ways proved) to become conformally invariant in the scaling limit (e.g., its interfaces and correlation functions converge to conformally invariant or covariant quantities [HS13, CHI15, CDCH+14, Izy17, BPW21]).

To study the geometry of the critical Ising model, one can study *interfaces* between "+" spins and "−" spins. Some of these interfaces are macroscopic, so they survive in the scaling limit. For example, one can force the system to have a macroscopic interface via imposing boundary conditions: take $G = D^\delta = D \cap \delta\mathbb{Z}^2$ for some bounded simply connected domain $D \subsetneq \mathbb{C}$, split the boundary $\partial D = \partial^+ \sqcup \partial^-$ into two segments ∂^+ and ∂^-, and consider the Ising model with the constraint that the vertices in ∂^+ all equal $+1$ and the vertices in ∂^- all equal -1. (See Figure 1.1(left).) This setup is called *Dobrushin boundary conditions.*

Then for topological reasons, there must exist a macroscopic path traversing between "+" and "−" spins and connecting the two boundary points where the segments ∂^+ and ∂^- touch. (More generally, one could consider alternating boundary conditions with more "+" and "−" segments on the boundary. In that case, there are several macroscopic boundary-to-boundary interfaces as in Figure 1.1(right).) At the *critical temperature* $T = T_c$, the interfaces have interesting self-similar behavior. Indeed, in the scaling limit $\delta \to 0$, such interfaces have been proven to converge to random conformally invariant curves, called Schramm-Loewner evolution (SLE(3))

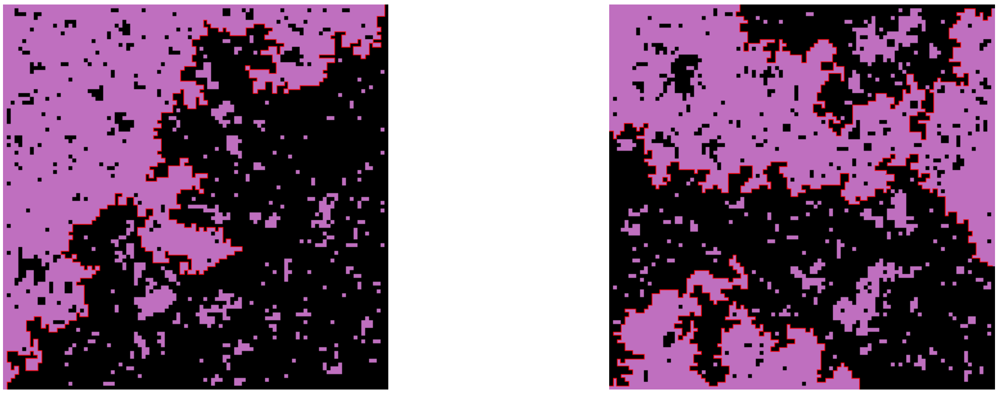

Figure 1.1: Critical Ising model configurations on a square lattice with alternating boundary conditions (that is, some boundary segments have spins equal to +1 and the other segments have spins equal to −1). Interfaces connecting boundary points are highlighted. (Figure from [Pel19].)

curves [CDCH+14, Izy17, BPW21]. Also, loops (domain walls) in the interior of the domain separating "+" spins and "−" spins have been shown to converge to the so-called conformal loop ensemble (CLE(3)) [BH19].

To motivate how one could describe scaling limits of critical Ising interfaces, there are a few natural properties that the limit should satisfy. In addition to *conformal invariance*, one would expect a Markovian property (which holds for many lattice models with local interactions) in the following sense.

Consider the Ising model on G as above, with its boundary divided into the two segments ∂^+ and ∂^-. An exploration process on G, started from one of the boundary points where the segments ∂^+ and ∂^- touch and ending at the other such boundary point, is defined by following the interface between the opposite spins step by step[1]. Let $\gamma(k)$, for $k = 0, 1, \ldots, n$, denote this exploration process (in discrete time). Explore it up to some time k_0. Consider the exploration process $\tilde{\gamma}$ for the model on the smaller grid $\tilde{G} = G \smallsetminus \gamma[0, k_0]$, started from the tip $\gamma(k_0)$, where the boundary conditions are taken as before on ∂G and naturally continued to both sides of the segment $\gamma[0, k_0]$ of $\partial\tilde{G}$. Then, the distribution of the exploration process $\tilde{\gamma}$ associated to the model on the grid $\tilde{G}$ equals the conditional law of the original process γ on the original graph G given the initial segment $\gamma[0, k_0]$. This is called the *(domain) Markov property*.

1.2 Schramm-Loewner evolution, SLE(κ)

The Schramm-Loewner evolutions, originally called "stochastic" Loewner evolutions, were introduced at the turn of the millennium by O. Schramm [Sch00], who argued that they are the only possible random curves that could describe scaling limits of critical lattice interfaces in two-dimensional systems. Schramm's definition was inspired by the classical theory of C. Loewner [Loe23] for dynamical description of the growth of hulls, encoded in conformal maps. Schramm's revolutionary input was that such maps could also be *random*. Aiming at the construction of scaling limits of critical lattice interfaces, the law of the SLE curve should be manifestly conformally invariant. Schramm observed in [Sch00] that when requiring in addition the *domain Markov property* for the growth of the curve (analogously to the discrete interfaces), there is, in fact, only a one-parameter family of such random curves, that he labeled by $\kappa \geq 0$ and called the SLE(κ). Physically, κ describes the *universality class* of the corresponding critical model, or equivalently, the *central charge* of the corresponding conformal field

[1] A careful reader may notice a caveat (that becomes clear by drawing a small figure on the square grid): on the square grid, one might encounter an indetermination for the exploration step, which can be resolved by picking a preferred choice for the direction of each exploration step.

theory [Car96, Car05]. Mathematically, κ is the "speed" of the Brownian motion associated to the growth of the SLE(κ) curve. Also, for example the Hausdorff dimension of the SLE(κ) curves is a function of κ.

Definition 1.1 (SLE(κ) random curve)**.** For $\kappa \geq 0$, the (chordal) Schramm-Loewner evolution SLE(κ) is a family of probability measures $\mathbb{P}_{D;x,y}$ on curves, indexed by simply connected domains $D \subsetneq \mathbb{C}$ with two distinct boundary points $x, y \in \partial D$. Each measure $\mathbb{P}_{D;x,y}$ is supported on continuous unparameterized curves in $\overline{D}$ from x to y. This family is uniquely determined by the following two properties:

- ▷ ***Conformal invariance***: Fix two simply connected domains $D, D' \subsetneq \mathbb{C}$ and boundary points $x, y \in \partial D$ and $x', y' \in \partial D'$, with $x \neq y$ and $x' \neq y'$. According to the Riemann mapping theorem[2], there exists a conformal bijection $f : D \to D'$ such that $f(x) = x'$ and $f(y) = y'$. With any choice of such a map, we have $f(\eta) \sim \mathbb{P}_{D';x',y'}$ if $\eta \sim \mathbb{P}_{D;x,y}$.
- ▷ ***Domain Markov property***: Given an initial segment $\eta[0,\tau]$ of the SLE(κ) curve $\eta \sim \mathbb{P}_{D;x,y}$ up to a stopping time τ (parameterizing η by $[0,\infty)$, say), the conditional law of the remaining piece $\eta[\tau,\infty)$ is the law $\mathbb{P}_{D_\tau;\eta(\tau),y}$ of the SLE(κ) from the tip $\eta(\tau)$ to y in the component D_τ of the complement $D \setminus \eta[0,\tau]$ of the initial segment containing the target point y on its boundary.

The existence and uniqueness of SLE(κ) was proved by Schramm [Sch00] — see also the relatively recent book by Kemppainen [Kem17] for basics on SLE, Loewner theory, and the necessary background in stochastic analysis and complex analysis.

Remark 1.2. Using properties of Bessel processes, one can show [RS05] that

- ▷ when $0 \leq \kappa \leq 4$, the SLE(κ) curve is simple;
- ▷ when $4 < \kappa < 8$, the SLE(κ) curve is not simple, nor space-filling;
- ▷ when $\kappa \geq 8$, the SLE(κ) curve is space-filling.

Definition 1.3 (Curve space)**.** For convergence of curves, an appropriate topological space is the (Polish: complete separable metric) space $\mathcal{X}(D; x, y)$ of all unparameterized curves in $\overline{D}$ from x to y with metric

$$d_{\mathcal{X}}(\eta, \tilde{\eta}) := \inf_{\psi,\tilde{\psi}} \sup_{t\in[0,1]} |\eta(\psi(t)) - \tilde{\eta}(\tilde{\psi}(t))|,$$

where $\eta, \tilde{\eta} : [0,1] \to \overline{D}$ are representatives of curves, and the infimum is taken over all reparameterizations, that is, increasing bijections $\psi, \tilde{\psi} : [0,1] \to [0,1]$.

Explicitly, SLE(κ) curves can be generated using random Loewner evolutions. Thanks to its conformal invariance, it suffices to construct the SLE(κ) curve $\eta \sim \mathbb{P}_{\mathbb{H};0,\infty}$ in the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ from 0 to ∞. In its construction as a growth process, the time evolution of η is encoded in a solution of the Loewner differential equation: a collection $(g_t)_{t\geq 0}$ of conformal maps $z \mapsto g_t(z)$. Such maps were first considered by C. Loewner in the 1920s while studying the Bieberbach conjecture [Loe23]. He managed to describe certain growth processes by a single ordinary differential equation, now known as the Loewner equation. In the upper half-plane $\mathbb{H} \ni z$, it has the form

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z \tag{LE}$$

[2]This is a basic result in complex analysis (a special case of uniformization), see for example [Ahl79].

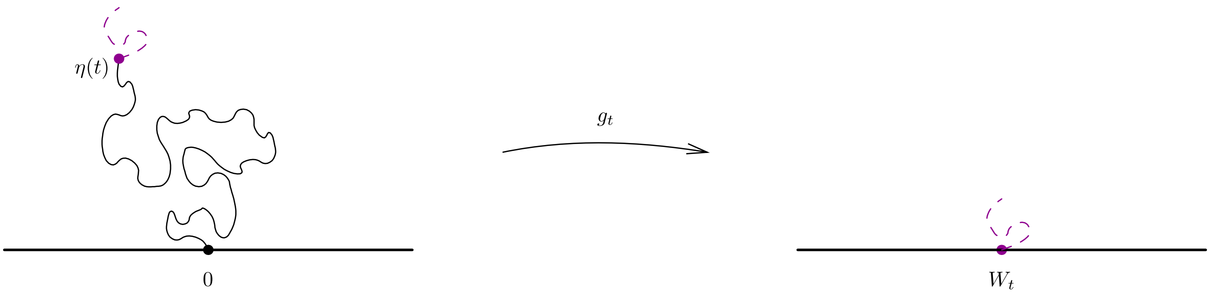

Figure 1.2: Illustration of the Loewner maps $g_t : \mathsf{H}_t \to \mathbb{H}$ for the SLE(κ) curve η, where H_t is the unbounded component of the curve's complement $\mathbb{H} \smallsetminus \eta[0,t]$ at time t. The image of the tip $\eta(t)$ of the SLE(κ) curve is the driving process $W_t = \sqrt{\kappa} B_t$. (Figure from [Pel19].)

where $t \mapsto W_t$ is a real-valued continuous function, called the *driving function*. Note that, for each $z \in \mathbb{H}$, this equation is only well-defined up to a blow-up time, called the *swallowing time* of z,

$$T_z := \sup \Big\{ t > 0 \;\Big|\; \inf_{s \in [0,t]} |g_s(z) - W_s| > 0 \Big\}.$$

The hulls $K_t := \overline{\{z \in \mathbb{H} \mid T_z \le t\}}$, for $t \ge 0$, define a growth process, called a *Loewner chain*. For each $t \in [0, T_z)$, the map $z \mapsto g_t(z)$ is the unique conformal bijection from $\mathsf{H}_t := \mathbb{H} \smallsetminus K_t$ onto $\mathbb{H}$ with normalization chosen as $\lim_{z\to\infty} |g_t(z) - z| = 0$. Figure 1.2 illustrates the Loewner chain associated to the SLE(κ) process.

Note that the growing hulls $(K_t)_{t\ge0}$ do not necessarily form a continuous curve. It is in fact highly non-trivial to show this property for the SLE(κ) process, see [RS05].

Definition 1.4. We say that growing sets $(K_t)_{t\ge0}$ in $\overline{\mathbb{H}}$ are *generated by a curve* if there exists a (continuous) curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ such that for each time $t \ge 0$, the set $\mathbb{H} \smallsetminus K_t$ is the unbounded connected component of $\mathbb{H} \smallsetminus \gamma[0,t]$.

To guarantee the two properties in Definition 1.1, the driving process W has to be a multiple of Brownian motion (plus possibly a drift). Indeed, we expect that the following properties hold:

1. $t \mapsto W_t$ is continuous (which we obviously expect),
2. the increments $W_{t+s} - W_t$ are independent (by the domain Markov property) and stationary (only depend on s),
3. and there is an additional symmetry in law $W \leftrightarrow -W$.

A result in stochastic analysis shows that a process with properties 1 & 2 must have the form [Kem17, Theorem 2.1]

$$W_t = \sqrt{\kappa} B_t + \alpha t,$$

where B is a one-dimensional Brownian motion (see Definition 1.5), $\kappa \ge 0$, and $\alpha \in \mathbb{R}$. Property 3 then implies that $\alpha = 0$. For more general variants of SLE(κ), the drift α can be a (nice enough) function.

Definition 1.5 (Brownian motion)**.** Recall (e.g. from [MP10]) that one-dimensional *Brownian motion* started at a point $B_0 = x \in \mathbb{R}$ is a continuous-time real-valued stochastic process $B = (B_t)_{t\ge0}$ satisfying the following properties (that determine it uniquely):

- ▷ (Independent increments): For any partition $0 \le t_0 < t_1 < t_2 < \cdots < t_n$, the increments $\{B_{t_{j+1}} - B_{t_j} \mid j = 0, 1, \ldots, n-1\}$ are independent random variables.

▷ (Stationary, Gaussian increments): For each $0 \le s < t$, the increment $B_t - B_s$ has the Gaussian distribution: $B_t - B_s \sim N(0, t-s)$, that only depends on the time difference.

▷ (Continuous sample paths): The map $t \mapsto B_t$ is continuous almost surely.

Theorem 1.6 (Brownian motion generates SLE curve)**.** *The growing sets $(K_t)_{t\ge 0}$ obtained from solving the Loewner equation* (LE) *with $W_t = \sqrt{\kappa} B_t$ are almost surely generated by a curve.*

Proof idea. For $\kappa \neq 8$, this was proven by Rohde & Schramm [RS05] by an elaborate argument relying on estimates for the derivative of the inverse conformal map g_t^{-1} near the driving function W_t. This estimate breaks down when $\kappa = 8$, but the result still holds (this is just a limitation of the proof). Lawler, Schramm & Werner proved that the case $\kappa = 8$ gives the scaling limit of the Peano curve for the uniform spanning tree, and the proof in particular implies that the limiting object is a curve. To date, there is no direct analytical proof[3] for the case of $\kappa = 8$. □

For more background on SLEs and related topics, see, e.g., the books [Law05, Kem17] and the original papers [Sch00, RS05].

1.3 Interface convergence

Theorem 1.7 (Scaling limit of Ising interface is SLE(3))**.** *Let D be a simply connected domain. Consider the critical Ising interface γ^δ in $D^\delta = D \cap \delta\mathbb{Z}^2$ started from x^δ and ending at $y^\delta \in \partial D^\delta$, where $x^\delta \to x \in D$ and $y^\delta \to y \in \partial D$ as $\delta \to 0$. Then, $\gamma^\delta \to \gamma$ in distribution (weakly as probability measures on the curve space $\mathcal{X}(D; x, y)$), where γ has the law of the chordal SLE(3) in $\overline{D}$ from x to y.*

Proof idea. The proof (summarized in [CDCH⁺14]) has two main steps:

1. First, one proves that the sequence $(\gamma^\delta)_{\delta>0}$ of lattice interfaces on D^δ is *relatively compact* in the space $\mathcal{X}(D; x, y)$ of curves. Thus, one deduces that there exist convergent subsequences as $\delta \to 0$. For the Ising model, the relative compactness is established using topological crossing estimates (a priori estimates ruling out pathological behavior of the curves), see in particular [KS17].

2. Second, one has to prove that all of the subsequences in fact converge to a *unique* limit, identified as the chordal SLE(κ) with $\kappa = 3$. For the identification of the limit, Smirnov used[4] a discrete holomorphic martingale observable [Smi06, Smi10], that is, a solution to a discrete boundary value problem on D^δ, converging as $\delta \to 0$ to the solution of the corresponding boundary value problem on D. Using the martingale observable, he identified the Loewner driving function of the scaling limit curve as $\sqrt{3} B_t$. □

(For the critical Ising model, a similar result holds for a quite general collection of graphs by techniques developed in particular by D. Chelkak, see [CS11, Che20].)

For multiple curves, the relative compactness follows from the one-curve case [Kar19, Wu20]. For the identification, one can use either a multipoint discrete holomorphic observable cf. [Izy15, Izy17], or the classification of multiple SLE probability measures, cf. [BPW21].

[3] There is a recent proof relying on the Gaussian free field [KMS21].

[4] Such an idea was also implemented by others (especially Schramm, Kenyon, and Chelkak).

Remark 1.8. A similar strategy to address scaling limits of interfaces in other critical planar lattice models has been carried out in some cases, but both Steps 1 and 2 in this strategy require some model-specific tools. For example, the precompactness is known for a wide class of bond-percolation models (known as random-cluster models) [DCST17, Theorem 6] and [DCMT21, Section 1.4], while the identification step only for a special case of these models (also called FK-Ising model [CDCH⁺14]).

Interestingly, for one of the very first examples, critical Bernoulli site-percolation[5], a version of Theorem 1.7 only has been proven rigorously for the case of the triangular lattice in Smirnov's first celebrated work in this area [Smi01]. All other reasonable setups are believed to have the same limit, but the identification step 2 is still missing:

Conjecture 1.9 (Scaling limit of percolation interface is SLE(6))**.** *Consider critical Bernoulli site- or bond-percolation with Dobrushin boundary conditions on a discrete approximation of a simply connected domain D. The interface γ^δ between the two boundary points from x^δ to y^δ where the boundary conditions change converges in distribution: $\gamma^\delta \to \gamma$ (weakly as probability measures on the curve space $\mathcal{X}(D; x, y)$), where where γ has the law of the chordal SLE(6) in $\overline{D}$ from x to y.*

[5]That is, color vertices of a graph black (with probability p_c) or white (with probability $1-p_c$) all independently, where $p_c \in (0,1)$ is a well-chosen "critical" probability.

2 Second Lecture: CFT à la BPZ, and Virasoro algebra

Next, we briefly describe some aspects of 2D conformal field theory (CFT). There are many textbooks on CFT from different viewpoints, see, e.g, [DFMS97, Sch08, Mus10]. Here, we aim to only give some rough ideas, in order to motivate the connection of SLEs with CFT and to illustrate how it could be understood. The philosophy is that from the CFT heuristics, we can *predict* some aspects that one could *expect* the scaling limit objects to have, which we can then *use as a guiding principle* for mathematically rigorous investigations. (Already for the definition of SLE, the conformal invariance property was really just a guess based on physics heuristics.)

Let us emphasize that in CFT, the *fields* themselves might not be analytically well-defined objects, but nevertheless, their *correlation functions* are well-defined functions of several complex variables. As a concrete example, the so-called "Liouville CFT" was recently constructed completely rigorously [DKRV16, KRV20], while for some objects in the critical Ising model (that we discussed above), the CFT description is not completely clear. For the Ising spin (magnetization) field $\sigma^\delta : V^\delta \to \{\pm 1\}$, that is a random variable on the vertices, it is known that when suitably renormalized (by a power of δ), in the scaling limit $\delta \to 0$ the function σ^δ does converge — but not as a random function, rather, as a random distribution [CGN15]. However, no such convergence is expected for the the energy field, defined via the interaction of spins across each edge: $\varepsilon^\delta(x,y) := \sigma^\delta(x)\sigma^\delta(y)$, for $(x,y) \in E^\delta$. Note that a scaling limit of $\varepsilon^\delta : E^\delta \to \{\pm 1\}$ should morally be a product of two random distributions — but how does one multiply distributions? The situation is even more unclear for the CFT corresponding to SLE curves.

2.1 Correlation functions in CFT

Thus, we will be mainly interested in correlation functions in CFT, which describe — in some sense — the physical observables in the models of interest. Correlation functions are analytic (multi-valued) functions $F : \mathfrak{W}_n \to \mathbb{C}$ (also called n-point functions) defined on the configuration space

$$\mathfrak{W}_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}. \tag{2.1}$$

Physicists speak of correlation functions as "vacuum expectation values" of fields $\Phi_{\iota_i}(z_i)$ and denote them by

$$F_{\iota_1,\ldots,\iota_n}(z_1, \ldots, z_n) = \big\langle \Phi_{\iota_1}(z_1) \cdots \Phi_{\iota_n}(z_n) \big\rangle. \tag{2.2}$$

Because of the conformal symmetry, the correlation functions are assumed to be covariant under (global) conformal transformations. In a CFT on the full Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, this means that under all Möbius transformations[6] $f \in \mathrm{PSL}(2,\mathbb{C})$, we have

$$F_{\iota_1,\ldots,\iota_n}(z_1, \ldots, z_n) = \prod_{i=1}^{n} |f'(z_i)|^{\Delta_{\iota_i}} \times F_{\iota_1,\ldots,\iota_n}(f(z_1), \ldots, f(z_n)), \tag{2.3}$$

with some *conformal weights* $\Delta_{\iota_i} \in \mathbb{R}$ associated to the fields Φ_{ι_i}. In these notes, we are mainly concerned with so-called "primary fields", to which we return later — they can be thought of as building blocks for all other fields in the CFT.

Upshot 2.1. As input, we consider a formal collection of objects, "fields" $\Phi_\iota(z)$ for $z \in \hat{\mathbb{C}}$ and indexed by some collection $I \ni \iota$ of indices. Each field comes with a number $\Delta_\iota \in \mathbb{R}$ postulated to give its scaling behavior:

$$\big\langle \Phi_\iota(z) \big\rangle = \lambda^{\Delta_\iota} \big\langle \Phi_\iota(\lambda z) \big\rangle, \qquad \lambda > 0,$$

upgraded to the above type of conformal covariance (2.3) assuming the postulates in physics.

[6] Of specific interest to us will be CFT in the domain $\mathbb{H}$ with boundary $\partial\mathbb{H} = \mathbb{R}$, where the global conformal transformations are also Möbius maps, $f \in \mathrm{PSL}(2,\mathbb{R})$.

Notably, global conformal invariance only results in *finitely* many (three) constraints for the physical system. However, A. Belavin, A. Polyakov, and A. Zamolodchikov (BPZ) observed in the 1980s that, in two dimensions, imposing *local* conformal invariance yields *infinitely many independent symmetries* [BPZ84a, BPZ84b]. This was originally argued in the physics level of rigor, but it has now been completely rigorously verified for the Liouville CFT [KRV19]. On $\hat{\mathbb{C}}$, the local conformal transformations are just the locally invertible holomorphic and anti-holomorphic maps — see, e.g., [Sch08, Chapters 1,2,5] for details.

2.2 Conformal symmetry and Virasoro algebra

Roughly speaking, in CFT à la BPZ, one regards the local conformal invariance as invariance under infinitesimal transformations (or vector fields which generate the local conformal mappings): for instance, the infinitesimal holomorphic[7] transformations are written as Laurent series,

$$z \mapsto z + \sum_{n \in \mathbb{Z}} a_n z^n,$$

which can be seen to be generated by the vector fields

$$\ell_n := -z^{n+1} \frac{\partial}{\partial z}, \qquad n \in \mathbb{Z},$$

constituting a Lie algebra isomorphic to the *Witt algebra* $\mathfrak{Witt}$ with commutation relations

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}.$$

In quantized systems, the symmetry groups and algebras often are *central extensions* of their classical counterparts. In particular, in conformally invariant quantum field theory (i.e., CFT), the conformal symmetry algebra is the unique central extension[8] of the Witt algebra by the one-dimensional abelian Lie algebra $\mathbb{C}$, namely the *Virasoro algebra* $\mathfrak{Vir}$.

Definition 2.2 (Virasoro algebra)**.** $\mathfrak{Vir}$ is the infinite-dimensional Lie algebra generated by L_n, for $n \in \mathbb{Z}$, together with a central element C, with commutation relations

$$\begin{cases} [\mathrm{L}_n, \mathrm{L}_m] = (n-m)\mathrm{L}_{n+m} + \frac{1}{12} n(n^2-1)\delta_{n,-m}\mathrm{C}, & \text{for } n, m \in \mathbb{Z}, \\ [\mathrm{L}_n, \mathrm{C}] = 0. \end{cases} \tag{2.4}$$

We will use the same notation[9] $\mathfrak{Vir}$ also for the universal enveloping algebra of the Virasoro algebra, i.e., the associative algebra obtained by taking the quotient of polynomials in the generators of $\mathfrak{Vir}$ modulo the relation $[X, Y] = XY - YX$.

Algebraically, the basic objects in a CFT, the conformal fields, can be regarded as elements in *representations* of the symmetry algebra $\mathfrak{Vir}$, where the central element acts as a constant multiple of the identity, $\mathrm{C} = c\,\mathrm{id}$. The number $c \in \mathbb{C}$ is called the *central charge* of the CFT. For relation to SLEs and statistical physics that we will be discussing in these notes, real central charges $c \leq 1$ are relevant: we use the parameterization

$$c(\kappa) = \frac{(3\kappa - 8)(6-\kappa)}{2\kappa} \quad \leq \quad 1, \qquad\qquad \kappa > 0.$$

[7] We will ignore the anti-holomorphic sector for this discussion.

[8] The central part of $\mathfrak{Vir}$ represents a *conformal anomaly*, giving rise to a projective representation of $\mathfrak{Witt}$ — see, e.g. [Sch08, Sections 3-4] for details.

[9] Because there is a one-to-one correspondence between the representations of a Lie algebra and its universal enveloping algebra, we do not have to distinguish between them here.

Definition 2.3 (Representation, module, action)**.** Let $\mathfrak{g}$ be a Lie algebra and V a vector space. A pair (ρ, V) is called a *representation* of $\mathfrak{g}$ if

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

is a Lie algebra homomorphism from $\mathfrak{g}$ to the space $\mathfrak{gl}(V)$ of all linear operators on V.

- ▷ The vector space V is called a $\mathfrak{g}$-*module.*
- ▷ The $\mathfrak{g}$-*module* V is called *simple*, or *irreducible* if it has no non-trivial submodules: that is, if $W \subset V$ is a $\mathfrak{g}$-submodule, then either $W = V$ or $W = \{0\}$.

The assignment $\rho(a) : V \to V$ for each $a \in \mathfrak{g}$ is also called a $\mathfrak{g}$-*action* on V.

2.3 Primary fields

Primary fields are fields whose correlation functions also have a covariance property also under *local* conformal transformations, in an "infinitesimal" sense, see [Sch08, Chapter 9]. Other fields[10] are called descendant fields, obtained from the primary fields by action of the Virasoro algebra.

The rough idea is the following. A primary field $\Phi_\iota(z)$ of conformal weight Δ_ι generates a *highest-weight module* $\mathsf{V}_{c,\Delta_\iota}$ of the Virasoro algebra of *weight* Δ_ι and *central charge* c. In physics, it is called the *conformal family* of $\Phi_\iota(z)$, consisting of linear combinations of the *descendant fields* of $\Phi_\iota(z)$. The latter are obtained from $\Phi_\iota(z)$ via action of the Virasoro algebra. Here the space-time point $z \in \hat{\mathbb{C}}$ plays no role yet. Algebraically, we could identify $\Phi_\iota(z)$ with a highest-weight vector as in the following Definition 2.4:

$$\Phi_\iota(z) \text{ with weight } \Delta_\iota \qquad \Longleftrightarrow \qquad v_{c,h} \text{ with weight } h = \Delta_\iota.$$

We use the algebraic notation from the right-hand side when discussing representations of $\mathfrak{Vir}$, and the analytical notation from the left-hand side when discussing fields in a CFT. (Recall that these "fields" might not themselves be well-defined objects, but their correlation functions are.)

Definition 2.4 (Highest-weight module)**.** A $\mathfrak{Vir}$-module V is a *highest-weight module* if

$$V = \mathfrak{Vir}\, v_{c,h},$$

where $v_{c,h} \in V$ is a *highest-weight vector* of weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$, that is, a vector satisfying

$$\mathrm{L}_0 v_{c,h} = h v_{c,h}, \qquad \mathrm{L}_n v_{c,h} = 0, \qquad \text{for } n \geq 1, \qquad \text{and} \qquad \mathrm{C} v_{c,h} = c v_{c,h}.$$

Definition 2.5 (Verma module)**.** In particular, for any pair (c, h), there exists a unique (up to isomorphism) *Verma module*

$$\mathrm{M}_{c,h} = \mathfrak{Vir}/\mathrm{I}_{c,h}$$

where $\mathrm{I}_{c,h}$ is the left ideal generated by the elements $\mathrm{L}_0 - h1$, $\mathrm{C} - c1$, and L_n, for $n \geq 1$.

The Verma module $\mathrm{M}_{c,h}$ is a highest-weight module generated by a highest-weight vector $v_{c,h}$ of weight h and central charge c (given by the equivalence class of the unit 1). It has a Poincaré-Birkhoff-Witt type basis

$$\left\{ \mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k} v_{c,h} \mid n_1 \geq \cdots \geq n_k > 0,\ k \in \mathbb{Z}_{\geq 0} \right\}$$

given by the action of the Virasoro generators with negative index, ordered by applying the commutation relations (2.4). This is very analogous to the theory of classical Lie algebras, but note that $\mathfrak{Vir}$ is infinite-dimensional.

[10]There is also the special field called stress-energy tensor, that we won't discuss here.

Lemma 2.6. *The Verma modules* $\mathrm{M}_{c,h}$ *are* universal *in the sense that if* V *is any* $\mathfrak{Vir}$*-module containing a highest-weight vector* v *of weight* h *and central charge* c*, then there exists a canonical homomorphism* $\varphi : \mathrm{M}_{c,h} \to V$ *such that* $\varphi(v_{c,h}) = v$*. In other words, any highest-weight* $\mathfrak{Vir}$*-module is isomorphic to a quotient of some Verma module.*

See, e.g., the book [IK11] for more background.

2.4 Descendant fields and BPZ PDEs

Suppose that the primry field $\Phi_\iota(z)$ is given. In general, its descendants have the form

$$\mathrm{L}_{-n_1}\cdots\mathrm{L}_{-n_k}\Phi_\iota(z), \qquad \text{where} \quad n_1 \geq \cdots \geq n_k > 0 \text{ and } k \geq 1.$$

Their correlation functions are formally determined from the correlation functions of $\Phi_\iota(z)$ using linear differential operators which arise from the generators of the Virasoro algebra (this is quite complicated — see, e.g., [Mus10, Chapter 10]): for any primary fields $\{\Phi_{\iota_i}(z_i) \mid 1 \leq i \leq n\}$,

$$\big\langle \Phi_{\iota_1}(z_1)\cdots\Phi_{\iota_n}(z_n)\, \mathrm{L}_{-k}\Phi_\iota(z) \big\rangle \quad \overset{(\star)}{=} \quad \mathcal{L}_{-k}^{(z)} \, \big\langle \Phi_{\iota_1}(z_1)\cdots\Phi_{\iota_n}(z_n)\Phi_\iota(z) \big\rangle,$$

where

$$\mathcal{L}_{-k}^{(z)} \; := \; \sum_{i=1}^{n} \bigg(\frac{(k-1)\Delta_{\iota_i}}{(z_i - z)^k} \; - \; \frac{1}{(z_i - z)^{k-1}} \frac{\partial}{\partial z_i} \bigg), \qquad \text{for } k \in \mathbb{Z}_{>0}. \tag{2.5}$$

Here, the identity $(\star)$ should be thought of as a "black box", that is heuristically argued in the physics literature [Mus10, Chapter 10] via the "infinitesimal conformal symmetry" of the space-time, and can be a posteriori rigorously verified in some cases [KRV19].

Upshot 2.7. The conclusion from here is that the linear differential operators (2.5) relate the *purely algebraic content* in CFT, encoded in representations of the Virasoro algebra $\mathfrak{Vir}$, to its *analytical content* that includes the dependence of the space-time variables $z_1, \ldots, z_n, z \in \mathfrak{W}_{n+1}$.

Now, let's consider the $\mathfrak{Vir}$-module $\mathsf{V}_{c,\Delta_\iota}$ generated by the primary field $\Phi_\iota(z)$ with weight Δ_ι. By Lemma 2.6, we know that it is some quotient of a Verma module by some submodule J_ι:

$$\mathsf{V}_{c,\Delta_\iota} \cong \mathrm{M}_{c,\Delta_\iota}/\mathrm{J}_\iota. \tag{2.6}$$

Of course, the quotient structure needs to be determined from some information about $\Phi_\iota(z)$. We could have $\mathrm{J}_\iota = \{0\}$ or $\mathrm{J}_\iota = \mathrm{M}_{c,\Delta_\iota}$, in which case there's nothing to quotient by. However, in certain special cases we have a non-trivial quotient, which results in interesting information about correlations of $\Phi_\iota(z)$ with other fields. (See Section 2.5 for classification of those cases.)

Suppose that the conformal weight $\Delta_\iota = h_{r,s}$ belongs to the special class (2.11) discussed below, and denote $\Phi_\iota := \Phi_{r,s}$ accordingly. Then, by Theorem 2.10 (stated in the next Section 2.5), the Verma module $\mathrm{M}_{c,h_{r,s}}$ contains a so-called *singular* vector (see Definition 2.9 below)

$$v = P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots) v_{c,h_{r,s}} \quad \in \quad \mathrm{M}_{c,h_{r,s}}$$

at level rs, where P is a polynomial in the generators of the Virasoro algebra. Suppose furthermore that this vector is contained in $\mathrm{J}_{r,s}$ (which is the case, e.g., when $\mathsf{V}_{c,h_{r,s}}$ is irreducible):

$$v = P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots) v_{c,h_{r,s}} \quad \in \quad \mathrm{J}_{r,s}. \tag{2.7}$$

Then its equivalence class in the quotient module (2.6) is zero:

$$[v] = 0 \quad \in \quad \mathrm{M}_{c,h_{r,s}}/\mathrm{J}_{r,s} \cong \mathsf{V}_{c,h_{r,s}}. \tag{2.8}$$

In other words, the descendant field

$$P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots)\Phi_{r,s}(z) = 0$$

corresponding to the singular vector v is zero, a "null-field". In this case, we say that $\Phi_{r,s}(z)$ has a *degeneracy at level rs*. In particular, correlation functions containing the field $\Phi_{r,s}(z)$ then satisfy partial differential equations (known as "null-field equations") given by the polynomial

$$P(\mathcal{L}_{-1}^{(z)}, \mathcal{L}_{-2}^{(z)}, \ldots)$$

and the differential operators (2.5):

$$\begin{aligned} 0 &= \langle \Phi_{\iota_1}(z_1) \cdots \Phi_{\iota_n}(z_n) P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots)\Phi_{r,s}(z) \rangle && \text{[by (2.8, 2.7)]} \\ &\overset{(\star)}{=} P(\mathcal{L}_{-1}^{(z)}, \mathcal{L}_{-2}^{(z)}, \ldots) \langle \Phi_{\iota_1}(z_1) \cdots \Phi_{\iota_n}(z_n) \Phi_{r,s}(z) \rangle && \text{[by "black box" } (\star)] \end{aligned}$$

Upshot 2.8. For the correlation function (2.2) with $\Phi_\iota(z) = \Phi_{r,s}(z)$, we see that from certain *linear relations* on the Virasoro module side, we obtain the following (perfectly well-defined) *partial differential equation* (called BPZ PDE) on the correlation function side:

$$F_{\iota_1,\ldots,\iota_n,\iota} : \mathfrak{W}_{n+1} \to \mathbb{C}, \qquad P(\mathcal{L}_{-1}^{(z)}, \mathcal{L}_{-2}^{(z)}, \ldots)\, F_{\iota_1,\ldots,\iota_n,\iota}(z_1, \ldots, z_n, z) = 0. \tag{2.9}$$

We will see some concrete examples very soon (see Examples 2.11 and 2.12 and PDEs (2.14) and (2.15) in Section 2.6). Let us first summarize what is known about the structure of the universal Verma modules, which gives us the linear relations and the BPZ PDEs (Section 2.5).

2.5 Structure of Verma modules and singular vectors for $\mathfrak{Vir}$

Each Verma module $\mathrm{M}_{c,h}$ has a unique maximal proper submodule, and the quotient of $\mathrm{M}_{c,h}$ by this submodule is the unique irreducible highest-weight $\mathfrak{Vir}$-module of weight h and central charge c. In general, submodules of Verma modules were classified by B. Feĭgin and D. Fuchs [FF82, FF84, FF90], who showed that every non-trivial submodule of a Verma module $\mathrm{M}_{c,h}$ is generated by some singular vectors. We can use this information to learn properties of the correlation functions of interest.

Definition 2.9. A vector $v \in \mathrm{M}_{c,h} \setminus \{0\}$ is said to be *singular* at level $\ell \in \mathbb{Z}_{>0}$ if it satisfies

$$\mathrm{L}_0 v = (h + \ell)v \qquad \text{and} \qquad \mathrm{L}_n v = 0, \qquad \text{for } n \geq 1. \tag{2.10}$$

Note that the L_0-eigenvalue of a basis vector $v = \mathrm{L}_{-n_1} \cdots \mathrm{L}_{-n_k} v_{c,h} \in \mathrm{M}_{c,h}$ can be calculated using the commutation relations (2.4): we have

$$\mathrm{L}_0 v = (h + \sum_{i=1}^{k} n_i)v = (h + \ell)v.$$

The number $\ell := \sum_{i=1}^{k} n_i$ is called the *level* of the vector v.

In particular, Feĭgin and Fuchs found a characterization for the existence of singular vectors and thus for the irreducibility of $\mathrm{M}_{c,h}$. Indeed, the Verma module $\mathrm{M}_{c,h}$ is irreducible if and only if it contains no singular vectors. On the other hand, $\mathrm{M}_{c,h}$ contains singular vectors precisely when the numbers (c, h) belong to a special class:

Theorem 2.10. [FF84, Proposition 1.1 & Theorem 1.2] *The following are equivalent:*

1. *The Verma module* $\mathrm{M}_{c,h}$ *contains a singular vector.*
2. *There exist* $r, s \in \mathbb{Z}_{>0}$*, and* $\theta \in \mathbb{C} \setminus \{0\}$ *such that*

$$\begin{cases} h = h_{r,s}(\theta) := \frac{(r^2-1)}{4}\theta + \frac{(s^2-1)}{4}\theta^{-1} + \frac{(1-rs)}{2}, \\ c = c(t) = 13 - 6(\theta + \theta^{-1}). \end{cases} \tag{2.11}$$

In this case, the smallest such $\ell = rs$ *is the lowest level at which a singular vector occurs in* $\mathrm{M}_{c,h}$.

The special conformal weights $h_{r,s}$ are the roots of the Kac determinant [Kac79, Kac80], often called *Kac conformal weights.* The notation $h_{r,s}$ for them is very common historically.

Example 2.11 (Level 1)**.** $\mathrm{L}_{-1}v_{c,h}$ is a singular vector at level one if and only if $h = h_{1,1} = 0$.

Example 2.12 (Level 2)**.** As a more involved example, let us make an ansatz

$$v = (\mathrm{L}_{-2} + a\mathrm{L}_{-1}^2)v_{c,h} \tag{2.12}$$

for a singular vector at level two, with some $a \in \mathbb{C}$. Definition (2.10) implies that, in order for v to be singular, we must have

$$a = -\frac{3}{2(2h+1)}, \qquad h = \frac{1}{16}\Big(5 - c \pm \sqrt{(c-1)(c-25)}\Big),$$

which equals $h_{1,2}$ or $h_{2,1}$ depending on the choice of sign.

In general, explicit expressions for singular vectors are hard to find — one has to construct a suitable (complicated) polynomial P so that the vector $v = P(\mathrm{L}_{-1}, \mathrm{L}_{-2}, \ldots)v_{c,h}$ is singular. Remarkably, in the case when either $r = 1$ or $s = 1$, L. Benoit and Y. Saint-Aubin found a family of such vectors [BSA88]: for $r = 1$ and $s \in \mathbb{Z}_{>0}$, the singular vector at level $\ell = s$ has the formula

$$\sum_{k=1}^{s} \sum_{\substack{n_1,\ldots,n_k \geq 1 \\ n_1+\cdots+n_k=s}} \frac{(-\theta)^{k-s}\,(s-1)!^2}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(\sum_{i=j+1}^{k} n_i)} \times \mathrm{L}_{-n_1}\cdots\mathrm{L}_{-n_k}v_{c,h_{1,s}}. \tag{2.13}$$

The case $s = 1$ and $r \in \mathbb{Z}_{>0}$ is obtained by taking $\theta \mapsto \theta^{-1}$. Later, M. Bauer, P. Di Francesco, C. Itzykson, and J.-B. Zuber found the general singular vectors via a fusion procedure [BFIZ91]. The formulas for these expressions, however, are not explicit.

2.6 Examples of BPZ PDEs

As we noticed in Upshot 2.8, singular vectors give rise to kind of degeneracies in CFT — null-fields whose correlation functions solve BPZ PDEs (2.9) obtained from the Virasoro generators. Let us collect some concrete examples, that will appear in the theory of SLE curves.

Example 2.13 (Level 1)**.** From the singular vector at level one (Example 2.11), one obtains the null-field $\mathrm{L}_{-1}\Phi_{1,1}(z)$, whose correlation functions $F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) := \big\langle \Phi_{\iota_1}(z_1)\cdots\Phi_{\iota_n}(z_n)\Phi_{1,1}(z)\big\rangle$ satisfy the PDE

$$0 \quad = \quad \mathcal{L}_{-1}^{(z)}\, F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) \quad = \quad -\sum_{i=1}^{n} \frac{\partial}{\partial z_i} F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z).$$

Assuming that the correlation function F is translation-invariant, we can replace $\sum_{i=1}^{n} \frac{\partial}{\partial z_i}$ by the single derivative $\frac{\partial}{\partial z}$, so

$$\frac{\partial}{\partial z} F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) = 0, \tag{2.14}$$

i.e., the correlation function is *constant* in the variable z corresponding to $\Phi_{1,1}(z)$.

Example 2.14 (Level 2)**.** More interestingly, for the level two singular vectors (2.12) (Example 2.12), the corresponding null-fields are

$$\Big(\mathrm{L}_{-2} - \frac{3}{2(2h_{1,2}+1)}\mathrm{L}_{-1}^2\Big)\Phi_{1,2}(z)$$

and

$$\Big(\mathrm{L}_{-2} - \frac{3}{2(2h_{2,1}+1)}\mathrm{L}_{-1}^2\Big)\Phi_{2,1}(z).$$

In the former case, the correlation functions $F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) := \big\langle \Phi_{\iota_1}(z_1)\cdots\Phi_{\iota_n}(z_n)\Phi_{1,2}(z)\big\rangle$ satisfy the second order PDE

$$\left[-\frac{3}{2(2h_{1,2}+1)}\left(\sum_{i=1}^{n}\frac{\partial}{\partial z_i}\right)^2 - \sum_{i=1}^{n}\left(\frac{1}{z_i - z}\frac{\partial}{\partial z_i} - \frac{\Delta_{\iota_i}}{(z_i - z)^2}\right)\right]F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) = 0, \qquad (2.15)$$

where Δ_{ι_i} are the conformal weights of the fields Φ_{ι_i}, for $1 \le i \le n$. Assuming again translation invariance, this PDE simplifies to

$$\left[-\frac{3}{2(2h_{1,2}+1)}\frac{\partial^2}{\partial z^2} - \sum_{i=1}^{n}\left(\frac{1}{z_i - z}\frac{\partial}{\partial z_i} - \frac{\Delta_{\iota_i}}{(z_i - z)^2}\right)\right]F_{\iota_1,\ldots,\iota_n,\iota}(z_1,\ldots,z_n,z) = 0. \qquad (2.16)$$

Remark 2.15. Using the parameterization $\theta = \kappa/4$, we have $c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ and $h_{1,2} = \frac{6-\kappa}{2\kappa}$. Then the PDE (2.16) is the same as we will see in the next section — see Equations (3.7) and (3.10).

3 Third Lecture: Interacting SLEs

Natural variants of SLE(κ) curves can be obtained via change of measure (Girsanov transform). Indeed, one would expect that such variants are absolutely continuous with respect to the usual chordal SLE(κ) curve — at least at small times. We will see how such changes of measure can be related to CFT quantities, thereby bridging a relationship between interacting SLEs and CFT.

Let us summarize some preliminaries on stochastic analysis and Itô calculus — for background, there are many textbooks, e.g., [Dur96, RW00a, RW00b, Law05, RY05, Dur10]. Readers familiar with basic stochastic analysis can skip Sections 3.1–3.2. Section 3.3 discusses an example of interacting SLEs, and Section 3.4 concerns SLE martingale observables and interacting SLEs, and points out the relation to CFT via BPZ PDEs.

3.1 Martingales and Itô calculus

Martingales are processes $(M_t)_{t\geq 0}$ such that, given the history up to time t, the conditional expectation of M observed at time $s \geq t$ equals the present value M_t.

Definition 3.1 (Martingale)**.** Given a filtered probability space $(\Omega, \mathcal{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, a stochastic process $(M_t)_{t\geq 0}$ is a *martingale* if

(i) it is integrable: $\mathbb{E}|M_t| < \infty$ for all $t \geq 0$,

(ii) it is adapted: M_t is $\mathscr{F}_t$-measurable for all $t \geq 0$,

(iii) and it satisfies the martingale property $\mathbb{E}[M_s \mid \mathscr{F}_t] = M_t$ for all $s \geq t$.

One often wants to relax the condition (i) to hold only for up to some time, say. This can be done by *localization.* Local martingales are defined as processes $(M_t)_{t\geq 0}$ for which there is an increasing sequence of stopping times τ_n such that $\tau_n \nearrow \infty$ as $n \nearrow \infty$ almost surely, and the stopped processes

$$M_t^{\tau_n} := \begin{cases} M_{\min(t,\tau_n)}, & \tau_n > 0, \\ 0, & \tau_n = 0, \end{cases} \qquad t \geq 0,$$

are martingales, i.e., they satisfy (i)–(iii). Often the stopping times are chosen, e.g., so that

$$\tau_n := \inf\big\{ t \geq 0 \,\big|\, |M_t| \geq n \big\}, \qquad n \in \mathbb{N}.$$

The Optional Stopping Theorem [Dur10, Theorem 4.7.4] is a common tool in SLE proofs. It states that, under certain conditions, the martingale property (iii) holds for stopping times as well. Optional stopping can be applied for instance when $(M_t)_{t\geq 0}$ is uniformly bounded, as then it clearly is uniformly integrable.

Theorem 3.2 (Optional stopping)**.** *Let $(M_t)_{t\geq 0}$ be a continuous martingale and τ, σ two stopping times with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. Suppose $\sigma \leq \tau$. If the martingale $(M_t)_{t\geq 0}$ is uniformly integrable*[11]*, that is,*

$$\lim_{n\to\infty} \sup_{t\geq 0} \mathbb{E}\big[|M_t| \, \mathbb{1}\{|M_t| \geq n\} \big] = 0,$$

then we have $\mathbb{E}[M_\tau \mid \mathscr{F}_\sigma] = M_\sigma$. In particular, taking $\sigma = 0$, we have $\mathbb{E}[M_\tau] = M_0$.

[11] We also should assume certain "usual" conditions for the filtered probability space.

For a continuous function $F : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ which is continuously differentiable at least once in the time variable $t \in [0, \infty)$ and twice in the space variable $x \in \mathbb{R}$, we have the following Itô differential (see Theorem 3.4):

$$\mathrm{d}F(t, B_t) \;=\; \partial_t F(t, B_t)\,\mathrm{d}t \;+\; \partial_x F(t, B_t)\,\mathrm{d}B_t \;+\; \frac{1}{2}\partial_x^2 F(t, B_t)\,\mathrm{d}t.$$

The intuition behind this type of a truncating Taylor expansion is that one can think the differentials $\mathrm{d}t$ and $\mathrm{d}B_t$ of as satisfying the multiplication rules given by their quadratic variations:

$$\mathrm{d}t\,\mathrm{d}t = 0, \qquad \mathrm{d}t\,\mathrm{d}B_t = 0, \qquad \mathrm{d}B_t\,\mathrm{d}B_t = \mathrm{d}\langle B, B\rangle_t = \mathrm{d}t.$$

We will apply Itô differential to functions of several space-variables, and to more general stochastic processes than the Brownian motion. Let thus $(B_t^{(1)}, \ldots, B_t^{(n)})$ be a standard n-dimensional Brownian motion (i.e., a process whose components are 1-dimensional Brownian motions), and let $(\mathscr{F}_t)_{t\geq 0}$ be its natural completed filtration. Consider a stochastic processes Y_t (often called semimartingale) satisfying an stochastic differential equation (SDE) of the form[12]

$$\mathrm{d}Y_t \;=\; F(t)\,\mathrm{d}t \;+\; \sum_{k=1}^{n} G_k(t)\,\mathrm{d}B_t^{(k)}. \tag{3.1}$$

Amongst such processes, one can easily characterize the ones which are local martingales:

Lemma 3.3. *$(Y_t)_{t\geq 0}$ is a local martingale if and only if its drift term vanishes, i.e., $F \equiv 0$.*

For two semimartingales $Y_t^{(1)}, Y_t^{(2)}$, satisfying SDEs of the form (3.1), their covariation is defined as

$$\langle Y^{(1)}, Y^{(2)}\rangle_t \;:=\; \sum_{k,l} G_k^{(1)}(t)\, G_l^{(2)}(t)\, \langle B_t^{(k)} B_t^{(l)}\rangle_t \;=\; \sum_{k} G_k^{(1)}(t)\, G_k^{(2)}(t)\, t.$$

Itô's formula for semimartingales can now be written as follows, see e.g. [RW00b, Theorem (32.8)].

Theorem 3.4 (Itô's formula)**.** *Let $Y_t^{(1)}, \ldots, Y_t^{(N)}$ be semimartingales, and let $\psi : \mathbb{R}^N \to \mathbb{R}$ be a continuous function whose all partial derivatives up to second order exist and are continuous.*

Then, also $\psi(Y_t^{(1)}, \ldots, Y_t^{(N)})$ is a semimartingale, and we almost surely have

$$\begin{aligned}\mathrm{d}\psi(Y_t^{(1)}, \ldots, Y_t^{(N)}) \;=\; & \sum_{j=1}^{N} \partial_j \psi(Y_t^{(1)}, \ldots, Y_t^{(N)})\,\mathrm{d}Y_t^{(j)} \\ & + \frac{1}{2}\sum_{i,j=1}^{N} \partial_i\partial_j \psi(Y_t^{(1)}, \ldots, Y_t^{(N)})\,\mathrm{d}\langle Y^{(i)}, Y^{(j)}\rangle_t.\end{aligned}$$

Moreover, $\psi(Y_t^{(1)}, \ldots, Y_t^{(N)})$ is a local martingale if and only if the drift vanishes:

$$\sum_{j=1}^{N} F^{(j)}(t)\,\partial_j\psi(y_1, \ldots, y_N) \;+\; \frac{1}{2}\sum_{i,j=1}^{N}\sum_{k=1}^{n} G_k^{(i)}(t)\, G_k^{(j)}(t)\,\partial_i\partial_j\psi(y_1, \ldots, y_N) \;\equiv\; 0, \quad (y_1, \ldots, y_N) \in \mathbb{R}^N.$$

We will need the following example later in the context of changes of probability measures.

Example 3.5 (Exponential martingale)**.** Consider a continuous local martingale M_t. The process

$$\mathcal{E}_t := \exp\Big(M_t - \frac{1}{2}\langle M, M\rangle_t\Big) \tag{3.2}$$

is also a local martingale. This can be proven using Itô's formula: compute the Itô differential $\mathrm{d}\mathcal{E}_t$ and show that its drift vanishes: $\mathrm{d}\mathcal{E}_t = \mathcal{E}_t\,\mathrm{d}M_t$.

Lemma 3.6. *Any strictly positive local martingale has the exponential form* (3.2).

[12]Here, we assume that $G_1, \ldots, G_n$ are locally square-integrable functions adapted to $(\mathscr{F}_t)_{t\geq 0}$, and F is a Lebesgue-measurable function adapted to $(\mathscr{F}_t)_{t\geq 0}$ such that almost surely, we have $\int_0^t |F(s)|\,\mathrm{d}s < \infty$ for all $t \geq 0$.

3.2 Girsanov's theorem

Girsanov's theorem (see, e.g. [RY05, Chapter 8]) provides a way of changing the measure of Brownian motion B in an absolutely continuous manner. Let $(\mathscr{F}_t)_{t\geq 0}$ be the natural completed filtration for B and denote by P its probability measure. To change the probability measure P to another one, Q, which is absolutely continuous in the sense that all restrictions with respect to the filtration satisfy $\mathsf{Q}_t \leq \mathsf{P}_t$, can be obtained using the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathsf{Q}_t}{\mathrm{d}\mathsf{P}_t} = \exp\Big(M_t - \frac{1}{2}\langle M, M\rangle_t\Big), \qquad t < \tau, \tag{3.3}$$

where M_t is a continuous local P-martingale adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$, and $\langle M, M\rangle_t$ is its quadratic variation process, and τ is some stopping time (upon which the stopped process M is a martingale). Girsanov's theorem (see, e.g. [RY05, Chapter 8, Theorem 1.4]) shows that under this change of measure, the following process is a Q-Brownian motion:

$$\tilde{B}_t := B_t - \langle B, M\rangle_t. \tag{3.4}$$

Conversely, if we are given a (nice enough) local P-martingale M, we can *define* a new measure Q via (3.3) using the exponential martingale (3.2) associated to M,

$$\mathcal{E}_t := \exp\Big(M_t - \frac{1}{2}\langle M, M\rangle_t\Big).$$

In particular, if $\mathcal{E}_t$ is uniformly integrable, then we can normalize Q to a probability measure $\mathsf{Q}^\sharp$,

$$\mathsf{Q}^\sharp[A] := \frac{\mathsf{Q}[A]}{|\mathsf{Q}|} = \frac{\mathsf{Q}[A]}{\mathcal{E}_0}, \qquad A \in \mathcal{F},$$

where (by (3.3) and the Optional Stopping Theorem 3.2)

$$|\mathsf{Q}| := \mathsf{Q}[\Omega] = \mathsf{E}\big[\mathcal{E}_\tau\big] = \mathcal{E}_0$$

is the *total mass* of Q on the probability space Ω, and $\mathcal{F}$ denotes Q-measurable sets.

3.3 Interacting SLEs

Let $D \subsetneq \mathbb{C}$ be a simply connected Jordan domain with $2N$ distinct points $x_1, x_2, \ldots, x_{2N} \in \partial D$ appearing in counterclockwise order along the boundary (called *topological polygon*). We consider curves $\overline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ in D each of which connects two points among $\{x_1, x_2, \ldots, x_{2N}\}$. These curves can have various planar (non-crossing) connectivities, described in terms of planar pair partitions (planar *link patterns*), that we write in the form

$$\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{LP}_N,$$

where $\{a_1, b_1, \ldots, a_N, b_N\} = \{1, 2, \ldots, 2N\}$, and where LP_N denotes the set of all such planar link patterns. Note that for each fixed $N \in \mathbb{N}$, the total number of planar link patterns is the *Catalan number*

$$C_N = \frac{1}{N+1}\binom{2N}{N} = \#\mathrm{LP}_N.$$

An example of (discrete) curves $\overline{\gamma}$ is obtained from interfaces in the critical Ising model with alternating boundary conditions at the points $x_1, x_2, \ldots, x_{2N}$, see Figure 1.1. We wish to describe the scaling limits of these interfaces using SLE type curves.

From Girsanov's theorem, we expect that the curves in a multiple SLE(κ) can be described via a Loewner chain similar to the usual chordal case (LE), but where the Loewner driving function W_t has a *drift* given by the *interaction* with the other curves. The drift arises from a change of measure of the form (3.3):

$$\frac{\mathrm{d}\mathsf{Q}_t^\sharp}{\mathrm{d}\mathsf{P}_t} = \frac{\mathcal{E}_t}{\mathcal{E}_0},$$

where $\mathcal{E}_t$ is a local martingale that encodes the interaction (see [Dub07] for the derivation):

$$\mathcal{E}_t = \prod_{i \neq j} |g_t'(x_i)|^{h_{1,2}(\kappa)} \times \mathcal{Z}\big(g_t(x_1), \ldots, g_t(x_{j-1}), \sqrt{\kappa} B_t + x_j, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big), \tag{3.5}$$

where $h_{1,2}(\kappa) = \frac{6-\kappa}{2\kappa}$, and g_t is the solution to the Loewner equation (LE) with driving function $\sqrt{\kappa} B_t + x_j$. The drift for the Loewner driving function then can be obtained from (3.4).

It turns out that one can also write the drift in a quite convenient form [Dub07]. On the upper half-plane $\mathbb{H}$ with marked points $x_1 < \cdots < x_{2N}$, for the marginal law of the curve starting from x_j, with $j \in \{1, \ldots, 2N\}$, we have the SDEs[13]

$$\begin{cases} \mathrm{d}W_t = \sqrt{\kappa}\ \mathrm{d}B_t + \kappa\, \partial_j \log \mathcal{Z}\big(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), \ldots, g_t(x_{2N})\big)\ \mathrm{d}t, \\ \mathrm{d}g_t(x_i) = \frac{2\ \mathrm{d}t}{g_t(x_i) - W_t}, \qquad \text{for } i \neq j, \end{cases} \tag{3.6}$$

where $\mathcal{Z}$ is a so-called SLE(κ) *partition function*, and with initial conditions

$$\begin{cases} W_0 = x_j, \\ g_0(x_i) = x_i, \qquad \text{for } i \neq j. \end{cases}$$

Now, it is straightforward to formally calculate[14] the Itô differential of the local martingale (3.9) using Itô's formula (Theorem 3.4), the observation $g_t'(z) > 0$, and the relations

$$\mathrm{d}g_t(z) = \frac{2}{g_t(z) - W_t}\ \mathrm{d}t \qquad \text{and} \qquad \mathrm{d}g_t'(z) = -\frac{2g_t'(z)}{(g_t(z) - W_t)^2}\ \mathrm{d}t,$$

which follow from the Loewner equation (LE). By the martingale property, the drift term in the result should equal zero (cf. Lemma 3.3), which gives the following second order PDE:

$$\left[\frac{\kappa}{2}\frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}\left(\frac{2}{x_i - x_j}\frac{\partial}{\partial x_i} - \frac{2h_{1,2}(\kappa)}{(x_i - x_j)^2}\right)\right]\mathcal{Z}(x_1, \ldots, x_{2N}) = 0. \tag{3.7}$$

Such an equation holds symmetrically for all $j \in \{1, \ldots, 2N\}$ [Dub07].

Remark 3.7. This PDE (3.10) coincides with the second order BPZ PDE in CFT that is postulated to hold for correlation functions of the CFT field $\Phi_{1,2}(x)$ whose conformal weight $h_{r,s}$ belongs to the special class (2.11) with $r = 1$ and $s = 2$ and $\theta = \kappa/4$ (recall Remark 2.15).

This in part motivates the *prediction* that the growth of SLE curves from the boundary should be associated to the special CFT fields $\Phi_{1,2}$. Unfortunately, the mathematical meaning of the "fields" $\Phi_{1,2}$ is not really understood.

The PDEs (3.7) set the starting point for investigations of the interaction of SLE(κ) curves. Similar ideas can be used to study other SLE problems, where one wants to find out, e.g., some probabilities related to SLE via tautological martingales of type (3.9) appearing in the next Section 3.4.

[13] The system (3.6) of SDEs only makes sense locally, i.e., up to a certain stopping time.

[14] We cautiously note that it is not clear that M is smooth enough to apply Itô's formula — this has to be argued *a posteriori*.

3.4 Martingale observables and CFT boundary condition changing operators

Let us for a moment return to the critical Ising model on D^δ with Dobrushin boundary conditions $\oplus$ on the boundary arc $\partial^+ = (x^\delta\ y^\delta)$ and $\ominus$ on the complementary arc $\partial^- = (y^\delta\ x^\delta)$. (See Figure 1.1(left).) Let $(\mathscr{F}_t^\delta)_{t\geq 0}$ be the natural filtration for the exploration process $\gamma^\delta(t)$, for $t = 0, 1, \ldots$ (in discrete time, that we still denote t to make the connection to SLE curves obvious), started at $\gamma^\delta(0) = x^\delta$. Let $\mathbb{P}_{D^\delta}^{\mathrm{Dob}}$ denote the law of γ^δ (depending on the points x^δ, y^δ)).

The conditional expectation of an *observable* $\mathcal{O}^\delta$ (that is, a random variable) given the information $\mathscr{F}_t^\delta$ is trivially a local martingale, and thanks to the domain Markov property, we can rewrite such a conditional expectation as the usual expectation on the slitted[15] graph:

$$\mathbb{E}_{D^\delta}^{\mathrm{Dob}}[\mathcal{O}^\delta \mid \mathscr{F}_t^\delta] \quad = \quad \mathbb{E}_{D^\delta \smallsetminus \gamma^\delta[0,t]}^{\mathrm{Dob}}[\mathcal{O}^\delta].$$

Conjecturally, the expectation of the discrete observable $\mathcal{O}^\delta$ should converge in the scaling limit to a correlation function of some "continuum observable" (or quantum field) $\mathcal{O} = \Phi$. Thus, we expect that its conditional expectation converges in the scaling limit to a ratio of CFT correlation functions (where the denominator comes from normalizing the physics bracket operation $\langle\cdot\rangle_{D_t}^{\mathrm{Dob}}$ to be a probability measure, if possible): with some renormalization exponent Δ,

$$\delta^{-\Delta}\, \mathbb{E}_{D^\delta}^{\mathrm{Dob}}[\mathcal{O}^\delta \mid \mathscr{F}_t^\delta] \quad = \quad \delta^{-\Delta}\, \mathbb{E}_{D^\delta \smallsetminus \gamma^\delta[0,t]}^{\mathrm{Dob}}[\mathcal{O}^\delta] \quad \overset{\delta\to 0}{\longrightarrow} \quad \frac{\langle \Phi \rangle_{D_t}^{\mathrm{Dob}}}{\langle \mathbb{1} \rangle_{D_t}^{\mathrm{Dob}}} \tag{3.8}$$

where the random domain $D_t \subset \mathbb{C}$ is approximated by $D^\delta \smallsetminus \gamma^\delta[0,t]$ as $\delta \searrow 0$. Of course, the domain $D_t = D \smallsetminus \gamma[0,t]$ should be given by the complement of the scaling limit curve γ of the discrete exploration interface γ^δ, namely, the chordal SLE(3) curve [CDCH+14]. In particular:

Upshot 3.8. The limiting expression on the right side of (3.8) should be a local martingale for the chordal SLE(κ) curve γ. (Here, $\kappa = 3$ is the choice that works for the Ising model.)

Note that the domain D endowed with Dobrushin boundary conditions (induced from the discrete model) has two special boundary points[16]: the starting point $x \in \partial D$ and the end point $y \in \partial D$ of the curve γ. It is natural to think of these points carrying some "boundary condition changing operators", as argued by J. Cardy in the physics literature [Car03, Car05]. The idea is that to get from the measure with no boundary condition to a measure with given boundary condition (like Dobrushin), one makes a change of measure

$$\langle \cdot \rangle_D \qquad \longmapsto \qquad \langle \cdot \rangle_D^{\mathrm{Dob}} = \langle \cdot\, \Psi^{\mathrm{Dob}} \rangle_D$$

implemented by some special CFT field Ψ^{Dob} called "boundary condition changing operator". For the Dobrushin boundary conditions, it should depend on the two points $x, y \in \partial D$, so we would like to write

$$\Psi^{\mathrm{Dob}}(x,y) = \Phi_{1,2}(x)\Phi_{1,2}(y),$$

where on the right side we use a suggestive notation that will gain better meaning later. This gives rise to a *prediction* that certain conformal fields, denoted "$\Phi_{1,2}$", should be associated to the growth of SLE curves from the boundary.

[15] We assume here for simplicity that γ^δ is an injective path.

[16] Similarly, D_t has the two special boundary points $\gamma(t) \in D_t$ and $y \in \partial D$ (which are the starting and end points of the exploration $\tilde{\gamma}$ in D_t). This is also built into the definition of the chordal SLE(κ) curve.

To see what the martingale property gives us, suppose that:

- ▷ Our observable depends on some variables $z_1, \ldots, z_n \in \overline{D}$ and its limit[17] (if exists) $\mathcal{O} = \Phi$ has the form of a product of some CFT (primary) fields,

$$\Phi(z_1, \ldots, z_n) = \Phi_{\iota_1}(z_1) \cdots \Phi_{\iota_n}(z_n),$$

  with conformal weights $\Delta_{\iota_1}, \ldots, \Delta_{\iota_n} \in \mathbb{R}$.

- ▷ With the analogy from the Ising model in mind, let us write also the boundary condition changing operator in the form

$$\Psi^{\mathrm{Dob}}(x, y) = \Phi^{\ominus\oplus}(x)\Phi^{\oplus\ominus}(y),$$

  that is a product of some (primary) fields of some conformal weights $\Delta^{\ominus\oplus}$ and $\Delta^{\oplus\ominus}$.

Then, using conformal covariance postulate (2.3) for CFT correlation functions, we can write the local martingale (3.8) in the following form. After exploring up to time t, the martingale depends on the random slit domain $D_t = D \smallsetminus \gamma[0, t]$, and

$$\begin{aligned} & M_{D_t}(\gamma(t), y; z_1, \ldots, z_n) \\ := \; & \frac{\big\langle \Phi_{\iota_1}(z_1) \cdots \Phi_{\iota_n}(z_n) \; \Phi^{\ominus\oplus}(\gamma(t)) \Phi^{\oplus\ominus}(y) \big\rangle_{D_t}}{\big\langle \Phi^{\ominus\oplus}(\gamma(t)) \Phi^{\oplus\ominus}(y) \big\rangle_{D_t}} \\ = \; & \frac{|f'(\gamma(t))|^{\Delta^{\ominus\oplus}} |f'(y)|^{\Delta^{\oplus\ominus}} \; \prod_{i=1}^n |f'(z_i)|^{\Delta_{\iota_i}}}{|f'(\gamma(t))|^{\Delta^{\ominus\oplus}} |f'(y)|^{\Delta^{\oplus\ominus}}} \times \frac{\big\langle \Phi_{\iota_1}(f(z_1)) \cdots \Phi_{\iota_n}(f(z_n)) \; \Phi^{\ominus\oplus}(f(\gamma(t))) \Phi^{\oplus\ominus}(f(y)) \big\rangle_D}{\big\langle \Phi^{\ominus\oplus}(f(\gamma(t))) \Phi^{\oplus\ominus}(f(y)) \big\rangle_D} \\ = \; & \prod_{i=1}^n |f'(z_i)|^{\Delta_{\iota_i}} \times M_D(f(\gamma(t)), f(y); f(z_1), \ldots, f(z_n)), \end{aligned}$$

where $f : D_t \to D$ is a conformal map (and we assume that it extends to the boundary of D_t).

In particular, taking $D = \mathbb{H}$ to be the upper half-plane, $x = 0$, $y = \infty$, and $f = g_t : \mathsf{H}_t \to \mathbb{H}$ the solution to the Loewner equation (LE) for the SLE(κ) curve γ with driving function $W_t = \sqrt{\kappa} B_t$, and dropping $g_t(y) = y = \infty$, we have

$$M_{\mathsf{H}_t}(\gamma(t); z_1, \ldots, z_n) = \prod_{i=1}^n |g_t'(z_i)|^{\Delta_{\iota_i}} \times M_{\mathbb{H}}(W_t; g_t(z_1), \ldots, g_t(z_n)), \tag{3.9}$$

where $W_t = g_t(\gamma(t))$. Note that this has a similar form as the local martingale (3.5).

3.5 BPZ PDEs from martingale observables

Now, it is straightforward to formally calculate[18] the Itô differential of the local martingale (3.9) using Itô's formula (Theorem 3.4), the observation $g_t'(z) > 0$, and the relations

$$\mathrm{d}g_t(z) = \frac{2}{g_t(z) - W_t} \, \mathrm{d}t \qquad \text{and} \qquad \mathrm{d}g_t'(z) = -\frac{2 g_t'(z)}{(g_t(z) - W_t)^2} \, \mathrm{d}t,$$

which follow from the Loewner equation (LE). By the martingale property, the drift term in the result should equal zero, which gives the following second order PDE:

$$\left[\frac{\kappa}{2} \frac{\partial^2}{\partial x^2} + \sum_{i=1}^n \left(\frac{2}{z_i - x} \frac{\partial}{\partial z_i} - \frac{2\Delta_{\iota_i}}{(z_i - x)^2} \right) \right] M_{\mathbb{H}}(x; z_1, \ldots, z_n) = 0. \tag{3.10}$$

[17] For example, in the case of the CFT describing critical Ising model, Φ could be a product of spins (with $\Phi_{\iota_i}(z_i) = \sigma_{z_i}$ and $\Delta_{\iota_i} = 1/16$, for all i).

[18] We cautiously note that it is not clear that M is smooth enough to apply Itô's formula — this has to be argued *a posteriori*.

This PDE (3.10) coincides with the second order BPZ PDE in CFT that is postulated to hold for correlation functions of the boundary condition changing operator $\Phi^{\ominus\oplus}(x) = \Phi_{1,2}(x)$. In CFT parlance, this operator has a *degeneracy at level two*, with conformal weight of special type[19]: $\Delta^{\ominus\oplus} = h_{1,2}$. This motivates the *prediction* that the growth of SLE curves from the boundary should be associated to the special CFT fields $\Phi_{1,2}$. Unfortunately, the mathematical meaning of the "fields" $\Phi_{1,2}$ is not really understood even for the Ising model.

[19]Similarly, we have $\Phi^{\oplus\ominus}(y) = \Phi_{1,2}(y)$ and $\Delta^{\oplus\ominus} = h_{1,2}$.

References

[Ahl79] Lars V. Ahlfors. *Complex analysis: an introduction to the theory of analytic functions of one complex variable*. International Series in Pure and Applied Mathematics. McGraw-Hill Book Co., New York, 3rd edition, 1979.

[BFIZ91] Michel Bauer, Philippe Di Francesco, Claude Itzykson, and Jean-Bernard Zuber. Covariant differential equations and singular vectors in Virasoro representations. *Nucl. Phys. B*, 362(3):515–562, 1991.

[BH19] Stéphane Benoist and Clément Hongler. The scaling limit of critical Ising interfaces is CLE(3). *Ann. Probab.*, 47(4):2049–2086, 2019.

[BPW21] Vincent Beffara, Eveliina Peltola, and Hao Wu. On the uniqueness of global multiple SLEs. *Ann. Probab.*, 49(1):400–434, 2021.

[BPZ84a] Alexander A. Belavin, Alexander M. Polyakov, and Alexander B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nucl. Phys. B*, 241(2):333–380, 1984.

[BPZ84b] Alexander A. Belavin, Alexander M. Polyakov, and Alexander B. Zamolodchikov. Infinite conformal symmetry of critical fluctuations in two dimensions. *J. Stat. Phys.*, 34(5-6):763–774, 1984.

[BSA88] Louis Benoit and Yvan Saint-Aubin. Degenerate conformal field theories and explicit expressions for some null vectors. *Phys. Lett. B*, 215(3):517–522, 1988.

[Car84] John L. Cardy. Conformal invariance and surface critical behavior. *Nucl. Phys. B*, 240(4):514–532, 1984.

[Car96] John L. Cardy. *Scaling and renormalization in statistical physics*, volume 5 of *Cambridge Lecture Notes in Physics*. Cambridge University Press, 1996.

[Car03] John L. Cardy. Stochastic Loewner evolution and Dyson's circular ensembles. *J. Phys. A*, 36(24):L379–L386, 2003.

[Car05] John L. Cardy. SLE for theoretical physicists. *Ann. Physics*, 318(1):81–118, 2005.

[CDCH+14] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of Ising interfaces to Schramm's SLE curves. *C. R. Acad. Sci. Paris Sér. I Math.*, 352(2):157–161, 2014.

[CGN15] Federico Camia, Christophe Garban, and Charles M. Newman. Planar Ising magnetization field I. Uniqueness of the critical scaling limit. *Ann. Probab.*, 43(2):528–571, 2015.

[Che20] Dmitry Chelkak. Ising model and s-embeddings of planar graphs. Preprint in arXiv:2006.14559, 2020.

[CHI15] Dmitry Chelkak, Clément Hongler, and Konstantin Izyurov. Conformal invariance of spin correlations in the planar Ising model. *Ann. Math.*, 181(3):1087–1138, 2015.

[CS11] Dmitry Chelkak and Stanislav Smirnov. Discrete complex analysis on isoradial graphs. *Adv. Math.*, 228(3):1590–1630, 2011.

[DCMT21] Hugo Duminil-Copin, Ioan Manolescu, and Vincent Tassion. Planar random-cluster model: fractal properties of the critical phase. *Probab. Theory Related Fields*, 181(1-3):401–449, 2021.

[DCS12] Hugo Duminil-Copin and Stanislav Smirnov. Conformal invariance of lattice models. In *Probability and Statistical Physics in two and more dimensions*, volume 15, pages 213–276. Clay Mathematics Proceedings, American Mathematical Society, Providence, RI, 2012.

[DCST17] Hugo Duminil-Copin, Vladas Sidoravicius, and Vincent Tassion. Continuity of the phase transition for planar random-cluster and Potts models with $1 \leq q \leq 4$. *Comm. Math. Phys.*, 349(1):47–107, 2017.

[DF84] Vladimir S. Dotsenko and Vladimir A. Fateev. Conformal algebra and multipoint correlation functions in $2d$ statistical models. *Nucl. Phys. B*, 240(3):312–348, 1984.

[DFMS97] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory.* Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[DKRV16] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on the Riemann sphere. *Comm. Math. Phys.*, 342(3):869–907, 2016.

[dN83] Marcel den Nijs. Extended scaling relations for the magnetic critical exponents of the potts model. *Phys. Rev.*, B(27):1674–1679, 1983.

[DS87] Bertrand Duplantier and Hubert Saleur. Exact determination of the percolation hull exponent in two dimensions. *Phys. Rev. Lett.*, 58:2325–2328, 1987.

[Dub07] Julien Dubédat. Commutation relations for SLE. *Comm. Pure Appl. Math.*, 60(12):1792–1847, 2007.

[Dur96] Richard Durrett. *Stochastic calculus: a practical introduction.* Probability and Stochastics Series. CRC Press LLC, 1996.

[Dur10] Richard Durrett. *Probability: theory and examples.* Cambridge University Press, 4th edition, 2010.

[FF82] Boris L. Feĭgin and Dmitry B. Fuchs. Invariant skew-symmetric differential operators on the line and Verma modules over the Virasoro algebra. *Funct. Anal. Appl.*, 16(2):114–126, 1982.

[FF84] Boris L. Feĭgin and Dmitry B. Fuchs. Verma modules over the Virasoro algebra. In *Topology (Leningrad 1982)*, volume 1060 of *Lecture Notes in Mathematics*, pages 230–245, Berlin Heidelberg, 1984. Springer-Verlag.

[FF90] Boris L. Feĭgin and Dmitry B. Fuchs. Representations of the Virasoro algebra. In *Representation of Lie Groups and Related Topics*, volume 7 of *Advanced Studies in Contemporary Mathematics*, pages 465–554. Gordon and Breach, New York, 1990.

[HS13] Clément Hongler and Stanislav Smirnov. The energy density in the planar Ising model. *Acta Math.*, 211(2):191–225, 2013.

[IK11] Kenji Iohara and Yoshiyuki Koga. *Representation theory of the Virasoro algebra.* Springer Monographs in Mathematics. Springer-Verlag, London, 2011.

[Izy15] Konstantin Izyurov. Smirnov's observable for free boundary conditions, interfaces and crossing probabilities. *Comm. Math. Phys.*, 337(1):225–252, 2015.

[Izy17] Konstantin Izyurov. Critical Ising interfaces in multiply-connected domains. *Probab. Theory Related Fields*, 167(1-2):379–415, 2017.

[Kac79] Victor G. Kac. Contravariant form for the infinite-dimensional Lie algebras and superalgebras. In *Lecture Notes in Physics*, volume 94, pages 441–445. Springer-Verlag, Berlin, 1979.

[Kac80] Victor G. Kac. Highest weight representations of infinite dimensional Lie algebras. In *Proceedings of the ICM 1978, Helsinki, Finland*, volume 1, pages 299–304. Acad. Sci. Fenn., 1980.

[Kar19] Alex Karrila. Multiple SLE type scaling limits: from local to global. Preprint in arXiv:1903.10354, 2019.

[Kem17] Antti Kemppainen. *Schramm-Loewner evolution*, volume 24 of *SpringerBriefs in Mathematical Physics*. Springer International Publishing, 2017.

[KMS21] Konstantinos Kavvadias, Jason Miller, and Lukas Schoug. Regularity of the SLE_4 uniformizing map and the SLE_8 trace. Preprint in arXiv:2107.03365, 2021.

[KRV19] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Local conformal structure of Liouville Quantum Gravity. *Comm. Math. Phys.*, 371(3):1005–1069, 2019.

[KRV20] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Integrability of Liouville theory: proof of the DOZZ formula. *Ann. of Math.*, 191(1):81–166, 2020.

[KS17] Antti Kemppainen and Stanislav Smirnov. Random curves, scaling limits and Loewner evolutions. *Ann. Probab.*, 45(2):698–779, 2017.

[Law05] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2005.

[Loe23] Charles Loewner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I. *Math. Ann.*, 89:103–121, 1923.

[LSW01a] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents I: Half-plane exponents. *Acta Math.*, 187(2):237–273, 2001.

[LSW01b] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Values of Brownian intersection exponents II: Plane exponents. *Acta Math.*, 187(2):275–308, 2001.

[LSW02] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. One-arm exponent for $2D$ critical percolation. *Electron. J. Probab.*, 7(2):1–13, 2002.

[MP10] Peter Mörters and Yuval Peres. *Brownian motion.* Cambridge University Press, 2010.

[Mus10] Giuseppe Mussardo. *Statistical field theory: An Introduction to exactly solved models in statistical physics.* Oxford Graduate Texts. Oxford University Press, 2010.

[Nie87] Bernard Nienhuis. Coulomb gas formulation of two-dimensional phase transitions. In *Phase Transitions and Critical Phenomena*, volume 11, pages 1–53. Academic Press, London, 1987.

[Pel19] Eveliina Peltola. Towards a conformal field theory for Schramm-Loewner evolutions. *J. Math. Phys.*, 60(10):103305, 2019. Special issue (Proc. ICMP, Montreal, July 2018).

[RS05] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math.*, 161(2):883–924, 2005.

[RW00a] L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales*, volume 1: Foundations of *Cambridge Mathematical Library.* Originally published by John Wiley & Sons Ltd., 1979, reprinted by Cambridge University Press, 2000.

[RW00b] L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales*, volume 2: Itô calculus of *Cambridge Mathematical Library.* Originally published by John Wiley & Sons Ltd., 1979, reprinted by Cambridge University Press, 2000.

[RY05] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, 2005. Reprint of the 3rd ed. Berlin Heidelberg New York 1999.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118(1):221–288, 2000.

[Sch08] Martin Schottenloher. *A mathematical introduction to conformal field theory*, volume 759 of *Lecture Notes in Physics.* Springer-Verlag, Berlin Heidelberg, 2nd edition, 2008.

[Smi01] Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci.*, 333(3):239–244, 2001. (Updated 2009, see arXiv:0909.4499).

[Smi06] Stanislav Smirnov. Towards conformal invariance of $2D$ lattice models. In *Proceedings of the ICM 2006, Madrid, Spain*, volume 2, pages 1421–1451. European Mathematical Society, 2006.

[Smi10] Stanislav Smirnov. Conformal invariance in random cluster models I. Holomorphic fermions in the Ising model. *Ann. Math.*, 172(2):1435–1467, 2010.

[SW01] Stanislav Smirnov and Wendelin Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8(6):729–744, 2001.

[Wu20] Hao Wu. Hypergeometric SLE: conformal Markov characterization and applications. *Comm. Math. Phys.*, 374(1):433–484, 2020.