# Stochastic quantization of Yang-Mills 

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Part 1: quantum field theories (QFT); some examples
Part 2: Stochastic quantization of Yang-Mills

## Example 1: Yang-Mills theory (and 'fundamental forces')

$$
\langle\mathcal{Q}(A)\rangle=\int \mathcal{Q}(A) e^{-\mathcal{S}(A)} D A
$$

where $A$ are connections on a G-bundle, and

$$
\mathcal{S}(A)=\int\left\|F_{A}\right\|^{2} d v_{M} \quad F_{A}^{i j}=\partial_{i} A_{j}-\partial_{j} A_{i}+\left[A_{i}, A_{j}\right]
$$

Finite dimensional integrals $\int_{-\infty}^{\infty} Q(x) e^{-S(x)} d x$

- $\mathcal{Q}(A)$ can be product of Wilson loops $W_{\ell}(A)=\operatorname{Tr} \operatorname{hol}_{\ell}(A)$.

Mass gap:
$\left\langle W_{\ell_{1}}(A) W_{\ell_{2}}(A)\right\rangle-\left\langle W_{\ell_{1}}(A)\right\rangle\left\langle W_{\ell_{2}}(A)\right\rangle \sim \exp \left(-c \cdot \operatorname{dist}\left(\ell_{1}, \ell_{2}\right)\right)$

- Gravity, electro-magnetism, weak, strong.


## Example 2: Chern-Simons theory (and topology)

$$
\langle\mathcal{Q}(A)\rangle=\int \mathcal{Q}(A) e^{-\mathcal{S}(A)} D A
$$

on $G$-bundle over 3D manifold $M$, and

$$
\mathcal{S}(A)=\frac{k}{4 \pi} \int_{M} \operatorname{Tr}\left(A \wedge d A+\frac{2}{3} A \wedge A \wedge A\right)
$$

$\left\langle W_{\ell}(A)\right\rangle:$

- $G=U(2)$ : Jones polynomial
- $G=U(N)$ : HOMFLY
- $G=S O(N)$ : Kauffman polynomial.


## Example 3: 1D sigma model (and index formula)

$$
\operatorname{Tr} e^{-H}=\int_{\Pi T \mathcal{L}(M)} e^{-\mathcal{S}(\gamma, \theta)} D \gamma D \theta
$$

where $\mathcal{L}(M)=\left\{\gamma: S^{1} \rightarrow M\right\}$ is loop space, $(M, g)$ Riem.manifold

$$
\mathcal{S}(\gamma, \theta)=\int_{0}^{1}\|\dot{\gamma}\|_{g}^{2}+\left\langle\theta(t), \nabla_{\dot{\gamma}} \theta(t)\right\rangle_{g} d t
$$

- Tre ${ }^{-H}$ analytical index (e.g. McKean-Singer)
- RHS: topological index (localization)

Alvarez-Gaume, Atiyah, Bismut, Witten (80s); Hanisch, Ludewig (2022)....

## Example 4: Liouville conformal field theory

Surface $(M, g), g=e^{\gamma X} \hat{g}$ where $X: M \rightarrow \mathbf{R}$

$$
\begin{gathered}
\langle\mathcal{Q}(X)\rangle=\int \mathcal{Q}(X) e^{-\mathcal{S}(X)} D X \\
\mathcal{S}(X)=\int_{M}\left(\left|\partial^{\hat{g}} X\right|^{2}+Q R_{\hat{g}} X+4 \pi \mu e^{\gamma X}\right) d v_{\hat{\mathrm{g}}}
\end{gathered}
$$

where $\gamma \in \mathbf{R}, \mu \in \mathbf{R}_{+}, Q=\frac{2}{\gamma}+\frac{\gamma}{2}$

- arises from 2D quantum gravity
- can be rigorously defined with many interesting properties:

Guillarmou,Kupiainen,Rhodes,Vargas..... "Polyakov's formulation of 2d string" (IHES), "Integrability"
(Annals), "Conformal bootstrap" (Acta), many recent works by X.Sun et al...

- Yang-Mills $\int \mathcal{Q}(A) e^{-\int\left\|F_{A}\right\|^{2} d v_{M}} D A$
- Topological field theories $\int \mathcal{Q}(A) e^{-\frac{k}{4 \pi} \int_{M} \operatorname{Tr}\left(A \wedge d A+\frac{2}{3} A \wedge A \wedge A\right)} D A$
- $\sigma$-models $\int_{\Pi T \mathcal{L}(M)} e^{-\int_{0}^{1}\|\dot{\gamma}\|_{g}^{2}+\left\langle\theta(t), \nabla_{\dot{\gamma}} \theta(t)\right\rangle_{g} d t} D \gamma D \theta$ (or 2D versions)
- Conformal field theories $\int \mathcal{Q}(X) e^{-\int_{M}\left(|\partial \hat{\varepsilon} X|^{2}+Q R_{\hat{g}} X+4 \pi \mu e^{\gamma X}\right) d v_{\hat{g}}} D X$

Mathematical foundation?
(1) sample paths continuous,
e.g. $e^{-\int_{0}^{1}\|\dot{\gamma}\|_{g}^{2}} D \gamma$ is (manifold-valued) Brownian motion.
(2) Gaussian measures,
e.g. Gaussian free fields $e^{-\int_{\Omega}|\nabla \Phi|^{2} d x} D \Phi$ for $\Omega \subset \mathbf{R}^{d}, \Phi: \Omega \rightarrow \mathbf{R}$ (factors into independent Gaussians $e^{-k^{2} \Phi_{k}^{2}} d \Phi_{k}$ for each Fourier mode $\Phi_{k}$ )
"harder" cases: e.g. $\Omega \subset \mathbf{R}^{d \geq 2}$

$$
e^{-\int_{\Omega}\left(|\nabla \Phi|^{2}+\phi^{4}\right) d x} D \Phi
$$

$\Phi$ singular, not continuous (in fact, $\Phi^{4}$ needs renormalization)
"even harder" when: infinite dimensional (e.g. gauge) symmetry, target space nonlinear etc.
"Constructive field theorists"
Wightman axioms ('60-'70): Hilbert space, representation of the Poincare group, fields operators (to construct local observables).

Osterwalder-Schrader axioms ('70s): gives precise condition to perform the passage to/from Euclidean space.
(An Euclidean quantum field theory is a measure on $\mathcal{S}^{\prime}\left(\mathbf{R}^{d}\right)$ satisfying some axiomatic properties: regularity, Euclidean covariance, reflection positivity.)
[B.Simon, A.Jaffe, Glimm, T.Spencer, D.Brydges......]
Difficulties in constructing such measures:

- small scale ('ultraviolet') problems;
- large field problems;
- large scale i.e. long distance problems;
- infinite dimensional symmetries

Quantum field theory: functional integral w.r.t. measure

$$
\exp (-\mathcal{S}(\phi)) D \phi
$$

Stochastic quantization: stochastic dynamic

$$
\partial_{t} \phi=-\nabla \mathcal{S}(\phi)+\xi
$$

where $\xi$ is space-time white noise.
The measure is (formally) invariant under the dynamic.
Advantages of stochastic quantization approach:

1) more tools e.g. PDE, stochastic analysis;
2) separate difficulties (to some extent)

Example: Stochastic heat equation (SHE)
Gaussian free field (GFF)

$$
e^{-\frac{1}{2} \int(\nabla \Phi)^{2} d x} D \Phi
$$

Stochastic heat equation (SHE)

$$
\partial_{t} \Phi=\Delta \Phi+\xi \quad x \in \mathbf{R}^{d}
$$

Solution to SHE is Gaussian process. Regularity of SHE:

$$
\mathcal{C}^{\frac{1}{2}-}(d=1) ; \quad \mathcal{C}^{0-}(d=2) ; \quad \mathcal{C}^{-\frac{1}{2}-}(d=3)
$$

Stochastic quantization approach has been successful for $\Phi^{4}$ model

$$
\begin{gathered}
e^{-\int_{\Omega}\left(\frac{1}{2}|\nabla \Phi|^{2}+\frac{1}{4} \phi^{4}\right) d x} D \Phi \\
\partial_{t} \Phi=\Delta \Phi-\Phi^{3}+\xi
\end{gathered}
$$

Short time：Da Prato－Debussche（2d），Hairer（3d），etc．
Long time：Weber－Mourrat，Gubinelli－Hofmanova（2，3d），etc．

$$
\Phi=\mathrm{SHE}+v
$$

－SHE is Gaussian and singular；

## 概率性质好，分析性质差

－$v$ has unknown probability law but is a bit more regular．
概率性质不知道，分析性质略好一点
stochastic quantization of Yang-Mills in 2D and 3D

1. Hao Shen, Stochastic quantization of an abelian gauge theory, (2018). CMP.
2. Ajay Chandra, Ilya Chevyrev, Martin Hairer, Hao Shen, Langevin dynamic for the 2D Yang-Mills measure, (2020). P.IHES.
3. Ajay Chandra, Ilya Chevyrev, Martin Hairer, Hao Shen, Stochastic quantisation of Yang-Mills-Higgs in 3D, (2022).
4. Ilya Chevyrev, Hao Shen, Invariant measure and universality of the 2D Yang-Mills Langevin dynamic, (2023).

## Stochastic Yang-Mills

Let $G$ be a Lie group and $\mathfrak{g}$ be its Lie algebra, $A_{i} \in \mathfrak{g}$.

$$
\partial_{t} A_{i}=\Delta A_{i}+\left[A_{j}, 2 \partial_{j} A_{i}-\partial_{i} A_{j}+\left[A_{j}, A_{i}\right]\right]+\xi_{i} \quad \text { on } \mathbf{T}^{2} \text { or } \mathbf{T}^{3}
$$

$\mathcal{S}(A)=\int\left\|F_{A}\right\|^{2} d x$, with $F_{A}^{i j}=\partial_{i} A_{j}-\partial_{j} A_{i}+\left[A_{i}, A_{j}\right]$
Gauge symmetry: $g \in \mathfrak{G} \stackrel{\text { def }}{=} \mathcal{C}^{\infty}\left(\mathbf{T}^{d}, G\right), A \mapsto g A g^{-1}-(d g) g^{-1}$
Wilson loops: $\gamma:[0,1] \rightarrow \boldsymbol{T}^{d}, \gamma(0)=\gamma(1)$.
Solve $d h(s)=h(s)\langle A(\gamma(s)), d \gamma(s)\rangle, h(0)=\mathrm{id} \in G$.

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$$
W_{\gamma}(A) \stackrel{\text { def }}{=} \operatorname{Tr}(h(1))
$$

## Stochastic Yang-Mills

Orbit space: " $\{A\} / \mathfrak{G}$ " where $\mathfrak{G}$ is gauge group
Observable: Wilson loops $W_{\gamma}(A)$ (same value along each orbit)
Dynamic: Nonlinear SPDE - and project to orbit space.


## Construction of state space in 2D

- Support YM dynamics
- Wilson loops ( $\approx$ integrate along curve)

$$
\mathcal{C}^{0} \subset \Omega \subset \mathcal{C}^{0-}
$$

1. Functionals $A:\{$ Line segments $\} \rightarrow \mathbf{R}$ (This includes $\left.C^{0}\right)$
2. Norm: $\frac{|A(\ell)|}{|\ell|} \lesssim|\ell|^{\alpha-1}, \quad|A(\partial P)| \leq|P|^{\alpha / 2}, \quad \alpha<1$
3. Completion. ( $\Omega$ can be embedded in $\mathcal{C}^{\alpha-1}$ )
4. Kolmogorov ${ }^{1}$ theorem in $\Omega \Rightarrow$ SHE (Gaussian) $\in \Omega$.
5. We can extend gauge transformations to $\Omega$. Quotient space is Polish.

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## State space in 3D

$\mathcal{C}^{-\frac{1}{2}-}$ field (even Gaussian part) can't be integrated along curves
Q: regularized version of Wilson loops which is still gauge invariant? Regularize $\mathcal{C}^{-\frac{1}{2}-} \ni A \rightarrow A_{\delta} \in \mathcal{C}^{\infty}$. Define Wilson loop $W_{\gamma}\left(A_{\delta}\right)$.

- YM flow: $\mathcal{F}_{\delta} A \in \mathcal{C}^{\infty}$ and $\bar{A}=g \circ A \Rightarrow W_{\gamma}\left(\mathcal{F}_{\delta} A\right)=W_{\gamma}\left(\mathcal{F}_{\delta} \bar{A}\right)$ where $\mathcal{F}_{\delta}$ is YM heat flow $\partial_{\delta} A_{i}=\Delta A_{i}+2\left[A_{j}, \partial_{j} A_{i}\right]+\cdots$

State space: $\mathcal{S}=\left\{X \in C^{-\frac{1}{2}-}:\left\|\left(e^{s \Delta} X\right)\left(\partial e^{s \Delta} X\right)\right\|\right.$
(We can solve stochastic YM equation in $\mathcal{S}$.)

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State space: $\mathcal{S}=\left\{X \in C^{-\frac{1}{2}-}:\left\|\left(e^{s \Delta} X\right)\left(\partial e^{s \Delta} X\right)\right\|_{C^{-\frac{1}{3}-}} \lesssim s^{-\frac{5}{6}}\right\}$ (We can solve stochastic YM equation in $\mathcal{S}$.)

Solve the stochastic PDE in 2D

$$
\partial_{t} A_{i}=\Delta A_{i}+\left[A_{j}, 2 \partial_{j} A_{i}-\partial_{i} A_{j}+\left[A_{j}, A_{i}\right]\right]+\xi_{i}
$$

Toy model：$\partial_{t} u=\Delta u+u^{3}+\xi$
SHE $\partial_{t} \Psi=\Delta \Psi+\xi$ has regularity $\Psi \in \mathcal{C}^{0-}$ ．

Renormalization：$\Psi_{\varepsilon}^{2}-C_{\varepsilon}$ and $\Psi_{\varepsilon}^{3}-3 C_{\varepsilon} \Psi_{\varepsilon}$ with $C_{\varepsilon}=\mathbf{E}\left[\Psi_{\varepsilon}^{2}\right]$ converge in $\mathcal{C}^{0-}$ in probability． Renormalized equation：$\partial_{t} u_{\varepsilon}=\Delta u_{\varepsilon}+u_{\varepsilon}^{3}-3 C_{\varepsilon} u_{\varepsilon}+\xi_{\varepsilon}$重整化常数发散，使得方程解收敛。

Solve the stochastic PDE in 2D

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Toy model：$\partial_{t} u=\Delta u+u^{3}+\xi$
SHE $\partial_{t} \Psi=\Delta \Psi+\xi$ has regularity $\psi \in \mathcal{C}^{0-}$ ．

$$
\begin{aligned}
\{u: & \left.u=\Psi+v, \quad v \in \mathcal{C}^{1}\right\} \quad \subset \mathcal{C}^{0-} \\
u^{3}= & \Psi^{3}+3 \Psi^{2} \cdot v+3 \Psi \cdot v^{2}+v^{3}
\end{aligned}
$$

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Solve the stochastic PDE in 2D

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\partial_{t} A_{i}=\Delta A_{i}+\left[A_{j}, 2 \partial_{j} A_{i}-\partial_{i} A_{j}+\left[A_{j}, A_{i}\right]\right]+\xi_{i}
$$

Toy model：$\partial_{t} u=\Delta u+u^{3}+\xi$
SHE $\partial_{t} \Psi=\Delta \Psi+\xi$ has regularity $\Psi \in \mathcal{C}^{0-}$ ．

$$
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$$
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Toy model: $\partial_{t} u=\Delta u+u \partial u+\xi$
$\left\{u: u=\Psi+v, v \in \mathcal{C}^{1}\right\}$ is not good: $\Psi \in \mathcal{C}^{0-}, \partial v \in \mathcal{C}^{0}$
Regularity structures (started by Hairer'14): Let $P$ be heat kernel


- "Generalized Taylor" approximations by stochastic basis;
- The stochastic basis have explicit probability laws;
- Products of them require renormalization.

$$
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Toy model: $\partial_{t} u=\Delta u+u \partial u+\xi$
$\left\{u: u=\Psi+v, v \in \mathcal{C}^{1}\right\}$ is not good: $\Psi \in \mathcal{C}^{0-}, \partial v \in \mathcal{C}^{0}$
Regularity structures (started by Hairer'14): Let $P$ be heat kernel

$$
\begin{aligned}
v(y)=v(x) & +(P * \partial \Psi(y)-P * \partial \Psi(x)) v(x) \\
& +(P * \Psi(y)-P * \Psi(x)) w(x) \\
& +(P *(\Psi \partial \Psi)(y)-P *(\Psi \partial \Psi)(x)) \\
& +(y-x) w(x)+o(|y-x|) \quad \text { for } y \approx x
\end{aligned}
$$

- "Generalized Taylor" approximations by stochastic basis;
- The stochastic basis have explicit probability laws;
- Products of them require renormalization.

$$
\partial_{t} u=\Delta u+F(u)+\xi
$$

Theory of regularity structures

$\mathcal{M}$ : space of maps realizing formal Taylor basis as functions The terms in the "Taylor basis" form a Hopf algebra $\mathcal{H}$ The "renormalization group $\mathcal{G}$ " is the group of characters of $\mathcal{H}$.

## Solve the SPDE in 3D

- 100+ terms need renormalisation.
[Bruned,Chandra,Chevyrev,Hairer,Zambotti, 2014-2019] Systematic treatment of renormalisation, on the levels of algebra, diagrams, and equations.

In our work, using some category theory, we extend these results to SPDEs taking values in certain algebraic structures e.g. Lie algebra in the case of YM

- But it turns out that all these terms lead to $-C A_{i}$ in the renormalized equation.

We developed general results: renormalization should respect to symmetries such as spatial reflection

Gauge-covariant process and project to the orbit space

$$
\partial_{t} A_{i}=\Delta A_{i}+\left[A_{j}, 2 \partial_{j} A_{i}-\partial_{i} A_{j}+\left[A_{j}, A_{i}\right]\right]-C A_{i}+\xi_{i}
$$



$$
g^{-1} \partial_{t} g=\partial_{j}\left(g^{-1} \partial_{j} g\right)+\left[A_{j}, g^{-1} \partial_{j} g\right]
$$

Then $B:=g \circ A$ satisfies
$\partial_{t} B_{i}=\Delta B_{i}+\left[B_{j}, 2 \partial_{j} B_{i}-\partial_{i} B_{j}+\left[B_{j}, B_{i}\right]\right]-C\left(B_{i}-g \partial_{i} g^{-1}\right)+g \xi_{i} g^{-1}$

- finite shift of $C$ such that the limit is gauge covariant.


## Lattice Yang-Mills

Wilson, Villain, Manton etc.

[S.-Zhu-Zhu'22] "strong coupling regime", positive curvature group:

- The invariant measure for the dynamic is unique on entire $\mathbf{Z}^{d}$.
- Dynamic is exponentially ergodic.
- Log-Sobolev inequality $\Rightarrow$ correlations decay exponentially (mass gap).
(Our "strong coupling" condition is better than [Osterwalder-Seiler'78])
"Universality" in 2D (i.e. lattice $\rightarrow$ continuum)
[Chevyrev-S. 2023]
For a wide class of lattice YM models,
- Their dynamics converge to the same limiting dynamic.

$$
\partial_{t} A_{i}=\Delta A_{i}+\left[A_{j}, 2 \partial_{j} A_{i}-\partial_{i} A_{j}+\left[A_{j}, A_{i}\right]\right]-C A_{i}+\xi_{i}
$$

- Their invariant measures converge to the same limiting measure. An important step: unique $C$, such that $A$ is gauge covariant.

Uniqueness of $C$ (Abelian case $G=U(1)$ : topological argument)

$$
\partial_{t} A_{i}=\Delta A_{i}+C A_{i}+\xi_{i} \quad(i=1,2) \quad \text { on } \mathbf{R}_{+} \times \mathbf{T}^{2}
$$

Gauge transformation: $g \circ A=A-d g g^{-1}$ where $g$ is $U(1)$ valued. Wilson loop observable: $\exp \left(\int_{\ell} A\right)$ for a loop $\ell$.
It's gauge invariant, because $\int_{\ell} d g g^{-1} \in 2 \pi i \mathbf{Z}$
Claim: Solution is gauge covariant if and only if $C=0$.
Proof: $C \neq 0$ case: Consider $A(0)=2 \pi i d x_{1}$ and $\bar{A}(0)=0$
They are gauge equivalent $A(0)=\bar{A}(0)-d e^{-2 \pi i x_{1}} e^{2 \pi i x_{1}}$

$$
A(t)=\bar{A}(t)+e^{t C} A(0)
$$

Take $\ell(s)=(s, 0) \subset \mathbf{T}^{2}$. We have $\mathbf{E} \exp \left(\int_{\ell} \bar{A}(t)\right) \neq \mathbf{E} \exp \left(\int_{\ell} A(t)\right)$.

## Uniqueness in the case of non-abelian Lie groups

"Euler estimates": for small $t$, nonlinear effect is of next order comparing to the discrepancy created in the previous page

Need a curve $\zeta:[0,1] \rightarrow \mathfrak{g}$ with $\zeta(0)=0 \neq \zeta(1)$ such that its lift $L:[0,1] \rightarrow G$ is given by

$$
d L L^{-1}=d \zeta \quad L(0)=L(1)=i d
$$

This is done using sub-Riemannian geometry (Chow-Rashevsky).

## Global solution in 2D via Bourgain's argument

[Chevyrev-S. 2023]
This requires moments bound for invariant measure, which is achieved by gauge fixing and "rough Uhlenbeck estimates"
K.Uhlenbeck'82: "Connections with $L^{p}$ bounds on curvature"

- Assuming $A$ is small, one can bound $A$ by curvature $F_{A}$ in Coulomb gauge
- Piece together local bounds by "continuity argument".

1. Large scales: we fix axial gauge
2. At intermediate scales where $A$ becomes reasonably small, we fix Coulomb gauge (all the way down to the smallest scales).
Axial gauge: good probability properties; Coulomb gauge: good regularity properties.

Yang-Mills:

- Manifolds and bundles (in progress)
- 3 space dimension: we haven't obtained long time solution.
- 4 space dimension: not even local solution

Other models:

- Conformal field theories
[Dubedat,S.'18] Liouville CFT $\Rightarrow$ Stochastic Ricci flow
(motivated by [Osgood-Phillips-Sarnak'88])
- Progress in fermionic fields; supersymmetries;
- elliptic and hyperbolic stochastic PDEs

Thank you!


[^0]:    ${ }^{1}$ Kolmogorov: $\mathbf{E}\left[\left|X_{t}-X_{s}\right|^{\alpha}\right] \leq C|t-s|^{1+\beta}$ implies $X$ is Hölder $\frac{\beta}{\alpha}-$ continuous $\overline{\underline{\bar{~}}}$ Page $16 / 28$

