Fourier restriction type problems: New developments in the last 15 years

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Contents

Local smoothing for the wave equation

Problem

Suppose $u(x_1, x_2, t)$ *solves*

$$\begin{cases} u_{tt} = \Delta_x(u), \\ u|_{t=0} = g, u_t|_{t=0} = 0. \end{cases} \tag{1}$$

How large can $p > 2$ *be s.t.:*
$\|u\|_{L^p(\mathbb{R}^2 \times [0,1])}$ *is "almost bounded" by* $\|g\|_{L^p(\mathbb{R}^2)}$*?*

Conjecture (Sogge, 1991)

p *can go all the way up to* 4*.*

The Lindelöf hypothesis

Problem

How small can you make $\varepsilon > 0$ s.t.

$$\left|\zeta(\frac{1}{2} + \mathrm{i}t)\right| \lesssim_\varepsilon t^\varepsilon, t > 1.$$

Conjecture (Lindelöf hypothesis, 1908)

Every $\varepsilon > 0$ works.

Falconer's distance conjecture

Problem

$E \subset \mathbb{R}^2$ compact. How small can you make α s.t.

$$\dim E = \alpha \Rightarrow |\Delta(E)| = |\{|x-y|, x, y \in E\}| > 0?$$

Conjecture (Falconer, 1985)

$\alpha > 1$ suffices.

A connection

- Recent progress on all three.
- They all have to do with Fourier restriction type problems!

Fourier transform

- Fourier transform: $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi \mathrm{i} x\cdot\xi} \mathrm{d}x$.
- Fourier inversion: $f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi \mathrm{i} x\cdot\xi} \mathrm{d}\xi$.
- FT is an L^2-isometry (Plancherel, known as orthogonality).

Fourier restriction type problems

- A Fourier restriction type problem asks: If supp$\hat{f} \subset M$ (submanifold), then, for some $X \subset \mathbb{R}^n$, can you predict and prove a good estimate of $\|f\|_{L^p(X)}$?
- $p \geq 2$ in this talk.
- Harmless assumption: X in a large ball B_R, $R \to \infty$.
- Also harmless: Allowing an R^ε-loss.

More concretely:

In local smoothing,

- $M = \{|x_3| = |(x_1, x_2)|, |(x_1, x_2)| \le 1\}$, truncated unit cone.
- $X = B_R^3$. $R > 1$.
- Want:
$$\frac{1}{R}\|f\|_{L^4(B_R^3)}^4 \lesssim_\varepsilon R^\varepsilon \|f\|_{L^4(\mathbb{R}^2)}^4.$$

M can be other curved manifolds in other applications.

- E.g. $M = S^1$ (unit circle) $\subset \mathbb{R}^2$ in Falconer.

What's special about Fourier restriction type problems?

- If supp$\hat{f} \subset S^{n-1} \subset \mathbb{R}^n$, then in B_R,

$$f = \sum_T f_T.$$

- Each f_T morally supported inside a tube T of thickness $R^{\frac{1}{2}}$ and length R.
- Each $|f_T|$ morally a constant on T.
- Different T have different positions (due to curvature of M).
- Similar but different decomposition for the cone.

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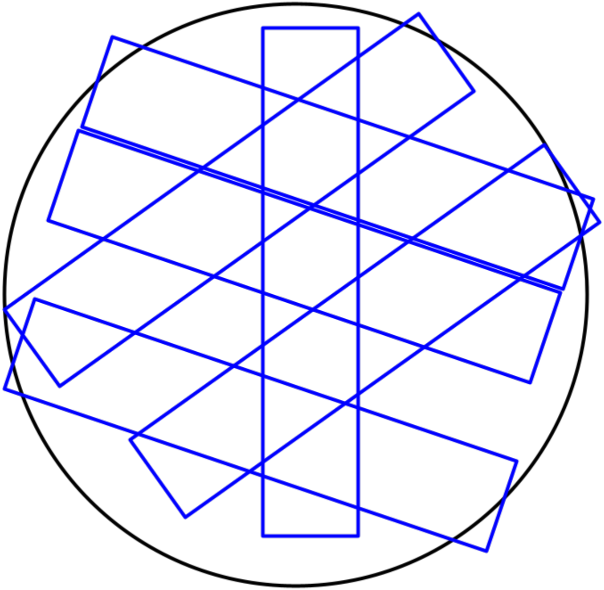
- Wave packet decomposition.

f = sum of wave packets



Combinatorics of tubes

- Wave packets are L^2 orthogonal where they meet.
- Many wave packets squeezed together can increase the L^p norm $(p > 2)$.
- Fourier restriction type problems are closely related to incidence problems.
- Incidence problems are about e.g. how tubes can (not) overlap (too much).

Incidence problems of tubes, etc.

- Incidence problems are about e.g. how tubes can (not) overlap (too much).
- Combinatorial.
- Source of many recent developments.
- Example of application: Local smoothing conjecture in dim $2+1$ is true (Guth-Wang-Z., 2020).

Central open problems in the field

Conjecture (Fourier Restriction)

If $1 < q < \frac{2n}{n+1}$*, then for every* $f \in L^q(\mathbb{R}^n)$*,* $\hat{f}$ *can be meaningfully restricted to* S^{n-1} *as an integrable function.*

Conjecture (Kakeya)

If a compact set in $\mathbb{R}^n$ *contains a unit line segment in every direction, then it has dimension* n*.*

New developments in recent 15 years

- Many new developments made, including ones on all three problems in the beginning.
- New ideas and tools behind them.
- We will take a look at three: two related to tube geometry and another exploiting different scales of the problem.

1. The polynomial method

- Influenced by computer science/combinatorics. Also: roots in number theory.
- Idea: Using zero sets of a polynomial to cut up the space to partition the function f evenly.
- Advantage: Because of algebra, tubes don't like to enter many pieces.
- Major applications: Finite field Kakeya (Dvir, 2009), new proof of multilinear Kakeya (Guth, 2010), new framework and progress for Restriction Conjecture (Guth, 2016 and 2018), ...

Multilinear Kakeya via polynomials

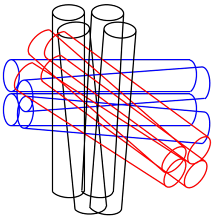
- Multilinear Kakeya (MK): n transverse families of tubes in $\mathbb{R}^n$ cannot overlap much. Sharp.
- Guth (2010) proved a slightly stronger version of MK by the polynomial method.
- Both Multilinear Kakeya and the polynomial method became very useful in the area.

Multilinear Kakeya

Theorem (Multilinear Kakeya, Bennett-Carbery-Tao, 2006)

$\mathbb{T}_1, \ldots, \mathbb{T}_n$: *families of* $1 \times \cdots \times 1 \times N$- *cylinders in* $\mathbb{R}^n$.
Direction of $T \in \mathbb{T}_j$ *is sufficiently close to* x_j*-axis. Then*

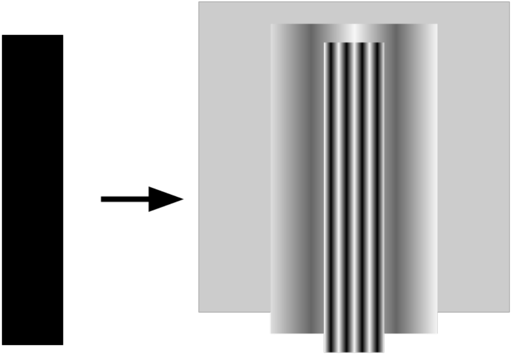
$$\|(\prod_{j=1}^{n}(\sum_{T\in\mathbb{T}_j} 1_T))^{\frac{1}{n}}\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim_\varepsilon N^\varepsilon \prod_{j=1}^{n} |\mathbb{T}_j|^{\frac{1}{n-1}}, \forall \varepsilon > 0.$$



2. The high/low method

- Influenced by incidence theorems over finite fields and geometric measure theory.
- Idea: Decompose 1_T by Fourier analysis. Low frequency part is spread out and high frequency part has orthogonality.
- Advantage: Being spread out and being orthogonal are both good for provable L^p bounds $(p > 1)$.
- Major applications: Local smoothing in $2 + 1$ dimensions (Guth-Wang-Z., 2020), ...

The high/low decomposition



Decomposing the characteristic function of a rectangle into oscillating parts + spread out parts.

3. New ways to induct on scales

- A key tool known to people around 2000. E.g. in Wolff's work.
- Idea: By (parabolic) rescaling, a small piece of a positively curved hypersurface looks like the standard paraboloid. Analyzing the contribution of that small piece often looks like the same problem in a smaller scale.
- Advantage: Finding the "important scale" for a function f in B_R. Gives us more assumptions for free.
- Major applications: Decoupling (Bourgain-Demeter, 2015), ...

Decoupling for the sphere

Theorem (Bourgain-Demeter, 2015)

$R > 1$. *Write the* $\frac{1}{R^2}$*-neighborhood* Ω *of* $S^{n-1} \subset \mathbb{R}^n$ *as union of (approx.) boxes* θ *of dimensions* $\frac{1}{R} \times \cdots \times \frac{1}{R} \times \frac{1}{R^2}$*.*
Suppose: $\hat{f} \subset \Omega$*. Contributions of* θ *to* f *is* f_θ*.*
Then

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon R^\varepsilon \left(\sum_\theta \|f_\theta\|^2_{L^p(\mathbb{R}^n)} \right)^{\frac{1}{2}}, \forall 2 \le p \le \frac{2(n+1)}{n-1}.$$

Decoupling leads to...

- Proof of main conjecture in Vinogradov' s Mean Value Theorem (Bourgain-Demeter-Guth, 2016).
- World record on Lindelöf (Bourgain, 2017).
- Affirmative answer to Carleson's problem for free Schrödinger solutions (Du-Guth-Li, 2017; Du-Z., 2019)
- World record on 2D Falconer (Guth-Iosevich-Ou-Wang, 2020).

Looking forward...

- The landscape of the subject has been rapidly evolving. Many cutting-edge discoveries. Some even sharp.
- We are still far from solving Restriction and Kakeya.
- More to be discovered in the future!

Possible direction 1: New methods for incidence estimates

- Geometry of tubes has been important to the subject, especially since Bourgain's work in 1991.
- Some very recent work on Restriction Conjecture (Wang, 2022; Wang-Wu, 2022) exploit tube geometry in new ways.
- A lot of new developments in geometric measure theory + discretized sum-product (dates back to Bourgain, 2003). Going from ε-improvement to quantitative gain.
- Exciting recent results (..., Orponen-Shmerkin-Wang, 2022; Wang-Zahl, 2022, ...). Great potential.

Possible direction 2: Connections to real algebraic geometry and o-minimal geometry

- Real algebraic geometry shows up naturally in the polynomial method and the restriction theory of algebraic varieties.
- Examples of applications: High dimensional Polynomial Wolff Axiom (Katz-Rogers, 2018). A stationary set method (Basu-Guo-Z.-Zorin-Kranich, 2023).
- Useful ideas from logic.

Possible direction 3: More algebra; unconventional thoughts and tools?

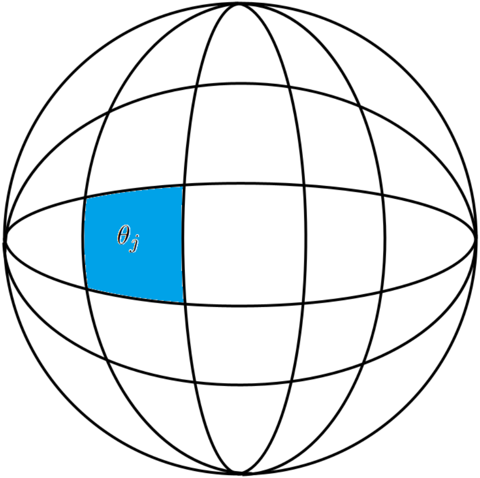
- Proof of p-adic Kakeya (Arsovski, 2023). "More advanced" algebra involved.
- New proof of finite field Kakeya (Dhar-Dvir, 2021).
- Current polynomial method is powerful but not good in many problems. Next level tools from algebra, differential geometry, etc. to boost its strength?
- Converse direction: Subs of alg. topology for finite fields?

An interesting question inspired by Dhar-Dvir's work

- $N > 1$. Cut S^2 into N^2 approx. sq. pieces (caps) $\theta_j, 1 \leq j \leq N^2$. Each has diameter $\sim \frac{1}{N}$.
- Define

$$|\theta_i \wedge \theta_j \wedge \theta_k| = \sup_{v_i \in \theta_i, v_j \in \theta_j, v_k \in \theta_k} |v_i \wedge v_j \wedge v_k|.$$

Caps on S^2



For example, $|\theta_j \wedge \theta_j \wedge \theta_j| \sim \frac{1}{N^2}$ by definition.

An open problem

Problem

If $a_{i,jk} \in \mathbb{R}^+$ *s.t. each*

$$a_{i,jk} \sim \frac{1}{|\theta_i \wedge \theta_j \wedge \theta_k|},$$

is it true that the rank of $(a_{i,jk})_{N^2 \times N^4}$ *is* $\gtrsim N^2$*?*

Thank you!